

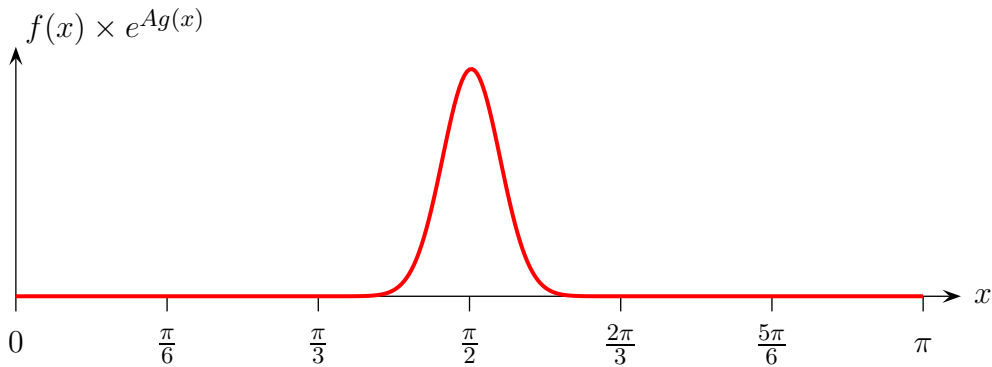
1. Saddle Point Method of Asymptotic Expansion

1.1 THE REAL CASE.

Consider an integral of the form

$$I(A) = \int_{x_1}^{x_2} dx f(x) e^{Ag(x)} \quad (1.1)$$

where f and g are some real functions of x and $A > 0$ is a parameter. For large values of A the integrand has narrow sharp peaks like this



(in this particular example $f(x) = x$, $g(x) = \sin x$ and $A = 100$), and the integral is completely dominated by the biggest peak. Each peak is located at a maximum of $g(x)$, and its width is $O(1/\sqrt{A})$. So let x_0 be the location of the biggest maximum of g between x_1 and x_2 , and let's change the integration variable from x to y according to

$$x = x_0 + \frac{y}{\sqrt{A}}. \quad (1.2)$$

Expanding $Ag(x)$ in powers of y , we have

$$Ag(x) = Ag(x_0) + \frac{1}{2}y^2g''(x_0) + \frac{y^3g'''(x_0)}{6\sqrt{A}} + \dots \quad (1.3)$$

(the first-derivative term is missing here because x_0 is a maximum of g). Treating this expansion as expansion in powers of $1/\sqrt{A}$ rather than y , we expand the exponential $e^{Ag(x)}$

as

$$e^{Ag(x)} = e^{Ag(x_0)} \cdot e^{y^2 g''(x_0)/2} \cdot \left(1 + \frac{y^3 g'''(x_0)}{6\sqrt{A}} + \frac{3y^4 g'''' + y^6 (g''')^2}{72A} + \dots \right). \quad (1.4)$$

Similarly, assuming $f(x_0) \neq 0$, we have

$$f(x) = f(x_0) \cdot \left(1 + \frac{y f'(x_0)}{f(x_0)\sqrt{A}} + \frac{y^2 f''(x_0)}{2f(x_0)A} + \dots \right). \quad (1.5)$$

Substituting eqs. (1.4) and (1.5) into (1.1) gives us

$$I(A) = \frac{f(x_0) e^{Ag(x_0)}}{\sqrt{A}} \int_{y_1}^{y_2} dy e^{y^2 g''(x_0)/2} \cdot \left(1 + \sum_{n=1}^{\infty} A^{-n/2} P_n(y) \right), \quad (1.6)$$

where P_n are some polynomial functions of y ; it is easy to show that $P_n(y)$ are odd polynomials for odd n and even polynomials for even n .

