QUANTA

Multiple Oscillators

Consider an N-dimensional harmonic oscillator. Or more generally, a system with N degrees of freedom $q_1, \ldots, q_n$, each being a harmonic oscillator. In terms of quantum operators, this means N position operators $\hat{q}_1, \ldots, \hat{q}_N$ and N momentum operators $\hat{p}_1, \ldots, \hat{p}_N$ obeying the canonical commutation relations

$$[\hat{q}_\alpha, \hat{q}_\beta] = 0, \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0, \quad [\hat{q}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta}\,, \quad \alpha, \beta = 1, \ldots, N, \tag{1}$$

and the Hamiltonian operator

$$\hat{H} = \sum_{\alpha=1}^{N} \left(\frac{1}{2m_\alpha} \hat{p}_\alpha^2 + \frac{m_\alpha \omega_\alpha^2}{2} \hat{q}_\alpha^2 \right). \tag{2}$$

To diagonalize this Hamiltonian, we define raising and lowering operators for each *mode* α,

$$\hat{a}_\alpha = \frac{m_\alpha \omega_\alpha \hat{q}_\alpha + i\hat{p}_\alpha}{\sqrt{2\hbar\omega_\alpha m_\alpha}}\,, \qquad \hat{a}_\alpha^\dagger = \frac{m_\alpha \omega_\alpha \hat{q}_\alpha - i\hat{p}_\alpha}{\sqrt{2\hbar\omega_\alpha m_\alpha}}\,, \tag{3}$$

then for each mode $[\hat{a}_\alpha, \hat{a}_\alpha^\dagger] = 1$ but the raising and lowering operators for different modes $\alpha \neq \beta$ commute with each other, thus,

$$\forall \alpha, \beta = 1, \ldots, N: \quad [\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0, \quad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}\,. \tag{4}$$

Consequently, the number operators

$$\hat{n}_\alpha = \hat{a}_\alpha^\dagger \hat{a}_\alpha \tag{5}$$

commute with each other, so we may diagonalize all of them in the same basis $\{|n_1, \ldots, n_N\rangle\}$. As we have learned for a single oscillator, the spectrum of each $\hat{n}_\alpha$ comprises non-negative integers $n_\alpha = 0, 1, 2, \ldots$. Moreover, each n_α can take any non-negative integer value *independently of all other* n_β.

To see this independence, note the raising and lowering operators $\hat{a}_\alpha^\dagger$ and $\hat{a}_\alpha$ commute with the numbering operators $\hat{n}_\beta$ for other modes $\beta \neq \alpha$,

$$[\hat{a}_\alpha^\dagger, \hat{n}_\beta] \;=\; +\delta_{\alpha\beta}\hat{a}_\alpha^\dagger\,, \quad [\hat{a}_\alpha, \hat{n}_\beta] \;=\; -\delta_{\alpha\beta}\hat{a}_\alpha\,, \tag{6}$$

hence in the $\{|n_1,\ldots,n_N\rangle\}$ basis the $\hat{a}_\alpha$ and the $\hat{a}_\alpha^\dagger$ leave all the $\hat{n}_{\beta\neq\alpha}$ unchanged while lowering or raising the n_α. Specifically,

$$\begin{aligned} \hat{a}_\alpha^\dagger \left|\{n_\beta\}\right\rangle \;&=\; \sqrt{n_\alpha+1}\,\left|\{n'_\beta = n_\beta + \delta_{\alpha\beta}\}\right\rangle, \\ \hat{a}_\alpha \left|\{n_\beta\}\right\rangle \;&=\; \begin{cases} \sqrt{n_\alpha}\,\left|\{n'_\beta = n_\beta - \delta_{\alpha\beta}\}\right\rangle & \text{for } n_\alpha > 0, \\ 0 & \text{for } n_\alpha = 0. \end{cases} \end{aligned} \tag{7}$$

Consequently, for *any combination of non-negative integers* $(n_1,\ldots,n_N)$ there does exist a quantum state with these eigenvalues of the $(\hat{n}_1,\ldots,\hat{n}_N)$ operators, namely

$$|n_1,\ldots,n_N\rangle \;=\; \prod_\alpha \frac{1}{\sqrt{n_\alpha!}} \prod_\alpha \left(\hat{a}_\alpha^\dagger\right)^{n_\alpha} |0,\ldots,0\rangle\,. \tag{8}$$

Note: in the wave-function language, the quantum states of this system is described by wave-functions $\psi(q_1,\ldots,q_N)$ depending on all N 'coordinates' $q_1,\ldots,q_N$ (but not on the momenta $p_1,\ldots,p_N$) rather than wave-functions $\psi_1(q_1),\ldots,\psi_N(q_N)$ of the individual coordinates. However, for the states (8) the wave-function of the whole state happens to factorize into wave-functions of the individual harmonic oscillators:

$$\begin{aligned} \langle q_1,\ldots,q_n|n_1,\ldots,n_N\rangle \;&=\; \prod_{\alpha=1}^N \langle q_\alpha|n_\alpha\rangle\,, \\ \textit{i.e.,}\quad \psi_{n_1,\ldots,n_N}(q_1,\ldots,q_N) \;&=\; \prod_{\alpha=1}^N \psi_{n_\alpha}^{\rm osc}(q_\alpha). \end{aligned} \tag{9}$$

Finally, in the Hamiltonian (2), for each mode α we have

$$\frac{1}{2m_\alpha}\,\hat{p}_\alpha^2 \;+\; \frac{m_\alpha\omega_\alpha^2}{2}\,\hat{q}_\alpha^2 \;=\; \hbar\omega_\alpha\left(\hat{n}_\alpha + \tfrac{1}{2}\right), \tag{10}$$

exactly as we had earlier for a single oscillator, hence

$$\hat{H} = \sum_{\alpha=1}^{N} \hbar\omega_\alpha \big(\hat{n}_\alpha + \tfrac{1}{2}\big). \tag{11}$$

Therefore, each of the $|n_1, \ldots, n_N\rangle$ states is an eigenstate of the Hamiltonian,

$$\hat{H}\,|n_1, \ldots, n_N\rangle = E(n_1, \ldots n_1)\,|n_1, \ldots, n_N\rangle \tag{12}$$

with energy

$$E(n_1, \ldots n_1) = \sum_{\alpha=1}^{N} \hbar\omega_\alpha \big(n_\alpha + \tfrac{1}{2}\big). \tag{13}$$

In particular, the ground state — *i.e.*, the lowest-energy eigenstate — of the multi-oscillator system is the state $|0, 0, \ldots, 0\rangle$ where all $n_\alpha = 0$; it's energy

$$E_0 = \sum_{\alpha=1}^{N} \frac{\hbar\omega_\alpha}{2} \tag{14}$$

is the sum of zero-point energies of all the oscillators.

Up to this point we have assumed a finite number N of the oscillator modes $\alpha = 1, \ldots, N$. However, all the above formulae apply just as well to the systems with practically infinite numbers of modes, like the vibrations of a macroscopically large crystal. Or even literally infinite numbers of modes, like a free quantum field — or several related fields like the electric and the magnetic fields. Later in these notes we shall see a couple examples of such infinite families of oscillator modes.

Phonons

As a toy model of free quantum field, consider the transverse waves on a string $y(x, t)$ tied at both ends, $x = 0$ and $x = L$. Classically, this string has an infinite series of standing wave modes,

$$y(x,t) \;=\; \sum_{\alpha=1}^{\infty} y_\alpha(t)\times\sqrt{2}\sin(k_\alpha x), \qquad k_\alpha \;=\; \alpha\times\frac{\pi}{L}\,, \tag{15}$$

where each mode $y_\alpha(t)$ oscillates harmonically with frequency

$$\omega_\alpha \;=\; v\times k_\alpha \;=\; \alpha\times\frac{\pi v}{L}\,, \tag{16}$$

$v=\sqrt{T/\mu}$ being the speed of waves on the string.

