1. This exercise is about the $SO(3)$ group of rotations in three space dimensions.

   (a) A vector $\mathbf{v}$ rotated through an infinitesimal angle $d\alpha$ around axis $\mathbf{n}$ becomes $\mathbf{v}' = \mathbf{v} + (d\alpha)\mathbf{n} \times \mathbf{v}$. Show that a rotation through a *finite* angle α results in

   $$\mathbf{v}' \;=\; \cos\alpha\,\mathbf{v} \;+\; \sin\alpha\,\mathbf{n}\times\mathbf{v} \;+\; (1-\cos\alpha)\mathbf{n}(\mathbf{n}\cdot\mathbf{v}). \tag{1}$$

   Hint: $\mathbf{n}\times\mathbf{n}\times\mathbf{v} = -\mathbf{v} + \mathbf{n}(\mathbf{n}\cdot\mathbf{v})$ and hence $\mathbf{n}\times\mathbf{n}\times\mathbf{n}\times\mathbf{v} = -\mathbf{n}\times\mathbf{v}$.

   (b) Re-express formula (1) as $v'_i = R_{ij}(\alpha,\mathbf{n})v_j$ and write down the explicit form of the rotation matrix $R_{ij}(\mathbf{n},\alpha)$ and show that it is an $SO(3)$ matrix — real, orthogonal 3×3 matrix with determinant $\det(R) = +1$. (Orthogonality is the real-number analogue of unitarity: An orthogonal matrix satisfies $R\circ R^{\top} = R^{\rightarrow}\circ R = 1$.)

   (c) Optional exercise: Show that any $SO(3)$ matrix R is a rotation matrix $R(\alpha,\mathbf{n})$ for some angle α and some axis $\mathbf{n}$.
   Hint: Show that an $SO(3)$ matrix has eigenvalues $(e^{+i\alpha}, e^{-i\alpha}, +1)$, then identify α as the rotation angle and the eigenvector for the $+1$ eigenvalues as the axis of rotations.

   In the $SO(3)$ matrix language, the multiplication law for successive rotations is simply the matrix product $R_3 = R_2\circ R_1$, or in index terms $R^{(3)}_{ik} = R^{(2)}_{ij}R^{(1)}_{jk}$. Thus, if we first rotate through angle α_1 around axis $\mathbf{n}_1$ and then rotate through angle α_2 around axis $\mathbf{n}_2$, then the net effect is the $SO(3)$ matrix

   $$R(\alpha_2,\mathbf{n}_2)\circ R(\alpha_1,\mathbf{n}_1) \;=\; R_3 \;=\; R(\alpha_3,\mathbf{n}_3) \quad \text{for some } \alpha_3 \text{ and } \mathbf{n}_3\,, \tag{2}$$

   where the second equality follows from part (c). Alas, calculating the net rotation's angle $\mathbf{n}_3$ and axis α_3 directly from this formula is painfully tedious.

   Instead, there is a simpler Cayley–Klein method; originally, it involved quaternions, but later was rephrased in terms of the $SU(2)$ matrices, — *i.e.*, complex unitary 2×2 matrices of unit determinant. Here is how it works: For any rotation $R(\alpha,\mathbf{n})$, let's define an $SU(2)$ matrix

   $$Q(\alpha,\mathbf{n}) \;=\; \exp\bigl(-i\tfrac{\alpha}{2}\,\mathbf{n}\cdot\vec{\sigma}\bigr) \;=\; \cos\tfrac{\alpha}{2} \;-\; i\sin\tfrac{\alpha}{2}\,\mathbf{n}\cdot\vec{\sigma}. \tag{3}$$

   where $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices, *cf.* homework set#3 (problem#1).

(d) Show that

$$Q^\dagger(\alpha,\mathbf{n})\sigma_i Q(\alpha,\mathbf{n}) \;=\; R_{ij}(\alpha,\mathbf{n})\,\sigma_j\ . \tag{4}$$

(e) Now suppose the angles $\alpha_{1,2,3}$ and the unit vectors $\mathbf{n}_{1,2,3}$ satisfy the Cayley–Klein equation

$$Q(\mathbf{n}_3,\alpha_3) \;=\; Q(\mathbf{n}_2,\alpha_2)\,Q(\mathbf{n}_1,\alpha_1). \tag{5}$$

Use eq. (4) to show that the corresponding 3×3 rotation matrices satisfy eq. (2).

(f) Finally, solve the Cayley-Klein equation (5) for the $(\mathbf{n}_3,\alpha_3)$ in terms of the $(\mathbf{n}_2,\alpha_2)$ and the $(\mathbf{n}_1,\alpha_1)$.