We assume that $x_1 < x_0 < x_2$ — *i.e.*, the maximum of g occurs strictly between x_1 and x_2 and not at one of the end points. Then, in the large A limit, $y_1 \rightarrow -\infty$ and $y_2 \rightarrow +\infty$ as $O(\sqrt{A})$, and since the Gaussian factor $e^{y^2 g''(x_0)/2}$ decreases very rapidly for $y \rightarrow \pm\infty$ (note $g''(x_0) < 0$ because x_0 is a maximum of g), we may extend the integration range of the integral (1.6) to the entire real axis. This is in-exact for a finite A , but the relative error in $I(A)$ due to this extension decrease with A faster than any power of A . Therefore,

$$\begin{aligned} I(A) &\approx \frac{f(x_0) e^{Ag(x_0)}}{\sqrt{A}} \int_{-\infty}^{+\infty} dy e^{y^2 g''(x_0)/2} \cdot \left(1 + \sum_{n=1}^{\infty} A^{-n/2} P_n(y) \right) \\ &= f(x_0) e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \times \left(1 + \sum_{n=1}^{\infty} \frac{C_{2n}}{A^n} \right), \end{aligned} \quad (1.7)$$

where

$$C_{2n} = \int_{-\infty}^{+\infty} dy P_{2n}(y) e^{g''(x_0)y^2/2} \quad (1.8)$$

while the similar integrals $C_{2n-1} = 0$ because P_{2n-1} are odd polynomials of y . Working out explicit formulae for the C_{2n} in terms of the derivatives of the f and g functions at x_0 is a

straightforward (but rather boring) exercise; for example

$$C_2 = \left(-\frac{f''}{2fg''} + \frac{f'g'''}{2f(g'')^2} + \frac{g^{(4)}}{8(g'')^2} - \frac{5(g''')^2}{24(g'')^3} \right) @x_0. \quad (1.9)$$

Fortunately, we won't need such formulae in these notes, so let me skip their derivation.

The series in eq. (1.7) usually has zero radius of convergence, so it cannot be actually summed up for any finite value of A . Instead, it's an *asymptotic series* whose partial sums have the right asymptotic behavior in the large A limit,

$$1 + \sum_{n=1}^{\infty} \frac{C_{2n}}{A^n} = 1 + O(1/A) = 1 + C_2/A + O(1/A^2) = 1 + C_2/A + C_4/A^2 + O(1/A^3) = \dots \quad (1.10)$$

in the strict mathematical sense of $O(1/A^n)$ — it's no larger than const/A^n but only for a large enough A . Thus the precise meaning of eq. (1.7) is

$$\begin{aligned} I(A) &= f(x_0) e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \cdot (1 + O(1/A)) \\ &= f(x_0) e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \cdot (1 + C_2/A + O(1/A^2)) \\ &= \dots \end{aligned} \quad (1.11)$$

in the large A limit.

1.2 THE COMPLEX CASE.

Now consider the case of complex $f(x)$ and $g(x)$. Again, in the large A limit the integrand is sharply peaked near the maximum of $\text{Re} g(x)$, so it seems like we could proceed similarly to the real case. There is however one crucial difference — the maximum of $\text{Re} g(x)$ is not necessarily the stationary point of the phase $\text{Im} g(x)$, so we have to add a purely imaginary term $\sqrt{A} y g'(x_0)$ to the expansion (1.3) for the $Ag(x)$. Consequently, the integral (1.6) becomes

$$I(A) = \frac{f(x_0) e^{Ag(x_0)}}{\sqrt{A}} \int dy e^{y^2 g''(x_0)/2} e^{y\sqrt{A}g'(x_0)} \cdot \left(1 + \sum_{n=1}^{\infty} A^{-n/2} P_n(y) \right), \quad (1.12)$$

and the rapidly (in the large A limit) oscillating phase factor $e^{y\sqrt{A}g'(x_0)}$ severely suppresses the asymptotic behavior of the integral. Specifically, the leading term in the expansion now

gives us

$$I(A) \xrightarrow[A \rightarrow \infty]{?} f(x_0) e^{Ag(x_0)} \sqrt{\frac{2\pi}{-Ag''(x_0)}} \cdot \exp\left(-Ag'^2(x_0)/g''(x_0)\right), \quad (1.13)$$

and the last factor here is very small because the real part of $g'^2(x_0)/g''(x_0)$ is always positive. Consequently, a maximum of $\operatorname{Re}g(x)$ does not contribute at full strength unless it also happens to be a stationary point of the phase $\operatorname{Im}g(x)$. The suppression is so strong that the region around a maximum of $\operatorname{Re}g$ that is not a stationary point of the phase may no longer dominate the large A asymptotic behavior of the integral. This calls for a different approach in the complex case.

