1. This problem is about the oribital angular momentum $\hat{\mathbf{L}} = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$ and the *spherical harmonics* $Y_{\ell m}(\theta, \phi)$ — the angular wave functions of quantum states with definite values of $\hat{\vec{L}}^2$ and $\hat{L}_z$. In order to eliminate irrelevant degrees of freedom, let us consider a spinless particle living on a sphere; its quantum state is completely described by the angular wave function $\Psi(\vec{n}) \equiv \Psi(\theta, \phi)$, while the net angular momentum $\hat{\mathbf{J}}$ generating the rotational symmetry of the sphere is simply the orbital angular momentum $\hat{\mathbf{L}}$.

   (a) In the Cartesian coordinate basis, the angular momentum operator acts as $\hat{\mathbf{L}} = -i\hbar \mathbf{x} \times \nabla$. Show that in the spherical coordinates, the $\hat{L}_Z$ and $\hat{L}_\pm$ components of $\hat{\mathbf{L}}$ become

   $$\hat{L}_z \;=\; -i\hbar \frac{\partial}{\partial \phi}\,, \qquad \hat{L}_\pm \;\equiv\; \hat{L}_x \pm i\hat{L}_y \;=\; \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right). \tag{1}$$

   Note the decoupling of these formulae from the radial coordinate r, hence the legitimacy of restriction on a sphere of fixed r.

   (b) Show that for any integer m there is a unique wave function Ψ satisfying $\hat{L}_z \Psi = \hbar m \Psi$ and either $\hat{L}_+ \Psi = 0$ (for $m > 0$) or $\hat{L}_- \Psi = 0$ (for $m < 0$) or both (for $m = 0$).

   Give explicit solution $\Psi(\theta, \phi)$, normalized to $\iint d^2\vec{n}\, |\Psi(\vec{n})|^2 = 1$.

   (c) Use general properties of the angular momentum operators to prove that the wave functions you have just obtained belong to states $|\ell, m\rangle$ with $\ell = |m|$ and that for all pairs of integers (ℓ, m) with $\ell \geq |m|$ there should be a unique state $|\ell, m\rangle$.

   (d) Without performing any explicit calculations, argue that together $Y_{\ell m}(\mathbf{n}) \equiv \langle \mathbf{n} | \ell, m \rangle$ ($\mathbf{n}$ being a unit vector in some direction (θ, ϕ)) form a complete orthonormal basis for the angular wave functions. In other words,

   $$\iint d^2\mathbf{n}\, Y^*_{\ell m}(\mathbf{n}) Y_{\ell' m'}(\mathbf{n}) \;=\; \delta_{\ell \ell'} \delta_{m m'} \quad \text{and} \quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\mathbf{n}) Y^*_{\ell m}(\mathbf{n}') \;=\; \delta^{(2)}(\mathbf{n} - \mathbf{n}') \tag{2}$$

   Note: $d^2\mathbf{n} = \sin\theta d\theta d\phi$ and hence $\delta^{(2)}(\mathbf{n} - \mathbf{n}') = \delta(\theta - \theta')\delta(\phi - \phi')/\sin\theta$.

(e) Applying general formulae for the matrix elements of the angular momentum operators to the case of $\hat{L}_-$, we have

$$\hat{L}_- \, |\ell, m\rangle \;=\; \hbar\sqrt{(\ell+m)(\ell+1-m)}\, |\ell, m-1\rangle \,. \tag{3}$$

Use this formula recursively to show that

$$Y_{\ell,m}(\theta,\phi) \;=\; \sqrt{\frac{2\ell+1}{4\pi}}\,\frac{(-1)^\ell}{2^\ell\,\ell!}\,\sqrt{\frac{(\ell+m)!}{(\ell-m)!}} \times \frac{e^{im\phi}}{\sin^m\theta} \times \left[\frac{d^{\ell-m}}{dx^{\ell-m}}(1-x^2)^\ell\right]_{x=\cos\theta} \tag{4}$$

for any integer ℓ and m with $|m| \leq \ell$. In particular, for $m = 0$,

$$Y_{\ell,0}(\theta,\phi) \;=\; \sqrt{\frac{2\ell+1}{4\pi}} \times P_\ell(\cos\theta) \tag{5}$$

where

$$P_\ell(x) \;=\; \frac{1}{2^\ell\,\ell!}\,\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \tag{6}$$

is the ℓ^{th} Legendre polynomial.