Dynamically, the Lagrangian for the string vibrations is

$$\mathcal{L} \;=\; \int\limits_0^L dx\left(\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2 \;-\; \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2\right), \tag{17}$$

which after Fourier transforming to the standing wave modes $y_\alpha(t)$ becomes

$$\mathcal{L} \;=\; \sum_{\alpha=1}^{\infty}\left(\frac{\mu L}{2}\,\dot{y}_\alpha^2 \;-\; \frac{TLk_\alpha^2}{2}\,y_\alpha^2\right). \tag{18}$$

Consequently, in the Hamiltonian formalism the canonical momenta conjugate to the y_α are $p_\alpha=\mu L\dot{y}_\alpha$ and the Hamiltonian function is

$$\mathcal{H}(q_1,\ldots;p_1,\ldots) \;=\; \sum_{\alpha=1}^{\infty}\left(\frac{p_\alpha^2}{2\mu L} \;+\; \frac{TLk_\alpha^2}{2}\,y_\alpha^2\right) \tag{19}$$

where

$$\frac{TLk_\alpha^2}{2} \;=\; \frac{\mu v^2 Lk_\alpha^2}{2} \;=\; \frac{\mu L\omega_\alpha^2}{2}\,. \tag{20}$$

In other words, the Hamiltonian (19) describes an infinite series of harmonic oscillators with frequencies ω_α.

In the quantum theory the the vibrating string, the canonical positions $y_\alpha(t)$ and momenta $p_\alpha(t)$ becomes Hermitian operators $\hat{y}_\alpha$ and $\hat{p}_\alpha$ obeying the canonical commutation relations

$$[\hat{y}_\alpha, \hat{y}_\beta] \;=\; 0, \quad [\hat{p}_\alpha, \hat{p}_\beta] \;=\; 0, \quad [\hat{y}_\alpha, \hat{p}_\beta] \;=\; i\hbar\delta_{\alpha\beta}\,, \tag{21}$$

and the Hamiltonian operator for the string follows from the classical Hamiltonian function (19),

$$\hat{H} \;=\; \sum_{\alpha=1}^{\infty} \left(\frac{\hat{p}_\alpha^2}{2\mu L} \,+\, \frac{\mu L \omega_\alpha^2}{2}\, \hat{y}_\alpha^2 \right). \tag{22}$$

To diagonalize this Hamiltonian, we proceed exactly as we had for a finite number of modes α: We build the lowering and raising operators $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$ for each $\alpha = 1, 2, \ldots, \infty$, let $\hat{n}_\alpha = \hat{a}_\alpha^\dagger \hat{a}_\alpha$, diagonalize all the $\hat{n}_\alpha$ operators at once since they all commute with each other, and end up with the basis of states

$$|\text{infinite list } n_1, n_2, \ldots\rangle \tag{23}$$

where *each* n_α *runs over non-negative integers* $0, 1, 2, \ldots$ *independently from all other* n_β. In terms of the $\hat{n}_\alpha$ operators, the Hamiltonian is

$$\hat{H} \;=\; \sum_{\alpha=0}^{\infty} \hbar\omega_\alpha(\hat{n}_\alpha + \tfrac{1}{2}), \tag{24}$$

so its eigenstates are all the states (23),

$$\hat{H}\, |n_1, n_2, \ldots\rangle \;=\; E(n_1, n_2, \ldots)\, |n_1, n_2, \ldots\rangle\,, \tag{25}$$

with energies

$$E(n_1, n_2, \ldots) \;=\; \sum_{\alpha=0}^{\infty} \hbar\omega_\alpha(n_\alpha + \tfrac{1}{2}). \tag{26}$$

Physically, each n_α — the excitation level of the oscillator mode α — can be thought as the number of quanta of that mode, so let's define *the net number of quanta in all the modes*

$$\mathcal{N} = \sum_{\alpha=0}^{\infty} n_\alpha \tag{27}$$

and the operator

$$\widehat{\mathcal{N}} = \sum_{\alpha=0}^{\infty} \hat{n}_\alpha \tag{28}$$

measuring this net number of quanta. Also, let's split the Hilbert space $\mathcal{H}$ of the whole multi-oscillator system into subspaces $\mathcal{H}_\mathcal{N}$ of definite $\mathcal{N}$, *i.e.* into spaces of $\widehat{\mathcal{N}}$'s eigen-spaces for each eigenvalue $\mathcal{N}$. As for any Hermitian operators, such eigen-spaces add up to the whole Hilbert space in the tensor sum sense,

$$\mathcal{H} = \bigoplus_{\mathcal{N}=0}^{\infty} \mathcal{H}_\mathcal{N} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots, \tag{29}$$

meaning that any vector $|\Psi\rangle$ in $\mathcal{H}$ decomposes into a sum of $|\Psi_N\rangle \in \mathcal{H}_\mathcal{N}$. In a moment, we shall see that each $\mathcal{H}_\mathcal{N}$ can be re-interpreted as a Hilbert space on its own right, namely the Hilbert space of N quasi-particles.

To see how that works, let's take a closer look at the individual $\mathcal{H}_\mathcal{N}$ subspaces, especially for the low $\mathcal{N} = 0, 1, 2$. But first, notice that we may build a complete orthonormal basis of each $\mathcal{N}_\mathcal{H}$ by simply grouping the states $|n_1, n_2, \ldots\rangle$ according to their net numbers of quanta:

$$\text{the states } |n_1, n_2, \ldots\rangle \text{ with } \sum_\alpha n_\alpha = \text{given } N \text{ form basis of the } \mathcal{H}_\mathcal{N}, \tag{30}$$

or equivalently,

$$\text{each } \mathcal{H}_\mathcal{N} \ \underline{\text{spans}} \text{ the states } |n_1, n_2, \ldots\rangle \text{ with } \sum_\alpha n_\alpha = \mathcal{N}. \tag{31}$$

Now let's focus on the $\mathcal{N} = 0$ subspace. The only way for non-negative integers n_α to add up to $\mathcal{N} = 0$ is to have all $n_\alpha = 0$. Consequently, the $\mathcal{H}_0$ is a one-dimensional Hilbert space spanning the ground state $|0,0,\ldots\rangle$ of the quantum vibrating string. The energy (14) of this ground state is

$$E_0 \;=\; \sum_{\alpha=1}^{\infty}\frac{\hbar\omega_\alpha}{2} \quad \text{for} \quad \omega_\alpha \;=\; \frac{\pi v}{L}\times\alpha, \tag{32}$$

where the infinite sum is badly divergent, so it needs to be regularized.

In real life, the regularization follows from the string being made from atoms. Consequently, it does not have a literally infinite number of vibration modes, just a very large number $O(\#\text{atoms})$, so the ground state energy (32) is not really quite infinite but merely macroscopic rather than microscopic. Physically, we may treat it as a part of the chemical binding energy of the string, and since it does not depend on the vibrational state we may disregard it from our analysis of the vibrational quanta. Indeed, we may always add a constant (times a unit operator) to the Hamiltonian without affecting the systems dynamics in any meaningful way, so for the quantum string we simply re-define

$$\hat{H}' \;=\; \hat{H} \;-\; E_0 \;=\; \sum_{\alpha=1}^{\infty}\hbar\omega_\alpha\times\hat{n}_\alpha\,. \tag{33}$$

Consequently, the ground state has $E'(0,0,\ldots)=0$ while all the excited states of the Hamiltonian have positive energies

$$E'(n_1,n_2,\ldots) \;=\; \sum_{\alpha=1}^{\infty}\hbar\omega_\alpha\times n_\alpha\,. \tag{34}$$

On the other hand, if we treat the vibrating string as a toy model of a one-dimensional QFT which leaves in a truly continuous space, then there is a truly infinite number of modes α and the zero-point energy (32) is truly divergent. Alas, quantum field theories are full of divergences, and over the years learned how to regulate and subtract such divergences to obtain physically correct finite results, although the techniques for doing so are beyond the scope of this quantum mechanics class. For the present purposes, let me simply say that the E_0 is a constant which commutes with any operators relevant to the waves on the string, so we are free to subtract in from the Hamiltonian as in eqs. (33) and (34) even if E_0 happens to be a divergent constant.