Indeed, proper complexification of the integral (1.1) goes beyond making f and g complex functions of a real variable x . Instead, we should take f and g to be *complex analytic functions of a complex variable*, and write a contour integral

$$I(A) = \int_{\Gamma} dz f(z) \times e^{Ag(z)} \quad (1.14)$$

over some contour Γ in the complex plane. A fundamental theorem of complex analysis states that contour integrals of analytic functions are invariant under any continuous deformation of the contour that does not affect its end points (if any) and does not drag contour over any singularities of the integrand. Thus for the problem at hand, we deform the contour until the maximum of $\operatorname{Re}g$ along the contour is also a stationary point of the phase $\operatorname{Im}g$. Often, such deformation turns a real interval from x_1 to x_2 into a non-real contour in the complex plane. This may seem like making the problem even more complex (in both senses of the word) than it is, but in fact this leads to an easily obtainable large A asymptotics.

The points in the complex plane where a maximum of $\operatorname{Re}g$ (along some contour) coincides with a stationary point of the phase $\operatorname{Im}g$ are the zeros of the complex derivative $g'(z)$. Near such a point $\operatorname{Re}g(z)$ looks like a saddle or the top of a mountain pass — it has a maximum along some directions in the complex plane and a minimum along other directions — hence the two names for the asymptotic method described here: *the saddle point method* or *the mountain pass method*. The mountain pass analogy is particularly apt, for a properly routed contour not only goes through a zero of g' , but crosses that zero in the manner of a highway

crossing a mountain pass, by starting in a valley (of low $\text{Re } g$), going up till it reaches the top of the pass, then going down into another valley. Although a mountain goat might think of a pass as a low point on a trail from one hill to another, thinking like a goat does not work for computing integrals.

Once you have the right contour Γ , obtaining the large A asymptotics of the integral (1.14) becomes analogous to the real case. First, we change the integration variable from z to a new complex variable y related to z via

$$z = z_0 + \frac{\eta y}{\sqrt{A}}, \quad (1.15)$$

where z_0 is a zero of $g'(z)$ and η is a unimodular complex number, $|\eta| = 1$. Second, we expand the integrand of (1.14) into powers of $1/\sqrt{A}$ in the same manner as we did in the real case; this gives us

$$I(A) = \frac{\eta f(z_0) e^{Ag(z_0)}}{\sqrt{A}} \int_{\Gamma'} dy e^{y^2 \eta^2 g''(z_0)/2} \cdot \left(1 + \sum_{n=1}^{\infty} A^{-n/2} P_n(\eta y) \right), \quad (1.16)$$

where Γ' is the integration contour in the y plane. The Γ' always crosses the $y = 0$ point, but the direction of that crossing depends on η ; to simplify our arguments, let us choose the η that would make the direction of the Γ' contour — *i.e.*, the direction of dy along the Γ' — real and positive at $y = 0$. Consequently, for the original contour Γ (in the z plane) which crosses the saddle point z_0 as a mountain highway rather than as a goat trail, the coefficient of $y^2/2$ in the exponent in eq. (1.16) has a negative real part,

$$\text{Re}(\eta^2 \times g''(z_0)) < 0. \quad (1.17)$$

Also, the Γ' contour in the y plane is tangent to the real axis at $y = 0$, so in the $A \rightarrow \infty$ limit becomes the real axis itself, plus some appendages at infinity. Similar to the real case, contribution of these appendages to the integral (1.16) has relative magnitude smaller than any negative power of A , so we may safely neglect it. In other words, we may replace Γ'