(f) Prove $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell,m}(\theta,\phi)$.
Hint: prove and use

$$\frac{1}{(\ell-m)!}\,\frac{d^{\ell-m}}{dx^{\ell-m}}(1-x^2)^\ell \;=\; (-1)^m(1-x^2)^m \times \frac{1}{(\ell+m)!}\,\frac{d^{\ell+m}}{dx^{\ell+m}}(1-x^2)^\ell. \tag{7}$$

(g) Write down explicit formulae for $Y_{\ell,m}(\theta,\phi)$ for $\ell = 0, 1, 2$ and all legitimate m.

(h) Finally, for extra credit, show that

$$\sum_{m=-l}^{l} Y^*_{\ell,m}(\mathbf{n}_1)\, Y_{\ell,m}(\mathbf{n}_2) \;=\; \frac{2l+1}{4\pi}\, P_\ell(\mathbf{n}_1\cdot\mathbf{n}_2)\,. \tag{8}$$

Hint: First show that the left hand side is invariant under simultaneous rotations of $\mathbf{n}_1$ and $\mathbf{n}_2$ and use this invariance to rotate $\mathbf{n}_1$ into the north pole of the sphere ($\theta'_1 = 0$). Then show that $Y_{\ell m}(\theta' = 0) = 0$ for $m \neq 0$ and use this fact to simplify the sum.

2. We have seen in class the the $SO(3)$ rotation group has both single-valued and double-valued representations, corresponding to integral and half-integral values of j, respectively. Both kinds of representations become single-valued in terms of the Spin(3) group — the double cover of the $SO(3)$; as discussed in class, Spin(3) is isomorphic to $SU(2)$.

   The $SU(2)$ picture of the spin group is also more convenient for deriving the explicit rotation matrices

$$\mathcal{D}^{(j)}_{m'm}(\varphi,\mathbf{n}) \stackrel{\text{def}}{=} \langle j,m' \big| \hat{\mathcal{R}}(\varphi,\mathbf{n}) \,|j,m\rangle \tag{9}$$

   for all representations (j). In this problem, we are going to construct the $\mathcal{D}^{(j)}_{m'm}$ matrix elements as explicit polynomials of the matrix elements $U_{\alpha\beta}$ of the $SU(2)$ matrix

$$U(\varphi,\mathbf{n}) \;=\; \exp\left(-i\varphi\mathbf{n}\cdot\vec{\sigma}/2\right) \;=\; \cos\tfrac{\varphi}{2} \;-\; i\sin\tfrac{\varphi}{2}\,\mathbf{n}\cdot\vec{\sigma}. \tag{10}$$

   Our starting point is a system of two independent harmonic oscillators whose creation and annihilation operators $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$, $\hat{a}_1$ and $\hat{a}_2$ obey the canonical commutation relations

$$[\hat{a}_\alpha,\hat{a}_\beta] \;=\; 0 \;=\; [\hat{a}_\alpha^\dagger,\hat{a}_\beta^\dagger]\,, \qquad [\hat{a}_\alpha,\hat{a}_\beta^\dagger] \;=\; \delta_{\alpha\beta}\,, \qquad \alpha,\beta \;=\; 1,2, \tag{11}$$

   and a trio of model angular momentum operators

$$\hat{J}^i \;=\; \frac{\hbar}{2}\sum_{\alpha,\beta}\sigma^i_{\alpha\beta}\,\hat{a}_\alpha^\dagger\hat{a}_\beta\ , \tag{12}$$

   where $\sigma^i_{\alpha\beta}$ are the matrix elements of the Pauli matrices σ^i; this model was invented by Julian Schwinger.