Next, consider the $\mathcal{H}_1$ subspace of states with one quantum in any mode. Indeed, if the non-negative integers n_β add up to $\mathcal{N} = 1$ then there is one mode α with $n_\alpha = 1$ and every other mode $\beta \neq \alpha$ has $n_\beta = 0$. Let's re-label such a state

$$\big|n_\alpha = 1, \text{all other } n_\beta = 0\big\rangle \;=\; |\alpha\rangle\,, \tag{35}$$

so we may re-interpret it as a quantum state of a some particle or quasiparticle of energy

$$E(\alpha) \;=\; \sum_{\beta=1}^{\infty} \hbar\omega_\beta \times \big(n_\beta = \delta_{\beta,\alpha}\big) \;=\; \hbar\omega_\alpha. \tag{36}$$

To make better sense of this quasiparticle state, let's treat the x-dependence of the mode α,

$$y_\alpha \times \text{const} \times \sin(k_\alpha x), \quad k_\alpha \;=\; \frac{\pi\alpha}{L}\,, \tag{37}$$

as a wave-function of the quasiparticle,

$$\psi_\alpha(x) \;=\; \text{const} \times \sin\left(k_\alpha x \;=\; \frac{\pi\alpha x}{L}\right) \tag{38}$$

with energy

$$E(\alpha) \;=\; \hbar\omega_\alpha \;=\; \hbar v k_\alpha\,. \tag{39}$$

Physically, the wave-functions (38) describe a 1d (quasi)particle with momentum $P = \pm\hbar k_\alpha$ bouncing back and forth off the reflecting walls at $x = 0$ and $x = L$, hence the discrete spectrum of $k_\alpha = (\pi/L) \times \text{integer}\,\alpha$ and having both $P = +\hbar k_\alpha$ and $P = -\hbar k_\alpha$ components being present at equal strengths. Moreover, in terms of the momentum P, the energy (39) is $E = v|P|$, so each state $|\alpha\rangle$ is an eigenstate of a single-(quasi)particle Hamiltonian

$$\hat{H}_1 \;=\; v\,\text{abs}(\hat{P}) \;=\; v\sqrt{\hat{P}^2}\,. \tag{40}$$

This Hamiltonian belongs to a quasi-particle which moves left or right with a constant speed v, namely the speed of waves on the string. Treating these waves as a kind of transverse sound waves, we identify the quanta waves' as one-dimensional *phonons*.

From the phonon point of view, the $\mathcal{H}_1$ is the Hilbert space of the one-phonon states, $\hat{H}_1$ in the Hamiltonian operator for such single phonons, and the states $|\alpha\rangle$ are the eigenstates of that Hamiltonian $\hat{H}_1$.

Next, consider the $\mathcal{H}_2$ subspace, which we are going to re-interpret as a two-phonon Hilbert space. The basis of this subspace is made from states $|n_1, n_2, \ldots\rangle$ where the n_γ add up to 2. Since all the n_γ are non-negative integers, there are two possibilities:

(1) $$|\alpha,\alpha\rangle = |n_\alpha = 2,\ \text{all other } n_\gamma = 0\rangle\,,$$

(2) $$|\alpha,\beta\rangle \;=\; \big|n_\alpha = n_\beta = 1,\ \text{all other } n_\gamma = 0\big\rangle\,,$$

or in terms of raising operators acting on the ground state,

$$|\alpha,\alpha\rangle \;=\; \frac{1}{\sqrt{2}}\hat{a}_\alpha^\dagger\hat{a}_\alpha^\dagger\,|\text{ground}\rangle\,, \tag{41}$$

or

$$|\alpha,\beta\rangle \;=\; \hat{a}_\alpha^\dagger\hat{a}_\beta^\dagger\,|\text{ground}\rangle \quad \text{for } \alpha\neq\beta. \tag{42}$$

Either way — for $\alpha \neq \beta$ or for $\alpha = \beta$, — the state $|\alpha,\beta\rangle = \big|\{n_\gamma = \delta_{\gamma,\alpha} + \delta_{\gamma,\beta}\}\big\rangle$ is an eigenstate of the Hamiltonian (33) with energy

$$E'(\alpha,\beta) \;=\; \sum_\gamma \hbar\omega\gamma \times (n_\gamma = \delta_{\gamma,\alpha} + \delta_{\gamma,\beta}) \;=\; \hbar\omega_\alpha \;+\; \hbar\omega_\beta\,, \tag{43}$$

so we interpret it as a state of two independent phonons, one in state $|\alpha\rangle$ and the other in state $|\beta\rangle$, and their respective energies $\hbar\omega_\alpha$ and $\hbar\omega_\beta$ add up to the net energy (43). Thus, the $\mathcal{H}_2$ subspace becomes the two-phonon Hilbert space, with the net Hamiltonian

$$\hat{H}_2 \;=\; \hat{H}_1(1^{\text{st}}\text{ phonon}) \;+\; \hat{H}_1(2^{\text{nd}}\text{ phonon}) \;=\; \sqrt{\hat{P}_1^2} \;+\; \sqrt{\hat{P}_2^2}\,. \tag{44}$$

However, the two phonons in this Hilbert space are not completely independent because we cannot tell which phonon is which: for $\alpha \neq \beta$

$$|\alpha,\beta\rangle \;=\; \hat{a}_\alpha^\dagger\hat{a}_\beta^\dagger\,|\text{ground}\rangle \;=\; \hat{a}_\beta^\dagger\hat{a}_\alpha^\dagger\,|\text{ground}\rangle \;=\; |\beta,\alpha\rangle\,. \tag{45}$$

Please note: the states $|\alpha,\beta\rangle$ and $|\beta,\alpha\rangle$ are not just similar, but are literally the same single state in the Hilbert space, so it's utterly impossible to tell which phonon is first and which is second. Thus, the phonons are identical bosons, so the $\mathcal{H}_2$ is not just a two-particle Hilbert space but the Hilbert space of two identical bosons.

Note: by *identical* bosons I do not means particles which have to be in the same state but merely particles of the same species, so we cannot tell them apart except by their quantum states. Thus, we can have two phonons in different quantum states $|\alpha\rangle \neq |\beta\rangle$, and we can say that the phonon in state $|\alpha\rangle$ has energy $\hbar\omega_\alpha$ while the phonon in state $|\beta\rangle$ has energy $\hbar\omega_\beta$. But we cannot say which of the 2 phonons is in state $|\alpha\rangle$ and which is in state $|\beta\rangle$.

For $\mathcal{N} \geq 2$ the situation is similar to $\mathcal{N} = 2$: The $\mathcal{H}_\mathcal{N}$ is the $\mathcal{N}$–phonon Hilbert space where all $\mathcal{N}$ phonons are identical bosons. For example, for $\mathcal{N} = 3$ the $\mathcal{H}_3$ space spans the states

$$\begin{aligned} |\alpha,\beta,\gamma\rangle &= C_{\alpha,\beta,\gamma}\,\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}^\dagger_\gamma\,|\text{ground}\rangle\,, \\ \text{where}\quad C_{\alpha,\beta,\gamma} &= \begin{cases} \frac{1}{\sqrt{3!}} & \text{if } \alpha=\beta=\gamma, \\ \frac{1}{\sqrt{2!}} & \text{if two of the } \alpha,\beta,\gamma \text{ coincide but the third is different,} \\ 1 & \text{if } \alpha,\beta,\gamma \text{ are all different from each other,} \end{cases} \end{aligned} \tag{46}$$

and all such states have definite energies

$$E'(\alpha,\beta,\gamma) = \hbar\omega_\alpha + \hbar\omega_\beta + \hbar\omega_\gamma\,. \tag{47}$$

This allows us to re-interpret the $\mathcal{H}_3$ as a Hilbert space of 3 quasiparticles, each quasiparticle being a phonon, with the net Hamiltonian

$$\hat{H}_3 = \hat{H}_1(1^{\text{st}}\text{ phonon}) + \hat{H}_1(2^{\text{nd}}\text{ phonon}) + \hat{H}_1(3^{\text{rd}}\text{ phonon}) = \sqrt{\hat{P}_1^2} + \sqrt{\hat{P}_2^2} + \sqrt{\hat{P}_3^2}\,. \tag{48}$$

However, the states (46) do not distinguish which phonon is first, which is second and which is third; instead

$$|\text{any permutation of } \alpha,\beta,\gamma\rangle = |\alpha,\beta,\gamma\rangle\,, \tag{49}$$

so the 3 phonons in the $\mathcal{H}_3$ Hilbert space are 3 identical bosons rather than 3 distinguishable particles.