with the real axis, hence

$$\begin{aligned}
I(A) &\approx \frac{\eta f(z_0) e^{Ag(z_0)}}{\sqrt{A}} \int_{-\infty}^{+\infty} dy e^{y^2 \eta^2 g''(z_0)/2} \times \left(1 + \sum_{n=1}^{\infty} A^{-n/2} P_n(\eta y) \right) \\
&\ll \text{where the integral converges thanks to the inequality (1.17)} \gg \\
&= \sqrt{\frac{2\pi}{A}} \exp(Ag(z_0)) \times \frac{\eta f(z_0)}{\sqrt{-\eta^2 g''(z_0)}} \times \left(1 + \sum_{n=1}^{\infty} \frac{C_{2n}}{A^n} \right), \\
(1.18)
\end{aligned}$$

in complete analogy to the real formula (1.11). Note that the η parameter in eq. (1.18) essentially cancels itself out, except that it helps determine the sign of the complex square root $\sqrt{-\eta^2 g''(z_0)}$ — the real part of this root should be positive.

Although formulae (1.11) and (1.18) differ very little, there is one important difference between large A asymptotics of real and complex integrals $I(A)$. The asymptotics of a real integral (1.1) is always dominated by the global maximum of $g(x)$ *within the integration range*, which can be either the biggest local maximum x_0 strictly within the range, or one of the end points of that range (in which case eq. (1.11) does not apply). Local maxima of $g(x)$ *outside* the integration range of (1.1) never play any role in the asymptotic expansion even if they are bigger than any maximum within the range.

For the complex integrals (1.14), determining which of the saddle points of $g(z)$ in the complex plane dominates the integral's asymptotics is not so straightforward. Given the freedom to deform the integration contour Γ , one cannot simply say that a saddle point z_0 is “within the integration range” while another saddle point is “outside the integration range”, because Γ can always be deformed to cross any point we like. Usually, the general direction of the original contour and the phases of g'' at saddle points which control the directions in which those saddle points should be traversed give sufficient clues to determine which saddle point is dominant and how to deform the contour to go through it. However, such determination is somewhat of a black art best explained on specific examples; one such example — the asymptotic behavior of Airy functions — shall be discussed in the next section.

2. Airy Functions

2.1 CONSTRUCTION.

Airy functions Ai and Bi are solutions of the linear differential equation

$$\Psi''(z) - z\Psi(z) = 0, \quad (2.1)$$

which is a scale-invariant form of the Schrödinger equation for a quantum particle subject to a constant force, *i.e.*, linear potential $V(x) = -Fx$,

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} - Fx \times \Psi(x) = E\Psi(x). \quad (2.2)$$

The relation between z and the particle's coordinate x is

$$z = -\text{sign}(F) \sqrt[3]{\frac{2m|F|}{\hbar^2}} \times (x - x_0) \quad (2.3)$$

where $x_0 = -E/F$ is the classical turning point where $V(x_0) = E$.

For Bessel functions experts, the easiest way to solve the equation (2.1) is to substitute

$$z = \frac{2i}{3}y^{3/2}, \quad \Psi(z) = y^{1/3}J(y). \quad (2.4)$$

In terms of $J(y)$ equation (2.1) becomes

$$J''(y) + \frac{J'(y)}{y} + \left(1 - \frac{1}{9y^2}\right) J(y), \quad (2.5)$$

which is the Bessel equation of the order $1/3$. Thus, $J(y)$ is a linear combination of the Bessel functions $J_{+1/3}(y)$ and $J_{-1/3}(y)$.