2. In the Heisenberg picture of the rotational symmetries, a rotation through angle α around axis $\mathbf{n}$ transforms an operator $\hat{A}$ into

$$\hat{A}' \;=\; \hat{\mathcal{R}}^\dagger(\mathbf{n},\alpha)\,\hat{A}\,\hat{\mathcal{R}}(\mathbf{n},\alpha). \tag{6}$$

Consequently, a scalar operator $\hat{S}$ must be invariant under all rotations,

$$\hat{\mathcal{R}}^\dagger(\mathbf{n},\alpha)\,\hat{S}\,\hat{\mathcal{R}}(\mathbf{n},\alpha) \;=\; \hat{S}, \tag{7}$$

while the 3 components $(\hat{V}_x,\hat{V}_y,\hat{V}_z)$ of a vector operator $\hat{\mathbf{V}}$ transform into each other as

$$\hat{\mathcal{R}}^\dagger(\mathbf{n},\alpha)\,\hat{V}_i\,\hat{\mathcal{R}}(\mathbf{n},\alpha) \;=\; R_{ij}(\mathbf{n},\alpha)\,\hat{V}_j\,. \tag{8}$$

If fact, eqs. (7) is a definition of a scalar operator while eq. (8) is a definition of a vector operator.

These definitions of scalar and vector operators can be restated in terms of commutation relations with the angular momentum operators $\hat{J}_{x,y,z}$:

$$\hat{S}\text{ is a scalar iff }[\hat{S},\hat{J}_i] \;=\; 0, \tag{9}$$

$$\hat{\mathbf{V}}\text{ is a vector iff }[\hat{V}_i,\hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\hat{V}_k\,. \tag{10}$$

(a) Show that eqs. (7) and (9) for a scalar operator are equivalent to each other: is $\hat{S}$ remains invariant under all rotations as in eq. (7) then it must commute with the $\hat{J}_{x,y,z}$, and conversely if $\hat{S}$ commutes with all 3 $\hat{J}_{x,y,z}$ then it's invariant under all rotations.

For the vector operators, eqs. (8) and (10) are also equivalent to each other, but this takes a bit more work to prove. In particular, we need the Baker–Hausdorff lemma: For any two operators $\hat{B}$ and $\hat{C}$,

$$e^{\hat{B}}\hat{C}e^{-\hat{B}} \;=\; \hat{C} \;+\; [\hat{B},\hat{C}] \;+\; \tfrac{1}{2}[\hat{B},[\hat{B},\hat{C}]] \;+\; \cdots \;+\; \tfrac{1}{n!}[\hat{B},[\hat{B},\ldots,[\hat{B},\hat{C}]\ldots]]_n \;+\; \cdots. \tag{11}$$

(b) Prove this lemma. Hint: Let $\hat{C}_\lambda = e^{\lambda\hat{B}}\hat{C}e^{-\lambda\hat{B}}$, show that $\frac{d}{d\lambda}\hat{C}_\lambda = [\hat{B},\hat{C}_\lambda]$, iterate for the higher derivatives, and then expand into a series in powers of λ.

(c) Show that if the 3 components $\hat{V}_i$ of a vector operator $\hat{\mathbf{V}}$ transform into each other under space rotations according to eq. (8), then their commutation relations with the angular momentum components should be as in eq. (10). Hint: consider infinitesimally small α and work to the first order in α.

(d) Now, suppose three component operators $\hat{V}_{x,y,z}$ satisfy the commutation relations (10). Use the Baker-Hausdorff lemma to verify eq. (8) for any finite rotation.
Hint: $\hat{\mathcal{R}}(\mathbf{n},\alpha) = \exp(-i\frac{\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}})$.

Note that for spin-half system with no other degrees of freedom, $\hat{\mathbf{J}}$ is simply the spin $\hat{\mathbf{S}} = \frac{\hbar}{2}\vec{\sigma}$ and hence $\hat{\mathcal{R}}(\mathbf{n},\alpha) = U(\mathbf{n},\alpha)$ (*cf.* eq. (3)). Thus, eq. (4) is simply a special case of the general formula (8).

(e) Finally, consider the tensor operators. By definition, the component operators $\hat{T}_{i_1,i_2,\ldots,i_n}$ form an n-index tensor operator if and only if for any rotation,

$$\hat{\mathcal{R}}^\dagger\hat{T}_{i_1,i_2,\ldots,i_n}\hat{\mathcal{R}} \;=\; R_{i_1j_1}R_{i_2j_2}\cdots R_{i_nj_n}\,\hat{T}_{j_1,j_2,\ldots,j_n}\;. \tag{12}$$

Show that $\hat{T}_{i_1,i_2,\ldots,i_n}$ form a tensor if and only if

$$\left[\hat{T}_{i_1,i_2,\ldots,i_n},\hat{J}_j\right] \;=\; i\hbar\epsilon_{i_1jk}\hat{T}_{k,i_2,\ldots,i_n} \;+\; \cdots \;+\; i\hbar\epsilon_{i_njk}\hat{T}_{i_1,\ldots,i_{n-1},k}\;. \tag{13}$$

(f) Use $RR^\top = 1$ and $\det(R) = 1$ to show that for any two-index tensor operator $\hat{A}_{ij}$, $\hat{B} = \delta_{ij}\hat{A}_{ij}$ is a scalar and $\hat{C}_i = \epsilon_{ijk}\hat{A}_{jk}$ is a vector. Then use these facts to show that for any two vector operators $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ — regardless of whether they commute with each other or not, — the dot product $\hat{\mathbf{F}}\cdot\hat{\mathbf{G}}$ is a scalar operator and the cross product $\hat{\mathbf{F}}\times\hat{\mathbf{G}}$ is a vector operator.