

1. This problem is about the orbital angular momentum $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$ and the *spherical harmonics* $Y_{\ell m}(\theta, \phi)$ — the angular wave functions of quantum states with definite values of \hat{L}^2 and \hat{L}_z . In order to eliminate irrelevant degrees of freedom, let us consider a spinless particle living on a sphere; its quantum state is completely described by the angular wave function $\Psi(\vec{n}) \equiv \Psi(\theta, \phi)$, while the net angular momentum $\hat{\mathbf{J}}$ generating the rotational symmetry of the sphere is simply the orbital angular momentum $\hat{\mathbf{L}}$.

(a) In the Cartesian coordinate basis, the angular momentum operator acts as $\hat{\mathbf{L}} = -i\hbar \mathbf{x} \times \nabla$. Show that in the spherical coordinates, the \hat{L}_z and \hat{L}_{\pm} components of $\hat{\mathbf{L}}$ become

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}, \quad \hat{L}_{\pm} \equiv \hat{L}_x \pm i\hat{L}_y = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right). \quad (1)$$

Note the decoupling of these formulae from the radial coordinate r , hence the legitimacy of restriction on a sphere of fixed r .

(b) Show that for any integer m there is a unique wave function Ψ satisfying $\hat{L}_z \Psi = \hbar m \Psi$ and either $\hat{L}_+ \Psi = 0$ (for $m > 0$) or $\hat{L}_- \Psi = 0$ (for $m < 0$) or both (for $m = 0$).

Give explicit solution $\Psi(\theta, \phi)$, normalized to $\iint d^2\vec{n} |\Psi(\vec{n})|^2 = 1$.

(c) Use general properties of the angular momentum operators to prove that the wave functions you have just obtained belong to states $|\ell, m\rangle$ with $\ell = |m|$ and that for all pairs of integers (ℓ, m) with $\ell \geq |m|$ there should be a unique state $|\ell, m\rangle$.

(d) Without performing any explicit calculations, argue that together $Y_{\ell m}(\mathbf{n}) \equiv \langle \mathbf{n} | \ell, m \rangle$ (\mathbf{n} being a unit vector in some direction (θ, ϕ)) form a complete orthonormal basis for the angular wave functions. In other words,

$$\iint d^2\mathbf{n} Y_{\ell m}^*(\mathbf{n}) Y_{\ell' m'}(\mathbf{n}) = \delta_{\ell\ell'} \delta_{mm'} \quad \text{and} \quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\mathbf{n}) Y_{\ell m}^*(\mathbf{n}') = \delta^{(2)}(\mathbf{n} - \mathbf{n}') \quad (2)$$

Note: $d^2\mathbf{n} = \sin \theta d\theta d\phi$ and hence $\delta^{(2)}(\mathbf{n} - \mathbf{n}') = \delta(\theta - \theta') \delta(\phi - \phi') / \sin \theta$.

- (e) Applying general formulae for the matrix elements of the angular momentum operators to the case of \hat{L}_- , we have

$$\hat{L}_- |\ell, m\rangle = \hbar \sqrt{(\ell + m)(\ell + 1 - m)} |\ell, m - 1\rangle. \quad (3)$$

Use this formula recursively to show that

$$Y_{\ell, m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \frac{(-1)^\ell}{2^\ell \ell!} \sqrt{\frac{(\ell + m)!}{(\ell - m)!}} \times \frac{e^{im\phi}}{\sin^m \theta} \times \left[\frac{d^{\ell-m}}{dx^{\ell-m}} (1 - x^2)^\ell \right]_{x=\cos \theta} \quad (4)$$

for any integer ℓ and m with $|m| \leq \ell$. In particular, for $m = 0$,

$$Y_{\ell, 0}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \times P_\ell(\cos \theta) \quad (5)$$

where

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell \quad (6)$$

is the ℓ^{th} Legendre polynomial.

- (f) Prove $Y_{\ell, -m}(\theta, \phi) = (-1)^m Y_{\ell, m}^*(\theta, \phi)$.

Hint: prove and use

$$\frac{1}{(\ell - m)!} \frac{d^{\ell-m}}{dx^{\ell-m}} (1 - x^2)^\ell = (-1)^m (1 - x^2)^m \times \frac{1}{(\ell + m)!} \frac{d^{\ell+m}}{dx^{\ell+m}} (1 - x^2)^\ell. \quad (7)$$

- (g) Write down explicit formulae for $Y_{\ell, m}(\theta, \phi)$ for $\ell = 0, 1, 2$ and all legitimate m .

- (h) Finally, for extra credit, show that

$$\sum_{m=-l}^l Y_{\ell, m}^*(\mathbf{n}_1) Y_{\ell, m}(\mathbf{n}_2) = \frac{2\ell + 1}{4\pi} P_\ell(\mathbf{n}_1 \cdot \mathbf{n}_2). \quad (8)$$

Hint: First show that the left hand side is invariant under simultaneous rotations of \mathbf{n}_1 and \mathbf{n}_2 and use this invariance to rotate \mathbf{n}_1 into the north pole of the sphere ($\theta'_1 = 0$). Then show that $Y_{\ell, m}(\theta' = 0) = 0$ for $m \neq 0$ and use this fact to simplify the sum.

2. We have seen in class the the $SO(3)$ rotation group has both single-valued and double-valued representations, corresponding to integral and half-integral values of j , respectively. Both kinds of representations become single-valued in terms of the Spin(3) group — the double cover of the $SO(3)$; as discussed in class, Spin(3) is isomorphic to $SU(2)$.