However, it is more instructive to solve the equation (2.1) in a different way. Let us perform a Laplace-like transform and look for a solution $\Psi(z)$ in the form of a contour

integral

$$\Psi(z) = \int_{\Gamma} dt e^{tz} \Phi(t); \quad (2.6)$$

here Γ is some z -independent contour in the complex t plane and $\Phi(t)$ is an analytic function of t that does not have any singularities on the contour Γ . With these assumptions, the second derivative $\Psi''(z)$ is related to $t^2\Phi(t)$ via

$$\Psi''(z) = \int_{\Gamma} dt e^{tz} t^2 \Phi(t). \quad (2.7)$$

On the other hand, $z\Psi(z)$ is related to the first derivative of $\Phi(t)$ via

$$\begin{aligned} z\Psi(z) &= \int_{\Gamma} dt \frac{\partial e^{tz}}{\partial t} \times \Phi(t) \\ &\ll \text{integrating by parts} \gg \\ &= \left[e^{tz} \Phi(t) \right]_{\text{start}(\Gamma)}^{\text{end}(\Gamma)} - \int_{\Gamma} dt e^{tz} \Phi'(t). \end{aligned} \quad (2.8)$$

Therefore, in terms of $\Phi(t)$, the second order eq. (2.1) is equivalent to a first order equation

$$t^2\Phi(t) + \Phi'(t) = 0, \quad (2.9)$$

plus a boundary condition

$$e^{tz}\Phi(t) = 0 \quad \text{on the boundary of the contour } \Gamma. \quad (2.10)$$

The general solution of the equation (2.9) is

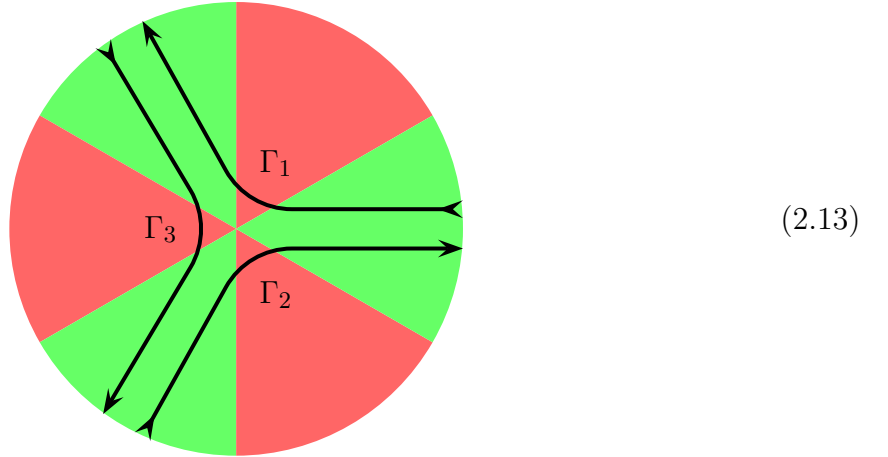
$$\Phi(t) = C \exp(-t^3/3), \quad (2.11)$$

where C is a constant; therefore, we have formally solved the Airy equation (2.1) in terms of the contour integral

$$\Psi(z) = C \int_{\Gamma} dt \exp\left(tz - \frac{1}{3}t^3\right), \quad (2.12)$$

and all we need now is to specify the integration contour Γ .

Since the integrand of eq. (2.12) has no singularities for any finite t , only the end points of the contour would affect the integral; in particular, any closed Γ would lead to the trivial solution $\Psi(z) \equiv 0$. On the other hand, an open Γ with finite end points would violate the boundary condition (2.10). Hence, both end points of the contour Γ must be at the complex ∞ , and the directions in which the two ends of the contour approach the ∞ would completely determine the integral (2.12) (the latter follows from the lack of finite singularities of the integrand). Those directions of approach are controlled by two considerations: First, one should approach the infinity along directions in which the integrand decreases rather than increases; for the problem at hand, this allows angles of approach between $-\pi/6$ and $+\pi/6$, between $+\pi/2$ and $+5\pi/6$, and between $-5\pi/6$ and $-\pi/2$, that is, within one of the three green sectors on the following diagram:



The second consideration is that all approaches within the same sector are equivalent; consequently, the two ends of the contour must belong to different sectors. These two considerations give us a choice of three contours — Γ_1 , Γ_2 and Γ_3 on figure (2.13) — corresponding to three different solutions $\Psi_1(z)$, $\Psi_2(z)$ and $\Psi_3(z)$ of the Airy equation (2.1). Since the combined contour $\Gamma_1 + \Gamma_2 + \Gamma_3$ is shrinkable, it follows that $\Psi_1(z) + \Psi_2(z) + \Psi_3(z) \equiv 0$, so only two of the solutions are independent. The Airy functions Ai and Bi are the two independent solutions which happen to be real for real z , namely

$$\begin{aligned}
 Ai(z) &= \frac{i}{2\pi} \int_{\Gamma_3} dt \exp\left(tz - \frac{1}{3}t^3\right), \\
 Bi(z) &= \frac{1}{2\pi} \int_{\Gamma_2 - \Gamma_1} dt \exp\left(tz - \frac{1}{3}t^3\right),
 \end{aligned}
 \tag{2.14}$$

where the integral over $\Gamma_2 - \Gamma_1$ is the integral over Γ_2 , plus the integral over $-\Gamma_1$, the latter being Γ_1 traversed in the direction opposite to the arrow on figure (2.13).

2.2 ASYMPTOTICS

For the purpose of matching WKB solutions on two sides of a classical turning point we need to know the asymptotic behaviors of the Airy functions for large real z , both positive and negative. The easiest way to obtain this information is to use the saddle point method described in the previous section. Although the integrals in eq. (2.14) do not have the exact form (1.14), changing the integration variable from t to $\tau \equiv t/\sqrt{|z|}$ brings them to the desired form:

$$\begin{aligned} Ai(z) &= \frac{i}{2\pi} \times |z|^{1/2} \int_{\Gamma_3} d\tau \exp\left(|z|^{3/2}\left(\frac{z}{|z|}\tau - \frac{1}{3}\tau^3\right)\right), \\ Bi(z) &= \frac{1}{2\pi} \times |z|^{1/2} \int_{\Gamma_2 - \Gamma_1} d\tau \exp\left(|z|^{3/2}\left(\frac{z}{|z|}\tau - \frac{1}{3}\tau^3\right)\right); \end{aligned} \quad (2.15)$$

note that the contours $\Gamma_{1,2,3}$ are essentially scale invariant and hence can be used without change in both t and τ planes.

For positive z , we have $g(\tau) = \tau - \tau^3/3$, which has saddle points at $\tau = \pm 1$. The positive saddle point at $\tau = +1$ has a higher value of $g(\tau)$ than the negative saddle point ($g(\tau = +1) = +\frac{2}{3}$ while $g(\tau = -1) = -\frac{2}{3}$), so it seems to be dominant. This dominance however would only work for contours that traverse that point in the mountain-pass-like fashion; given $g''(\tau = +1) = -2$, this means within $\pi/4$ of the horizontal axis. A quick look at figure (2.13) shows that contour deformation that would make Γ traverse the point $\tau = +1$ horizontally is an easy thing to do for $\Gamma = \Gamma_2$ or $\Gamma = -\Gamma_1$ ($\eta = +1$ in both cases) but is a very unnatural thing to do for the Γ_3 contour. Consequently, the integrals (2.15) over the contours Γ_2 and $-\Gamma_1$ — but not over the Γ_3 — are dominated by the positive saddle points $Z_0 = +1$, hence

$$\begin{aligned} \int_{\Gamma_2} d\tau \exp\left(|z|^{3/2}\left(\tau - \frac{1}{3}\tau^3\right)\right) &= \int_{-\Gamma_1} d\tau \exp\left(|z|^{3/2}\left(\tau - \frac{1}{3}\tau^3\right)\right) \\ &\approx \sqrt{\frac{2\pi}{2z^{3/2}}} \times \exp\left(+\frac{2}{3}z^{3/2}\right) \times (1 + O(z^{-3/2})), \end{aligned} \quad (2.16)$$