Likewise, for larger $\mathcal{N}>3$ the $\mathcal{H}_{\mathcal{N}}$ spans the $\mathcal{N}$ phonon states

$$|\alpha_1,\alpha_2,\ldots,\alpha_{\mathcal{N}}\rangle \;=\; \begin{pmatrix}\text{combinatorical}\\ \text{factor}\end{pmatrix} \times \hat{a}^\dagger_{\alpha_1}\hat{a}^\dagger_{\alpha_2}\cdots\hat{a}^\dagger_{\alpha_{\mathcal{N}}}|\text{ground}\rangle\,, \tag{50}$$

where in terms of the occupation numbers n_α, the list $(\alpha_1,\ldots,\alpha_{\mathcal{N}})$ includes all α with $n_\alpha>0$, and each such α appears n_α times in the list. The states (50) have energies

$$E(\alpha_i,\ldots,\alpha_{\mathcal{N}}) \;=\; \sum_\beta \hbar\omega_\beta \times \left(n_\beta=\sum_{i=1}^{\mathcal{N}}\delta_{\beta,\alpha_i}\right) \;=\; \sum_{i=1}^{\mathcal{N}}\hbar\omega_\alpha \;=\; \sum_{i=1}^{\mathcal{N}} v\hbar k_\alpha \tag{51}$$

which are simply sums of the individual phonons' energies, so we may interpret them as eigenstates of the $\mathcal{N}$–phonon Hamiltonian

$$\hat{H}_{\mathcal{N}} \;=\; \sum_{i=1}^{\mathcal{N}}\sqrt{\hat{P}_i^2} \tag{52}$$

where the phonons do not interact with each other.

Moreover, each phonon can be in any single-particle state $|\alpha\rangle$ independently from the other phonons, but given the list $(\alpha_1,\ldots,\alpha_{\mathcal{N}})$ of states taken by the $\mathcal{N}$ phonons, we cannot tell which of phonons take which state. Instead,

$$|\text{any permutation of }\alpha_1,\ldots,\alpha_{\mathcal{N}}\rangle \;=\; |\alpha_1,\ldots,\alpha_{\mathcal{N}}\rangle\,, \tag{53}$$

so the $\mathcal{N}$ phonons are $\mathcal{N}$ identical bosons.

Coherent states

Finally, let me go back to the complete Hilbert space $\mathcal{H}$ of the vibrating string. This space spans the states $|n_1,n_2,\ldots\rangle$ with all net numbers $\mathcal{N}$ of quanta. (Although if we wish to limit ourselves to states of finite energy E' relative to the ground state, then the net number of quanta should be finite, but it can be arbitrarily large without a limit.) From the phonon's point of view, this space

$$\mathcal{H} \;=\; \mathcal{H}_0 \,\oplus\, \mathcal{H}_1 \,\oplus\, \mathcal{H}_2 \,\oplus\, \mathcal{H}_3 \,\oplus\, \cdots \tag{54}$$

is *the Hilbert space of an arbitrary number of identical phonons.* By the superposition principle, it includes linear combinations of states with different phonon numbers. In particular, it includes the *coherent states* of the vibrating string which describe the almost-classical waves $\bar{y}(x,t) = \langle \hat{y}(x) \rangle (t)$.

The coherent states of a single harmonic oscillator are explained in detain in the current homework (set#5, problems 2 and 3). Basically, a coherent state is the quantum state which reproduces the classical harmonic oscillations as closely as possible in the quantum mechanics. Specifically, a coherent state has minimal uncertainties of position and momenta (same as the oscillator's ground state) which multiply to $\Delta q \times \Delta p = \frac{\hbar}{2}$, while the expectation values $\langle q \rangle (t)$ and $\langle p \rangle (t)$ oscillate like the $q(t)$ and $p(t)$ of a classical oscillator,

$$\langle q \rangle (t) \;=\; A\sin(\omega t - \phi), \quad \langle p \rangle (t) \;=\; A\omega m\cos(\omega t - \phi). \tag{55}$$

Also, a coherent state does not have a definite energy; instead, the average energy of a coherent state is the zero-point energy plus the classical energy of oscillations with amplitude A,

$$\langle E \rangle \;=\; E_0 \;+\; E_{\text{classical}}(A) \;=\; \frac{\hbar\omega}{2} \;+\; \frac{m\omega^2}{2} \times A^2 \tag{56}$$

while the energy uncertainty is

$$\Delta E \;=\; \sqrt{\hbar\omega \times E_{\text{classical}}} \quad\Longrightarrow\quad \text{for } E_{\text{classical}} \gg \hbar\omega, \quad \Delta E \ll \langle E \rangle\,. \tag{57}$$

In terms of the raising and lowering operators, the coherent states are

$$|\xi\rangle \;=\; e^{-|\xi|^2/2}\exp(\xi\hat{a}^\dagger)\,|0\rangle\,, \quad \hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle \tag{58}$$

for

$$\xi \;=\; \sqrt{\frac{2}{\hbar\omega m}}\big(\omega m\,\langle x\rangle \;+\; i\,\langle p\rangle\big). \tag{59}$$

Under classical harmonic oscillation (55), the complex parameter ξ evolves with time as

$$\xi(t) \;=\; iA\sqrt{\frac{2\omega m}{\hbar}} \times e^{-i\omega t}, \tag{60}$$

consequently, the coherent state $|\xi(t)\rangle$ — or rather $|\psi\rangle (t) \;=\; e^{-i\omega t/2}\,|\xi(t)\rangle$ — obeys the Schrödinger equation.

The multi-oscillator system like the vibrating string also have coherent states, which behave as classically as possible in quantum mechanics. Specifically, in a coherent state of a multi-oscillator system each oscillator mode α is in a coherent state $|\xi_\alpha\rangle$,

$$|\xi_1, \xi_2, \ldots\rangle \;=\; |\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \tag{61}$$

where $\otimes$ denotes direct product of the quantum states of separate degrees of freedom; in the wave-function language $\psi(q_1, q_2, \ldots) = \psi_1(q_1)\psi_2(q_2)\cdots$. Or in terms of the raising and lowering operators:

$$\begin{aligned} |\xi_1, \xi_2, \ldots\rangle \;&=\; \bigotimes_\alpha \left(e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} |0\rangle \right)_\alpha \;=\; \prod_\alpha \exp(-\tfrac{1}{2}|\xi_\alpha|^2) \exp(\xi_\alpha \hat{a}^\dagger_\alpha) \, |\text{ground}\rangle \\ &=\; \exp\left(-\frac{1}{2} \sum_\alpha |\xi_\alpha|^2 \right) \times \exp\left(\sum_\alpha \xi_\alpha \hat{a}^\dagger_\alpha \right) |\text{ground}\rangle \,, \end{aligned} \tag{62}$$

while

$$\forall\alpha: \quad \hat{a}_\alpha \, |\xi_1, \xi_2, \ldots\rangle \;=\; \xi_\alpha \, |\xi_1, \xi_2, \ldots\rangle \,. \tag{63}$$

In such a coherent state, each oscillator mode α has minimal uncertainties $\Delta y_\alpha \times \Delta p_\alpha = \frac{\hbar}{2}$ while the expectation values $\langle y_\alpha \rangle$ and $\langle p_\alpha \rangle$ follow from the real and imaginary parts of the complex parameter ξ_α,

$$\langle y_\alpha \rangle \;=\; \sqrt{\frac{2\hbar}{\omega_\alpha \mu L}} \times \operatorname{Re} \xi_\alpha \,, \quad \langle p_\alpha \rangle \;=\; \sqrt{2\hbar \omega_\alpha \mu L} \times \operatorname{Im} \xi_\alpha \,. \tag{64}$$

For each oscillator mode α, the ξ_α parameter evolves with time as

$$\xi_\alpha(t) \;=\; \xi_\alpha(0) \times e^{-i\omega_\alpha t}, \tag{65}$$

which makes the coherent state $|\xi_1(t), \xi_2(t), \ldots\rangle$ obey the time-dependent Schrödinger equation

$$i\hbar \frac{d}{dt} \, |\xi_1(t), \xi_2(t), \ldots\rangle \;=\; \hat{H}' \, |\xi_1(t), \xi_2(t), \ldots\rangle \tag{66}$$

while the expectation values (64) oscillate harmonically as in a classical oscillator. Or in terms of the vibrating string itself,

$$\begin{aligned}\langle\hat{y}(x)\rangle(t) &= 2\sqrt{\frac{\hbar}{\mu L}}\sum_{\alpha=1}^{n}\frac{1}{\sqrt{\omega_\alpha}}\times\operatorname{Re}\xi_\alpha(t)\times\sin(k_\alpha x),\\ \frac{d}{dt}\langle\hat{y}(x)\rangle &= 2\sqrt{\frac{\hbar}{\mu L}}\sum_{\alpha=1}^{n}\sqrt{\omega_\alpha}\times\operatorname{Im}\xi_\alpha(t)\times\sin(k_\alpha x),\end{aligned}\tag{67}$$

which obey the classical wave equation

$$\left(\frac{\partial^2}{\partial t^2}\,+\,v^2\frac{\partial^2}{\partial x^2}\right)\langle\hat{y}(x)\rangle(t)\;=\;0.\tag{68}$$