   (a) Calculate the commutators $[\hat{J}^i,\hat{a}_\alpha]$ and $[\hat{J}^i,\hat{a}_\alpha^\dagger]$.

   (b) Verify that $[\hat{J}^i,\hat{J}^j] = i\hbar\epsilon^{ijk}\hat{J}^k$; it is this relation that allows us to treat the $\hat{J}^i$ as model angular momenta.

(c) Prove that

$$\vec{\hat{J}}^2 \;=\; \hbar^2 \, \frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right) , \qquad \text{where} \quad \hat{N} \;\equiv\; \hat{a}_1^\dagger \hat{a}_1 \;+\; \hat{a}_2^\dagger \hat{a}_2 \,. \tag{13}$$

Hint: First express $\hat{J}_z$ and $\hat{J}_\pm$ explicitly in terms of $\hat{a}_{1,2}$ and $\hat{a}^\dagger_{1,2}$; then compute $\vec{\hat{J}}^2 = \hat{J}_z^2 + \frac{1}{2}\{\hat{J}_+, \hat{J}_-\}$.

(d) Show that for this model the states with definite values of j and m are precisely the states with definite numbers of oscillatorial quanta n_1 and n_2. Specifically,

$$|j, m\rangle \;=\; |n_1 = j + m, n_2 = j - m\rangle \;=\; \Big((j+m)!\,(j-m)!\Big)^{-1/2} (\hat{a}_1^\dagger)^{j+m} (\hat{a}_2^\dagger)^{j-m}\,|0\rangle\,, \tag{14}$$

where $|0\rangle$ is the ground state of the two-oscillator system.

Consequently, the Hilbert space of the model comprises one and only one copy of each allowed multiplet of the angular momentum algebra.

Now consider the rotation operators $\hat{\mathcal{R}}(\varphi, \mathbf{n}) = \exp(-i\varphi\mathbf{n} \cdot \hat{\mathbf{J}}/\hbar)$ generated by the model angular momentum operators (12).

(e) Show that for any such rotation $\hat{\mathcal{R}}\,|0\rangle = |0\rangle$.

Hint: Prove and use $\hat{\mathbf{J}}\,|0\rangle = 0$.

(f) Use commutation relations (a) and the Baker–Hausdorff lemma to show that

$$\hat{\mathcal{R}}(\varphi, \mathbf{n})\, \hat{a}_\alpha^\dagger\, \hat{\mathcal{R}}^\dagger(\varphi, \mathbf{n}) \;= \sum_\gamma \hat{a}_\gamma^\dagger\, U_{\gamma\alpha}(\varphi, \mathbf{n}) \tag{15}$$

where $U_{\gamma\alpha}(\varphi, \mathbf{n})$ are the martix elements of the matrix (10).

(g) Now comes the crucial step: Use the results of (d), (e) and (f) to show that for the Schwinger's model

$$\hat{\mathcal{R}}(\varphi, \mathbf{n})\,|j, m\rangle \;=\; \sum_{m'=-j}^{+j} \big|j, m'\big\rangle\, \mathcal{D}^{(j)}_{m'm}(\varphi, \mathbf{n}) \tag{16}$$

and express the matrix elements $\mathcal{D}^{(j)}_{m'm}(\varphi, \mathbf{n})$ as polynomials of degree $2j$ in the matrix elements $U_{\gamma\alpha}(\varphi, \mathbf{n})$ of the $SU(2)$ matrix (10). In particular, for $j = \frac{1}{2}$, you should get $\|\mathcal{D}^{(1/2)}\| = \|U\|$.

(h) Finally, explain why in any quantum system with a well-defined angular momentum, rotation operators must act according to

$$\hat{\mathcal{R}}(\varphi, \mathbf{n}) \left| j, m, n \right\rangle \; = \sum_{m'=-j}^{+j} \left| j, m', n \right\rangle \mathcal{D}^{(j)}_{m'm}(\varphi, \mathbf{n}) \tag{17}$$

with exactly the same rotation matrices $\left\| \mathcal{D}^{(j)}(\varphi, \mathbf{n}) \right\|$ as you have just computed for the Schwinger's model.