cf. eq. (1.18), and therefore *the irregular Airy function* Bi behaves like

$$\begin{aligned} Bi(z) &= (2 \text{ contours}) \times \frac{z^{1/2}}{2\pi} \times \sqrt{\frac{2\pi}{2z^{3/2}}} \times \exp\left(\frac{2}{3}z^{3/2}\right) \times (1 + O(z^{-3/2})) \\ &= \frac{1}{\pi^{1/2} z^{1/4}} \times \exp\left(\frac{2}{3}z^{3/2}\right) \times (1 + O(z^{-3/2})). \end{aligned} \quad (2.17)$$

On the other hand, the integral over the Γ_3 contours — and hence *the regular Airy function* Ai — is dominated by the other saddle point $\tau = -1$. At that saddle point $g''(\tau = -1) = +2$, so we want the contour to traverse it in a direction within $\pi/4$ of the vertical axis. Given the general downward direction of the Γ_3 contour, this suggests $\eta = -i$ for the deformed contour that crosses $\tau = -1$, hence

$$\int_{\Gamma_3} d\tau \exp\left(|z|^{3/2}\left(\tau - \frac{1}{3}\tau^3\right)\right) \approx \sqrt{\frac{2\pi}{z^{3/2}}} \times \frac{-i}{\sqrt{+2}} \times \exp\left(-\frac{2}{3}z^{3/2}\right) \times (1 + O(z^{-3/2})) \quad (2.18)$$

and therefore

$$Ai(z) = \frac{iz^{1/2}}{2\pi} \times (2.18) = \frac{1}{2\pi^{1/2} z^{1/4}} \times \exp\left(-\frac{2}{3}z^{3/2}\right) \times (1 + O(z^{-3/2})). \quad (2.19)$$

The asymptotic limits (2.19) and (2.17) of Airy functions are for large *positive* z . For large *negative* z , we have

$$g(\tau) = -\tau - \frac{\tau^3}{3}, \quad (2.20)$$

so the saddle points are imaginary $\tau = \pm i$ rather than real $\tau = \pm 1$. At both saddle points $g(\tau)$ has the same real part, namely $\text{Re}g(\tau = \pm i) = 0$, so we expect the two points to be co-dominant for both Airy functions $Ai(z)$ and $Bi(z)$.

For negative z , $g(\tau) = -\tau - \tau^3/3$ and the saddle points are at $\tau = \pm i$. Both saddle points have the same $\text{Re}g(\tau)$, namely zero, so we expect the two points to be co-dominant for both Ai and Bi . The second derivative $g''(\tau = \pm i) = \mp 2i$, so the mountain-path-like directions are NW to SE or SE to NW at $\tau = +i$ and SW to NE or NE to SW at $\tau = -i$. Again, we use the general direction of the contours on the diagram (2.13) to decide that the deformed Γ_3 should traverse the upper saddle point ($\tau = +i$) in the NW to SE direction ($\eta = e^{-\pi i/4}$)

and the lower saddle point ($\tau = -i$) from NE to SW ($\eta = e^{-3\pi i/4}$). Hence, using the general asymptotic formula (1.18), we obtain

$$\begin{aligned} \int_{\Gamma_3} dt \exp(|z|^{3/2}(-\tau - \frac{1}{3}\tau^3)) &\approx \sqrt{\frac{2\pi}{|z|^{3/2}}} \times \left(\begin{aligned} &\frac{e^{-\pi i/4}}{\sqrt{2}} \times \exp(-\frac{2}{3}i|z|^{3/2}) \times (1 + O(|z|^{-3/2})) \\ &+ \frac{e^{-3\pi i/4}}{\sqrt{2}} \times \exp(+\frac{2}{3}i|z|^{3/2}) \times (1 + O(|z|^{-3/2})) \end{aligned} \right) \\ &= -2i \sqrt{\frac{\pi}{|z|^{3/2}}} \times \sin(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}) \times (1 + O(|z|^{-3/2})) \end{aligned} \quad (2.21)$$

and therefore the regular Airy function Ai behaves as

$$Ai(z) = \frac{i|z|^{1/2}}{2\pi} \times (2.21) = \frac{1}{\pi^{1/2}|z|^{1/4}} \sin(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}) \times (1 + O(|z|^{-3/2})). \quad (2.22)$$