Finally, while the coherent states do not have definite energies, the average energy of a coherent state above the ground-state energy is equal to the classical energy of the oscillations with the same amplitude,

$$A_\alpha\;=\;\text{amplitude}[\langle y_\alpha\rangle(t)]\;=\;\sqrt{\frac{2\hbar}{\omega_\alpha\mu L}}\times|\xi_\alpha|,\tag{69}$$

$$\left\langle E'\right\rangle\;=\;\langle E\rangle\;-\;E_0\;=\;\sum_\alpha\hbar\omega_\alpha|\xi_\alpha|^2\;=\;\sum_\alpha\frac{\mu L\omega_\alpha^2}{2}\times A_\alpha^2.\tag{70}$$

As to the energy uncertainty,

$$(\Delta E)^2\;=\;\sum_\alpha\hbar^2\omega_\alpha^2|\xi_\alpha|^2,\tag{71}$$

so as long as *some* $|\xi_\alpha|$ *are large,* the energy uncertainty is relatively small, $\Delta E\ll\langle E'\rangle$.

Electromagnetic Fields and Photons

Just like the waves on a finite piece of string, the electromagnetic fields in a reflecting cavity also decompose into an infinite series of standing waves, each such wave acting as a harmonic oscillator. Thus, we may diagonalize the Hamiltonian of the whole series in terms of definite number of quanta for each mode, and when we focus on the subspace of states having $\mathcal{N}$ quanta altogether, we get a Hilbert space of $\mathcal{N}$ identical bosons, where each boson is a massless relativistic particle with 2 polarization states — the photon.

In this section of my notes, I show how this works in outline, but I skip some of the gory algebraic details. If you are seriously interested in the quantum EM fields, take a class in quantum field theory (I should be teaching it in 2022/23), although the 389 L class (second semester of the graduate quantum mechanics) should also cover some of this material.

The classical EM fields obey Maxwell equations, which in the absence of any charges and currents become

$$\nabla\cdot\mathbf{E} \;=\; \nabla\cdot\mathbf{B} \;=\; 0, \tag{72}$$

$$\frac{\partial\mathbf{B}}{\partial t} \;=\; -c\nabla\times\mathbf{E}, \qquad \frac{\partial\mathbf{E}}{\partial t} \;=\; +c\nabla\times\mathbf{B}. \tag{73}$$

In infinite space, these equations allow for EM waves with any wave-vectors $\mathbf{k}$, so to get a discrete spectrum of the wave modes we need to put the EM fields in a finite-size box. For simplicity, let's take a large cubic box of side L with periodic boundary conditions

$$\psi(x,y,z) \;=\; \psi(x+L,y,z) \;=\; \psi(x,y+L,z) \;=\; \psi(x,y,z+L) \tag{74}$$

for $\psi = E_x, E_y, E_z, B_x, B_y, B_z$. Consequently, the wave modes in this box become

$$\psi(\mathbf{x}) \;=\; \exp(i\mathbf{k}\cdot\mathbf{x}) \quad \text{for} \quad (k_x,k_y,k_z) \;=\; \frac{2\pi}{L}(\text{integer},\text{integer},\text{integer}) \tag{75}$$

where each integer can be positive negative or zero. This allows us to Fourier transform a continuous (but periodic) field to a discrete series of wave modes,

$$\psi(\mathbf{x},t) \;=\; L^{-3/2}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\times\psi_{\mathbf{k}}(t), \qquad \psi_{\mathbf{k}}(t) \;=\; L^{-3/2}\int d^3\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\times\psi(\mathbf{x},t). \tag{76}$$

Or for the vector fields $\mathbf{E}$ and $\mathbf{B}$,

$$\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= L^{-3/2}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\times\mathbf{E}_{\mathbf{k}}(t), \qquad \mathbf{E}_{\mathbf{k}}(t) = L^{-3/2}\int d^3\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\times\mathbf{E}(\mathbf{x},t),\\
\mathbf{B}(\mathbf{x},t) &= L^{-3/2}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\times\mathbf{B}_{\mathbf{k}}(t), \qquad \mathbf{B}_{\mathbf{k}}(t) = L^{-3/2}\int d^3\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\times\mathbf{B}(\mathbf{x},t).
\end{aligned} \tag{77}$$

In terms of these field modes, the time-independent Maxwell equation (72) become

$$\mathbf{k}\cdot\mathbf{E}_{\mathbf{k}} \;=\; \mathbf{k}\cdot\mathbf{B}_{\mathbf{k}} \;=\; 0, \tag{78}$$

so for each $\mathbf{k}$ we now need a basis of two polarization vectors $\perp\,\mathbf{k}$. Let's use the *helicity* basis of unit vectors $\mathbf{e}_{\mathbf{k},\lambda}$ obeying

$$-i\mathbf{k} \times \mathbf{e}_{\mathbf{k},\lambda} \;=\; \lambda|\mathbf{k}|\,\mathbf{e}_{\mathbf{k},\lambda}\,, \qquad \lambda = \pm 1 \text{ only;} \tag{79}$$

for example,

$$\text{for } \mathbf{k} \text{ in } +z \text{ direction}: \quad \mathbf{e}_{\mathbf{k},\lambda} \;=\; \frac{1}{\sqrt{2}}(1, \lambda i, 0). \tag{80}$$

These are complex unit vectors, so the orthonormality condition becomes

$$\mathbf{e}^*_{\mathbf{k},\lambda} \cdot \mathbf{e}_{\mathbf{k},\lambda'} \;=\; \delta_{\lambda,\lambda'}\,; \tag{81}$$

there are also commonly used phase conventions

$$\mathbf{e}_{\mathbf{k},\pm} \;=\; \mathbf{e}^*_{\mathbf{k},\mp} \quad \text{and} \quad \mathbf{e}_{-\mathbf{k},\lambda} \;=\; \mathbf{e}_{+\mathbf{k},-\lambda}\,. \tag{82}$$

Anyhow, having a basis of vectors $\perp \mathbf{k}$ allows us to decompose the vector modes $\mathbf{E_k}$ and $\mathbf{B_k}$ into two independent transverse polarizations,

$$\mathbf{E_k} \;=\; \sum_\lambda E_{\mathbf{k},\lambda}\,\mathbf{e}_{\mathbf{k},\lambda}\,, \qquad \mathbf{B_k} \;=\; \sum_\lambda B_{\mathbf{k},\lambda}\,\mathbf{e}_{\mathbf{k},\lambda}\,, \tag{83}$$

and hence the EM fields decompose into

$$\mathbf{E}(\mathbf{x},t) \;=\; L^{-3/2}\sum_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_{\mathbf{k},\lambda}\,E_{\mathbf{k},\lambda}(t) \quad \text{and} \quad \mathbf{B}(\mathbf{x},t) \;=\; L^{-3/2}\sum_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_{\mathbf{k},\lambda}\,B_{\mathbf{k},\lambda}(t). \tag{84}$$

Note: the EM fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are real, but due to complex coefficients in the mode expansion (84), the $E_{\mathbf{k},\lambda}(t)$ and $B_{\mathbf{k},\lambda}(t)$ are are complex but are related to each other by complex conjugation,

$$E^*_{\mathbf{k},\lambda} \;=\; E_{-\mathbf{k},\lambda} \quad\text{and}\quad B^*_{\mathbf{k},\lambda} \;=\; B_{-\mathbf{k},\lambda}\,. \tag{85}$$

In terms of the modes (84), the EM energy

$$H \;=\; \frac{1}{8\pi}\int\! d^3\mathbf{x}\big(\mathbf{E}^2+\mathbf{B}^2\big) \tag{86}$$

becomes

$$H \;=\; \frac{1}{8\pi}\sum_{\mathbf{k},\lambda}\big(E^*_{\mathbf{k},\lambda}E_{\mathbf{k},\lambda}\;+\;B^*_{\mathbf{k},\lambda}B_{\mathbf{k},\lambda}\big), \tag{87}$$

the time-independent Maxwell equations (72) are automatically satisfied, while the time-dependent Maxwell equations (73) become oscillator-like equations for each mode:

$$\frac{d}{dt}E_{\mathbf{k},\lambda} \;=\; -\lambda\omega_k B_{\mathbf{k},\lambda}\,,\quad \frac{d}{dt}B_{\mathbf{k},\lambda} \;=\; +\lambda\omega_k E_{\mathbf{k},\lambda}\,,\quad \text{for } \omega_k \;=\; c|\mathbf{k}|. \tag{88}$$