The $SU(2)$ picture of the spin group is also more convenient for deriving the explicit rotation matrices

$$\mathcal{D}_{m'm}^{(j)}(\varphi, \mathbf{n}) \stackrel{\text{def}}{=} \langle j, m' | \hat{\mathcal{R}}(\varphi, \mathbf{n}) | j, m \rangle \quad (9)$$

for all representations (j). In this problem, we are going to construct the $\mathcal{D}_{m'm}^{(j)}$ matrix elements as explicit polynomials of the matrix elements $U_{\alpha\beta}$ of the $SU(2)$ matrix

$$U(\varphi, \mathbf{n}) = \exp(-i\varphi \mathbf{n} \cdot \vec{\sigma}/2) = \cos \frac{\varphi}{2} - i \sin \frac{\varphi}{2} \mathbf{n} \cdot \vec{\sigma}. \quad (10)$$

Our starting point is a system of two independent harmonic oscillators whose creation and annihilation operators $\hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_1$ and \hat{a}_2 obey the canonical commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0 = [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger], \quad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, \quad (11)$$

and a trio of model angular momentum operators

$$\hat{J}^i = \frac{\hbar}{2} \sum_{\alpha, \beta} \sigma_{\alpha\beta}^i \hat{a}_\alpha^\dagger \hat{a}_\beta, \quad (12)$$

where $\sigma_{\alpha\beta}^i$ are the matrix elements of the Pauli matrices σ^i ; this model was invented by Julian Schwinger.

- (a) Calculate the commutators $[\hat{J}^i, \hat{a}_\alpha]$ and $[\hat{J}^i, \hat{a}_\alpha^\dagger]$.
- (b) Verify that $[\hat{J}^i, \hat{J}^j] = i\hbar \epsilon^{ijk} \hat{J}^k$; it is this relation that allows us to treat the \hat{J}^i as model angular momenta.

(c) Prove that

$$\vec{J}^2 = \hbar^2 \frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right), \quad \text{where } \hat{N} \equiv \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2. \quad (13)$$

Hint: First express \hat{J}_z and \hat{J}_\pm explicitly in terms of $\hat{a}_{1,2}$ and $\hat{a}_{1,2}^\dagger$; then compute $\vec{J}^2 = \hat{J}_z^2 + \frac{1}{2}\{\hat{J}_+, \hat{J}_-\}$.

(d) Show that for this model the states with definite values of j and m are precisely the states with definite numbers of oscillator quanta n_1 and n_2 . Specifically,

$$|j, m\rangle = |n_1 = j + m, n_2 = j - m\rangle = ((j+m)!(j-m)!)^{-1/2} (\hat{a}_1^\dagger)^{j+m} (\hat{a}_2^\dagger)^{j-m} |0\rangle, \quad (14)$$

where $|0\rangle$ is the ground state of the two-oscillator system.

Consequently, the Hilbert space of the model comprises one and only one copy of each allowed multiplet of the angular momentum algebra.

Now consider the rotation operators $\hat{\mathcal{R}}(\varphi, \mathbf{n}) = \exp(-i\varphi \mathbf{n} \cdot \hat{\mathbf{J}}/\hbar)$ generated by the model angular momentum operators (12).

(e) Show that for any such rotation $\hat{\mathcal{R}}|0\rangle = |0\rangle$.

Hint: Prove and use $\hat{\mathbf{J}}|0\rangle = 0$.

(f) Use commutation relations (a) and the Baker–Hausdorff lemma to show that

$$\hat{\mathcal{R}}(\varphi, \mathbf{n}) \hat{a}_\alpha^\dagger \hat{\mathcal{R}}^\dagger(\varphi, \mathbf{n}) = \sum_\gamma \hat{a}_\gamma^\dagger U_{\gamma\alpha}(\varphi, \mathbf{n}) \quad (15)$$

where $U_{\gamma\alpha}(\varphi, \mathbf{n})$ are the matrix elements of the matrix (10).

(g) Now comes the crucial step: Use the results of (d), (e) and (f) to show that for the Schwinger's model

$$\hat{\mathcal{R}}(\varphi, \mathbf{n}) |j, m\rangle = \sum_{m'=-j}^{+j} |j, m'\rangle \mathcal{D}_{m'm}^{(j)}(\varphi, \mathbf{n}) \quad (16)$$

and express the matrix elements $\mathcal{D}_{m'm}^{(j)}(\varphi, \mathbf{n})$ as polynomials of degree $2j$ in the matrix elements $U_{\gamma\alpha}(\varphi, \mathbf{n})$ of the $SU(2)$ matrix (10). In particular, for $j = \frac{1}{2}$, you should get $\|\mathcal{D}^{(1/2)}\| = \|U\|$.

(h) Finally, explain why in any quantum system with a well-defined angular momentum, rotation operators must act according to

$$\hat{\mathcal{R}}(\varphi, \mathbf{n}) |j, m, n\rangle = \sum_{m'=-j}^{+j} |j, m', n\rangle \mathcal{D}_{m'm}^{(j)}(\varphi, \mathbf{n}) \quad (17)$$

with exactly the same rotation matrices $\left\| \mathcal{D}^{(j)}(\varphi, \mathbf{n}) \right\|$ as you have just computed for the Schwinger's model.