1. In my notes on the spherical tensor operators and the Wigner–Eckart theorem, I made a few statements without proofs. This problem is where you prove those statements.

   By definition, a set of $2k+1$ component operators $\hat{T}^{(k)}_\mu$ (for $\mu=-k,\ldots,+k$) comprises a spherical tensor operator of rank k if and only if their commutators with the angular momentum operators $\hat{J}_i$ are

$$\left[\hat{J}_i,\hat{T}^{(k)}_\mu\right] \;=\; \sum_{\mu'}\hat{T}^{(k)}\times\left\langle j=k,m=\mu'\right|\hat{J}_i\left|j=k,m=\mu\right\rangle. \tag{1}$$

   Or in a more explicit form for $\hat{J}_z$ and $\hat{J}_\pm=\hat{J}_x\pm i\hat{J}_y$,

$$\begin{aligned}\left[\hat{J}_z,\hat{T}^{(k)}_\mu\right] &= \hbar\mu\times\hat{T}^{(k)}_\mu\,,\\ \left[\hat{J}_\pm,\hat{T}^{(k)}_\mu\right] &= \hbar\sqrt{(k\mp\mu)(k+1\pm\mu)}\times\hat{T}^{(k)}_{\mu\pm1}\,.\end{aligned} \tag{2}$$

   (a) Show that under finite rotations $R(\phi,\mathbf{n})$, the components of a spherical tensor operator $\hat{T}^{(k)}$ transform into linear combinations of each other according to

$$\hat{\mathcal{R}}(\phi,\mathbf{n})\hat{T}^{(k)}_\mu\hat{\mathcal{R}}^\dagger(\phi,\mathbf{n}) \;=\; \sum_{\mu'}T^{(k)}_{\mu'}\times\mathcal{D}^{(j=k)}_{\mu',\mu}(\phi,\mathbf{n}) \tag{3}$$

   where

$$\mathcal{D}^{(j)}_{m',m}(\phi,\mathbf{n}) \;=\; \left\langle j,m'\right|\hat{\mathcal{R}}(\phi,\mathbf{n})\left|j,m\right\rangle, \tag{4}$$

   cf. homework set#10, eqs. (16–17). Hint: use the Baker–Hausdorff lemma, *cf.* homework set#9, eq. (11).

   (b) Next, consider a vector operator $\hat{\mathbf{V}}$ obeying $[\hat{J}_i,\hat{V}_j]=i\hbar\epsilon_{ijk}\hat{V}_k$ (in Cartesian components of both $\hat{J}_i$ and $\hat{V}_j$). Show that the spherical components

$$\hat{V}_0 \;=\; \hat{V}_z,\qquad \hat{V}_{\pm1} \;=\; \frac{\mp\hat{V}_x-i\hat{V}_y}{\sqrt{2}} \tag{5}$$

   obey the commutation relations (2) for $k=1$.

   Please note that the $\hat{J}_\pm$ operators in eqs. (2) are defined using a different convention than the spherical components $\hat{J}_{\pm 1}$ of the vector operator $\hat{\mathbf{J}}$. Specifically, $\hat{J}_\pm \stackrel{\text{def}}{=} \hat{J}_x \pm i\hat{J}_y$ while $\hat{J}_{+1} = -\hat{J}_+/\sqrt{2}$ and $\hat{J}_{-1} = +\hat{J}_-/\sqrt{2}$.

(c) Finally, consider a 2-index symmetric traceless tensor operator $\hat{T}_{ij}$,

$$\hat{T}_{ji} = +\hat{T}_{ij}\,, \quad \hat{T}_{ii} = 0, \quad [\hat{J}_i, \hat{T}_{jk}] = i\hbar\epsilon_{ijm}\hat{T}_{mk} + i\hbar\epsilon_{ikm}\hat{T}_{jm}\,, \tag{6}$$

cf. homework set#9, eq. (13) for the commutator. This tensor operator is equivalent to the rank 2 spherical tensor operator $\hat{T}^{(2)}$ with components:

$$\begin{aligned} \hat{T}^{(2)}_{\pm 2} &= \tfrac{1}{2}(\hat{T}_{xx} - \hat{T}_{yy}) \pm i\hat{T}_{xy}\,, \\ \hat{T}^{(2)}_{\pm 1} &= \mp\hat{T}_{xz} - i\hat{T}_{yz}\,, \\ \hat{T}^{(2)}_{0} &= \sqrt{\tfrac{1}{6}}\,(2\hat{T}_{zz} - \hat{T}_{xx} - \hat{T}_{yy}). \end{aligned} \tag{7}$$

Verify that these component operator obey the commutation relations (2) for $k = 2$.

2. This second problem is about a completely different subject, the perturbation theory. Consider a particle in a two-dimensional infinitely deep square well; the un-perturbed Hamiltonian is

$$\hat{H}_0 = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + V_0(\hat{x}, \hat{y}) \quad \text{for} \quad V_0(x,y) = \begin{cases} 0 & \text{for } 0 \le x, y \le L, \\ +\infty & \text{everywhere else.} \end{cases} \tag{8}$$

(a) Diagonalize this un-perturbed Hamiltonian. That is, calculate all of its eigenvalues and write down the wave-functions of the corresponding eigenstates.

(b) Now let's perturb the Hamiltonian (8) by a small change to the potential

$$\delta V(x,y) = \lambda(x+y-L)^2 = \lambda(x-\tfrac{1}{2}L)^2 + \lambda(y-\tfrac{1}{2}L)^2 + 2\lambda(x-\tfrac{1}{2}L)(y-\tfrac{1}{2}L). \tag{9}$$

Focus on the 3 lowest-energy eigenstates of the perturbed Hamiltonian and calculate their energies to the first order in λ.

Hints: All 2d integrals you may need to evaluate for this problem can be written as are sums of products of 1d integrals. And here are some 1d integrals you may find useful:

$$
\begin{aligned}
&\text{for integer } k > 0: \\
&\int_0^L dx\, \cos\frac{k\pi x}{L} \;=\; 0, && (10) \\
&\int_0^L dx\, x \times \cos\frac{k\pi x}{L} \;=\; \frac{L^2}{k^2\pi^2} \times \big((-1)^k - 1\big), && (11) \\
&\int_0^L dx\, x^2 \times \cos\frac{k\pi x}{L} \;=\; \frac{2L^3}{k^2\pi^2} \times (-1)^k. && (12)
\end{aligned}
$$