In the quantum theory, the mode coefficients $E_{\mathbf{k},\lambda}(t)$ and $B_{\mathbf{k},\lambda}(t)$ become operators $\hat{E}_{\mathbf{k},\lambda}$ and $\hat{B}_{\mathbf{k},\lambda}$ obeying *skewed Hermiticity conditions*

$$\hat{E}^\dagger_{\mathbf{k},\lambda} \;=\; \hat{E}_{-\mathbf{k},\lambda}\,,\qquad \hat{B}^\dagger_{\mathbf{k},\lambda} \;=\; \hat{B}_{-\mathbf{k},\lambda}\,, \tag{89}$$

The classical energy (87) becomes the Hamiltonian operator

$$\hat{H} \;=\; \frac{1}{8\pi}\sum_{\mathbf{k},\lambda}\Big(\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\;+\;\hat{B}^\dagger_{\mathbf{k},\lambda}\hat{B}_{\mathbf{k},\lambda}\Big). \tag{90}$$

and to reproduce the time-dependent Maxwell equations (88) as Heisenberg equations in quantum mechanics — or as Heisenberg–Dirac equations in the Schrödinger picture — from the Hamiltonian (90), we need the commutation relations

$$\big[\hat{B}_{\mathbf{k},\lambda},\hat{B}^\dagger_{\mathbf{k}',\lambda'}\big]\ =\ 0,\qquad \big[\hat{E}_{\mathbf{k},\lambda},\hat{E}^\dagger_{\mathbf{k}',\lambda'}\big]\ =\ 0,\qquad \big[\hat{B}_{\mathbf{k},\lambda},\hat{E}^\dagger_{\mathbf{k}',\lambda'}\big]\ =\ 4\pi i\hbar c\lambda|\mathbf{k}|\times\delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}\,, \tag{91}$$

which provide for

$$\frac{1}{i\hbar}\big[\hat{E}_{\mathbf{k},\lambda},\hat{H}\big]\ =\ +\lambda c|\mathbf{k}|\times\hat{B}_{\mathbf{k},\lambda}\,,\qquad \frac{1}{i\hbar}\big[\hat{B}_{\mathbf{k},\lambda},\hat{H}\big]\ =\ -\lambda c|\mathbf{k}|\times\hat{E}_{\mathbf{k},\lambda}\,, \tag{92}$$

and hence Maxwell-like Heisenberg–Dirac equations

$$\frac{d}{dt}\big\langle E_{\mathbf{k},\lambda}\big\rangle\ =\ -\lambda c|\mathbf{k}|\times\big\langle B_{\mathbf{k},\lambda}\big\rangle\,,\qquad \frac{d}{dt}\big\langle B_{\mathbf{k},\lambda}\big\rangle\ =\ +\lambda c|\mathbf{k}|\times\big\langle E_{\mathbf{k},\lambda}\big\rangle\,, \tag{93}$$

cf. eqs. (88). To save time, let me skip the derivation of these formulae, or rather leave it as an *optional* extra exercise for the interested students.

Given the commutation relations (91), we define the lowering and raising operators for each mode as

$$\hat{a}_{\mathbf{k},\lambda}\ =\ \frac{\lambda\hat{B}_{\mathbf{k},\lambda}+i\hat{E}_{\mathbf{k},\lambda}}{\sqrt{8\pi\hbar\omega_k}}\,, \tag{94}$$

$$\hat{a}^\dagger_{\mathbf{k},\lambda}\ =\ \frac{\lambda\hat{B}^\dagger_{\mathbf{k},\lambda}-i\hat{E}^\dagger_{\mathbf{k},\lambda}}{\sqrt{8\pi\hbar\omega_k}}\ =\ \frac{\lambda\hat{B}_{-\mathbf{k},\lambda}-i\hat{E}_{-\mathbf{k},\lambda}}{\sqrt{8\pi\hbar\omega_k}}\,, \tag{95}$$

$$\hat{a}_{-\mathbf{k},\lambda}\ =\ \frac{\lambda\hat{B}_{-\mathbf{k},\lambda}+i\hat{E}_{-\mathbf{k},\lambda}}{\sqrt{8\pi\hbar\omega_k}}\ \neq\ \hat{a}^\dagger_{\mathbf{k},\lambda}\,, \tag{96}$$

$$\hat{a}^\dagger_{-\mathbf{k},\lambda}\ =\ \frac{\lambda\hat{B}_{\mathbf{k},\lambda}-i\hat{E}_{\mathbf{k},\lambda}}{\sqrt{8\pi\hbar\omega_k}}\ \neq\ a_{\mathbf{k},\lambda}\,. \tag{97}$$

Note: despite the skewed Hermiticity conditions (89) for the field modes $\hat{E}_{\pm\mathbf{k},\lambda}$ and $\hat{B}_{\pm\mathbf{k},\lambda}$, the lowering and the raising operators for opposite momenta $\pm\mathbf{k}$ are independent from each other.

As usual, the lowering and raising operators (94) through (97) obey the commutation relations

$$\big[\hat{a}_{\mathbf{k},\lambda},\hat{a}_{\mathbf{k}',\lambda'}\big]\ =\ 0,\quad \big[\hat{a}^\dagger_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\big]\ =\ 0,\quad \big[\hat{a}_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}\big]\ =\ \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}\,. \tag{98}$$

although verifying these relations takes a bit more work then usual due to non-Hermiticity of the $\hat{E}_{\mathbf{k},\lambda}$ and $\hat{B}_{\mathbf{k},\lambda}$ operators. For the same reason, it takes more work then usual rewrite the Hamiltonian (90) in the multi-oscillator form

$$\hat{H} = \sum_{\mathbf{k},\lambda} \hbar\omega_k \big(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \tfrac{1}{2}\big) = E_0 + \sum_{\mathbf{k},\lambda} \hbar\omega_k \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\,. \tag{99}$$

To save time, let me skip the derivation of eqs. (98) and (99) and leave them as an *optional* extra exercise to the students.

But once we got the commutation relations (98) and the Hamiltonian (99), we may proceed exactly as we did for the wave on a string. Thus, we define the number of quanta operators $\hat{n}_{\mathbf{k},\lambda} = \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}$ for each mode, diagonalize them all (since they all commute with each other), and build a basis of states $\big|\{n_{\mathbf{k},\lambda}\}\big\rangle$ where each $n_{\mathbf{k},\lambda}$ runs over all non-negative integers $0, 1, 2, \ldots$ independently of all the other $n_{\mathbf{k}',\lambda'}$. Thanks to eq. (99), all these states are eigenvalues of the Hamiltonian with energies

$$E(\{n_{\mathbf{k},\lambda}\}) = E_0 + \sum_{\mathbf{k},\lambda} \hbar\omega_k n_{\mathbf{k},\lambda} \tag{100}$$

where E_0 is the ground state energy. The E_0 is badly divergent, but it's a constant which does not affect the dynamics of the EM fields, so we may just as well subtract it from the Hamiltonian,

$$\hat{H}' = \hat{H} - E_0 = \sum_{\mathbf{k},\lambda} \hbar\omega_k \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \quad\Longrightarrow\quad E'(\{n_{\mathbf{k},\lambda}\}) = \sum_{\mathbf{k},\lambda} \hbar\omega_k n_{\mathbf{k},\lambda}\,. \tag{101}$$

Next, we reorganize the basic states $\big|\{n_{\mathbf{k},\lambda}\}\big\rangle$ by the net number of quanta

$$\mathcal{N} = \sum_{\mathbf{k},\lambda} n_{\mathbf{k},\lambda} \tag{102}$$

in all the modes. In other words, we split the Hilbert space of the EM fields in the box into a tensor sum of eigen-blocks of the $\widehat{\mathcal{N}} = \sum_{\mathbf{k},\lambda} \hat{n}_{\mathbf{k},\lambda}$ operator,

$$\mathcal{H} = \bigoplus_{\mathcal{N}=0}^{\infty} \mathcal{H}_{\mathcal{N}} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \tag{103}$$

where the states $\big|\{n_{\mathbf{k},\lambda}\}\big\rangle$ with $\sum_{\mathbf{k},\lambda} n_{\mathbf{k},\lambda} = \mathcal{N}$ serve as a basis of the $\mathcal{N}$–quanta Hilbert space $\mathcal{H}_{\mathcal{N}}$.

Similar to the phonon case, the $\mathcal{H}_0$ space spans a single quantum state $|\text{all } n_\alpha = 0\rangle$ which has no quanta at all. From the photonic point of view — which I shall explain in a moment — this is *the vacuum state* without any photons.

Next, the $\mathcal{H}_1$ space spans the states having one quantum in one mode $\mathbf{k}, \lambda$ and no quanta at all in all the other modes,

$$|(\mathbf{k}, \lambda)\rangle \;=\; \big|n_{\mathbf{k},\lambda} = 1,\ \text{all other } n_{\mathbf{k}',\lambda'} = 0\big\rangle\,. \tag{104}$$

These states are labeled by modes $(\mathbf{k}, \lambda)$, which me may interpret as quantum states of a single particle, then their energies

$$E'(\mathbf{k}, \lambda) \;=\; \hbar\omega_k \;=\; \hbar c|\mathbf{k}| \tag{105}$$

may be interpret as energy levels of that particle. Since the EM fields of the mode $\mathbf{k}, \lambda$ are proportional to

$$\mathbf{E}(\mathbf{x}) \;\propto\; e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{e}_{\mathbf{k},\lambda}, \qquad \mathbf{B}(\mathbf{x}) \;\propto\; e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathbf{e}_{\mathbf{k},\lambda}, \tag{106}$$

where

$$k_x, k_y, k_z \text{ run over } \frac{2\pi}{L} \times \text{integers}, \tag{107}$$

we may interpret $\mathbf{P} = \hbar k$ as the particle's momentum, while the conditions (107) on that momentum stem from the particle living in a cubic box with the periodic boundary conditions. Consequently, the energies (105) become

$$E(\mathbf{P}) \;=\; c|\mathbf{P}|, \tag{108}$$

which are appropriate for a massless relativistic particle — the photon. In Hamiltonian terms, they are eigenvalues of the

$$\hat{H}_1 \;=\; c\sqrt{\hat{\mathbf{P}}^2}. \tag{109}$$

Besides momentum $\mathbf{P} = \hbar\mathbf{k}$, the single photon states $|\mathbf{k}, \lambda\rangle$ are labeled by helicity $\lambda = \pm 1$ Thus, the photon has two distinct transverse polarization states. These polarization states play a similar role to the electron's spin states, but their relation to the angular momentum is different from the spin.

At the $\mathcal{N}=2$ level, the $\mathcal{H}_2$ space spans the states

$$\begin{aligned} \big|(\mathbf{k},\lambda),(\mathbf{k},\lambda)'\big\rangle &= C\times \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k}',\lambda'}\left|\text{vacuum}\right\rangle \\ \text{where } C &= \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \mathbf{k}=\mathbf{k}' \text{ and } \lambda=\lambda', \\ 1 & \text{otherwise.}\end{cases} \end{aligned} \tag{110}$$

These states have energies

$$E((\mathbf{k},\lambda),(\mathbf{k},\lambda)') = E(\mathbf{k},\lambda) + E(\mathbf{k}',\lambda') = \hbar c|\mathbf{k}| + \hbar c|\mathbf{k}'| \tag{111}$$

appropriate for two non-interacting photons with Hamiltonian

$$\hat{H}_2 = \hat{H}_1(1^{\text{st}}\text{ photon}) + \hat{H}_1(2^{\text{nd}}\text{ photon}) = c\sqrt{\hat{\mathbf{P}}_1^2} + c\sqrt{\hat{\mathbf{P}}_2^2}. \tag{112}$$

Moreover, each photon here can have any allowed momentum $\mathbf{P}=\hbar k$ and polarization λ independently of the other photon. However, we cannot tell which photon is the first and which is the second; instead,

$$\big|(\mathbf{k},\lambda),(\mathbf{k},\lambda)'\big\rangle = \big|(\mathbf{k},\lambda)',(\mathbf{k},\lambda)\big\rangle, \tag{113}$$

so the photons are identical bosons.

Likewise, for $\mathcal{N}>2$ the $\mathcal{H}_\mathcal{N}$ space is a space of $\mathcal{N}$ identical bosons, each boson being a photon — a massless relativistic particle with two distinct polarization states. Each photon in this space may have any momentum allowed by the boundary conditions and any polarization independently from the other $\mathcal{N}-1$ photons, but we cannot tell which photon is which because they are identical bosons.

Note: reorganizing the net Hilbert space $\mathcal{H}$ of the EM fields in the cavity into subspaces $\mathcal{H}_\mathcal{N}$ of definite net numbers $\mathcal{N}$ of quanta makes it easy to see the photons and the $\mathcal{N}$–photon states; that's how we have learned that the photon is a massless relativistic particle with two polarization states $|\lambda\rangle$, and that the multiple photons are identical bosons. On the other hand, the decomposition

$$\mathcal{H} \;=\; \bigoplus_{\mathcal{N}=0}^{\infty} \mathcal{H}_{\mathcal{N}} \;=\; \mathcal{H}_0 \,\oplus\, \mathcal{H}_1 \,\oplus\, \mathcal{H}_2 \,\oplus\, \cdots \tag{114}$$

makes it harder to analyze the states which do not have definite $\mathcal{N}$ but rather several components with different $\mathcal{N}$'s. So in the next section, we shall focus on the coherent states — which do not have definite $n_{\mathbf{k},\lambda}$ or definite $\mathcal{N}_{\text{total}}$ — but instead describe the semiclassical EM waves.

Coherent states and semiclassical EM waves

Similar to a single harmonic oscillator or the wave on a string, the best wave to reproduce the classical time-dependent electromagnetic fields in quantum mechanics is in terms of the coherent states. For example, consider the coherent state ξ of a particular move $(\mathbf{k},\lambda)$ while all the other modes are in the ground state,

$$|\mathbf{k},\lambda:\xi\rangle \;=\; e^{-|\xi|^2/2}\exp(\xi\hat{a}^\dagger_{\mathbf{k},\lambda})\,|\text{vacuum}\rangle\,; \tag{115}$$

this state obeys the Schrödinger equation

$$i\hbar\frac{d}{dt}\,|\mathbf{k},\lambda:\xi(t)\rangle \;=\; \hat{H}'\,|\mathbf{k},\lambda:\xi(t)\rangle \quad \text{for } \xi(t) \;=\; \xi(0)\times e^{-i\omega_k t}. \tag{116}$$

In this coherent state,

$$\left\langle \hat{E}_{\mathbf{k},\lambda}\right\rangle(t) \;=\; \sqrt{2\pi\hbar\omega_k}\times\bigl(-i\xi \;=\; -i\xi_0 e^{-i\omega_k t}\bigr), \tag{117}$$

$$\left\langle \hat{E}_{-\mathbf{k},\lambda}\right\rangle(t) \;=\; \sqrt{2\pi\hbar\omega_k}\times\bigl(+i\xi^* \;=\; +i\xi_0^* e^{+i\omega_k t}\bigr), \tag{118}$$

$$\left\langle \hat{E}_{\mathbf{k}',\lambda'}\right\rangle(t) \;=\; 0 \quad \text{for all other } (\mathbf{k}',\lambda') \neq (\pm\mathbf{k},\lambda), \tag{119}$$

and likewise

$$\left\langle \hat{B}_{\mathbf{k},\lambda} \right\rangle (t) \;=\; \sqrt{2\pi\hbar\omega_k} \times \left(\lambda\xi \;=\; \lambda\xi_0 e^{-i\omega_k t}\right), \tag{120}$$

$$\left\langle \hat{B}_{-\mathbf{k},\lambda} \right\rangle (t) \;=\; \sqrt{2\pi\hbar\omega_k} \times \left(\lambda\xi^* \;=\; \lambda\xi_0^* e^{+i\omega_k t}\right), \tag{121}$$

$$\left\langle \hat{B}_{\mathbf{k}',\lambda'} \right\rangle (t) \;=\; 0 \quad \text{for all other } (\mathbf{k}',\lambda') \;\neq\; (\pm\mathbf{k},\lambda). \tag{122}$$

Consequently, adding up the EM fields of all the modes in the cavity, we get

$$\begin{aligned}
\left\langle \hat{\mathbf{E}}(\mathbf{x}) \right\rangle (t) \;&=\; L^{-3/2} \sum_{\mathbf{k}',\lambda'} e^{i\mathbf{k}'\cdot\mathbf{x}}\, \mathbf{e}_{\mathbf{k}',\lambda'} \left\langle \hat{E}_{\mathbf{k}',\lambda'} \right\rangle (t) \\
&=\; \sqrt{\frac{2\pi\hbar\omega_k}{L^3}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \mathbf{e}_{\mathbf{k},\lambda}\, (-i\xi_0) e^{-i\omega_k t} \\
&\quad +\; \sqrt{\frac{2\pi\hbar\omega_k}{L^3}}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, (\mathbf{e}_{-\mathbf{k},\lambda} = \mathbf{e}^*_{\mathbf{k},\lambda})\, (+i\xi_0^*) e^{+i\omega_k t} \\
&=\; \sqrt{\frac{2\pi\hbar\omega_k}{L^3}} \left(-i\xi_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_k t}\, \mathbf{e}_{\mathbf{k},\lambda}\right) \;+\; \text{complex conjugate} \\
&=\; 2\sqrt{\frac{2\pi\hbar\omega_k}{L^3}}\, \mathrm{Re}\left(-i\xi_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_k t}\, \mathbf{e}_{\mathbf{k},\lambda}\right),
\end{aligned} \tag{123}$$

and likewise

$$\left\langle \hat{\mathbf{B}}(\mathbf{x}) \right\rangle (t) \;=\; 2\sqrt{\frac{2\pi\hbar\omega_k}{L^3}}\, \mathrm{Re}\left(\lambda\xi_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_k t}\, \mathbf{e}_{\mathbf{k},\lambda}\right). \tag{124}$$

Treating these expectation values of the quantum fields as classical fields, it is easy to check that they describe the plane wave with wave vector $\mathbf{k}$, circular polarization — right for $\lambda = +1$ or left for $\lambda = -1$, — and amplitude

$$A \;=\; 2\sqrt{\frac{2\pi\hbar\omega_k}{L^3}} \times |\xi_0|. \tag{125}$$

Moreover, the classical energy of this wave

$$E_{\text{cl}} \;=\; \int d^3\mathbf{x}\, \frac{\langle\mathbf{E}\rangle^2 + \langle\mathbf{B}\rangle^2}{8\pi} \;=\; \frac{L^3}{8\pi} \times A^2 = \; \hbar\omega_k \times |\xi_0|^2 \tag{126}$$

agrees with the expectation values of the coherent state's energy (counting from the zero point)

$$\left\langle E'\right\rangle \;=\; \hbar\omega_k\times|\xi|^2 \;=\; \hbar\omega_k\times|\xi_0|^2. \tag{127}$$

As to the energy uncertainty of the coherent state,

$$\Delta E \;=\; \hbar\omega_k\times|\xi| \;=\; \hbar\omega_k\times|\xi_0|, \tag{128}$$

it becomes relatively small, $\Delta E\ll\langle E'\rangle$, for waves with classical energies $E_{\rm cl}\gg\hbar\omega_k$, hence $|\xi_0|\gg 1$.

The above example explained a coherent state representing a single circularly polarized plane wave, but it is easy to generalize it to any classical configuration of the EM fields in the cavity as long as they obey the Maxwell equations and the cavity periodicity conditions. A most general coherent state of the EM fields is a tensor product of coherent states of each mode $(\mathbf{k},\lambda)$, thus

$$|\text{coherent}\rangle \;=\; \left|\{\xi_{\mathbf{k},\lambda}\}\right\rangle \;=\; \bigoplus_{\mathbf{k},\lambda}\left|\xi_{\mathbf{k},\lambda}\right\rangle_{\text{mode}\,\mathbf{k},\lambda}\,, \tag{129}$$

or in terms of the raising and lowering operators

$$\left|\{\xi_{\mathbf{k},\lambda}\}\right\rangle \;=\; \exp\left(-\frac{1}{2}\sum_{\mathbf{k},\lambda}|\xi_{\mathbf{k},\lambda}|^2\right)\times\exp\left(\sum_{\mathbf{k},\lambda}\xi_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k},\lambda}\right)|\text{vacuum}\rangle\,, \tag{130}$$

$$\forall(\mathbf{k},\lambda):\quad \hat{a}_{\mathbf{k},\lambda}\left|\{\xi_{\mathbf{k},\lambda}\}\right\rangle \;=\; \xi_{\mathbf{k},\lambda}\left|\{\xi_{\mathbf{k},\lambda}\}\right\rangle. \tag{131}$$

Such coherent states obey the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}\left|\{\xi_{\mathbf{k},\lambda}(t)\}\right\rangle \;=\; \hat{H}'\left|\{\xi_{\mathbf{k},\lambda}(t)\}\right\rangle \tag{132}$$

provided each $\xi_{\mathbf{k},\lambda}$ changes its phase with frequency ω_k,

$$\xi_{\mathbf{k},\lambda}(t) \;=\; \xi_{\mathbf{k},\lambda}(0)\times e^{-i\omega_k t}. \tag{133}$$

Consequently, the expectation values of the EM fields in a coherent state behave like the classical EM fields,

$$\left\langle \hat{\mathbf{E}}(\mathbf{x}) \right\rangle (t) \;=\; \sum_{\mathbf{k},\lambda} \sqrt{\frac{8\pi\hbar\omega_k}{L^3}}\; \mathrm{Re}\left(-i\xi_{\mathbf{k},\lambda}(0)\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_k t}\, \mathbf{e}_{\mathbf{k},\lambda}\right),$$
$$\left\langle \hat{\mathbf{B}}(\mathbf{x}) \right\rangle (t) \;=\; \sum_{\mathbf{k},\lambda} \sqrt{\frac{8\pi\hbar\omega_k}{L^3}}\; \mathrm{Re}\left(\xi_{\mathbf{k},\lambda}(0)\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega_k t}\, \lambda\mathbf{e}_{\mathbf{k},\lambda}\right). \tag{134}$$

In particular, these classical fields obey the Maxwell equations (and the periodicity conditions for the cavity), and their classical energy

$$E_{\mathrm{cl}} \;=\; \int d^3\mathbf{x}\, \frac{\langle \mathbf{E}\rangle^2 + \langle \mathbf{B}\rangle^2}{8\pi} \;=\; \sum_{\mathbf{k},\lambda} \hbar\omega_k \times |\xi_{\mathbf{k},\lambda}|^2 \tag{135}$$

agrees with the energy expectation value in the coherent state,

$$\langle E\rangle \;=\; \sum_{\mathbf{k},\lambda} \hbar\omega_k \times |\xi_{\mathbf{k},\lambda}|^2. \tag{136}$$

Finally, for any classical EM fields $\mathbf{E}_{\mathrm{cl}}(\mathbf{x},t)$ and $\mathbf{B}_{\mathrm{cl}}(\mathbf{x},t)$ in the cavity obeying free Maxwell equations (and the periodicity conditions), there is a coherent state $\left|\{\xi_{\mathbf{k},\lambda}\}\right\rangle$ with

$$\left\langle \hat{\mathbf{E}}(\mathbf{x}) \right\rangle (t) \;=\; \mathbf{E}_{\mathrm{cl}}(\mathbf{x},t) \quad \text{and} \quad \left\langle \hat{\mathbf{B}}(\mathbf{x}) \right\rangle (t) \;=\; \mathbf{B}_{\mathrm{cl}}(\mathbf{x},t), \tag{137}$$

specifically the state with

$$\xi_{\mathbf{k},\lambda}(t) \;=\; \frac{\lambda B^{\mathrm{cl}}_{\mathbf{k},\lambda}(t) \,+\, iEB^{\mathrm{cl}}_{\mathbf{k},\lambda}(t)}{\sqrt{8\pi\hbar\omega_k}} \;=\; \frac{1}{\sqrt{8\pi\hbar\omega_k L^3}} \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \mathbf{e}_{\mathbf{k},\lambda}\cdot\big(\lambda\mathbf{B}_{\mathrm{cl}}(\mathbf{x},t) \,+\, i\mathbf{E}_{\mathrm{cl}}(\mathbf{x},t)\big). \tag{138}$$

But to save time, let me skip verifying this statement and leave it an optional extra exercise for the interested students.