The irregular Airy function Bi can be studied in the same way: It is plain to see that the deformed Γ_2 contour should traverse only the $\tau = -i$ saddle point in the SW to NE direction ($\eta = e^{+\pi i/4}$) while the deformed $-\Gamma_1$ contour should traverse only the $\tau = +i$ saddle point in the direction NW to SE ($\eta = e^{-\pi i/4}$). Therefore,

$$\begin{aligned} \int_{\Gamma_2} dt \exp(|z|^{3/2}(-\tau - \frac{1}{3}\tau^3)) &\approx \sqrt{\frac{2\pi}{|z|^{3/2}}} \times \frac{e^{+\pi i/4}}{\sqrt{2}} \times \exp(+\frac{2}{3}i|z|^{3/2}) \times (1 + O(|z|^{-3/2})), \\ \int_{-\Gamma_1} dt \exp(|z|^{3/2}(-\tau - \frac{1}{3}\tau^3)) &\approx \sqrt{\frac{2\pi}{|z|^{3/2}}} \times \frac{e^{-\pi i/4}}{\sqrt{2}} \times \exp(-\frac{2}{3}i|z|^{3/2}) \times (1 + O(|z|^{-3/2})), \end{aligned} \quad (2.23)$$

and therefore

$$Bi(z) \approx \frac{1}{\pi^{1/2}|z|^{1/4}} \times \cos(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}) \times (1 + O(|z|^{-3/2})). \quad (2.24)$$

2.3 AIRY FUNCTIONS AND THE WKB APPROXIMATION.

Consider 1D motion of a quantum particle in some potential $V(x)$. In the WKB approximation, the particle's wave function in the classically allowed region (where $V(x) < E$) looks like

$$\Psi(x) = \sum_{\pm} \frac{C_{\pm}}{\sqrt{k(x)}} \times \exp\left(\pm i \int dx k(x)\right), \quad k(x) = \frac{\sqrt{2m(E - V(x))}}{\hbar}. \quad (2.25)$$

The approximation is valid when the potential is smooth so that $k(x)$ changes slowly on the $1/k$ scale of distance. A similar approximation exists in the classically forbidden region where $V(x) > E$, namely

$$\Psi(x) = \sum_{\pm} \frac{C_{\pm}}{\sqrt{\kappa(x)}} \times \exp\left(\pm \int dx \kappa(x)\right), \quad \kappa(x) = \frac{\sqrt{2m(V(x) - E)}}{\hbar}; \quad (2.26)$$

again, this approximation is valid as long as $\kappa(x)$ changes slowly on the $1/\kappa$ scale.

Near a classical turning point x_0 where $V(x_0) = E$, both approximations (2.25) and (2.26) break down. Instead, near x_0 we treat the force $F = -dV/dx$ as approximately constant, so the wave function is approximately an Airy function of

$$z = -\text{sign}(F) \sqrt[3]{\frac{2m|F|}{\hbar^2}} \times (x - x_0). \quad (2.3)$$

For $z \rightarrow \pm\infty$, x is far enough from the turning point x_0 so the Airy function of z should match the WKB approximation for a linear potential $V = -F(x - x_0)$. Specifically, $z \rightarrow +\infty$ corresponds to the classically forbidden side of the turning point x_0 , so the Airy-function solution should match eq. (2.26), while the $z \rightarrow -\infty$ corresponds to the classically allowed side so the Airy function should match the eq. (2.25).

Let's see how this works. For the sake of definiteness, let $F > 0$, so the classically allowed region is to the right of x_0 while the classically forbidden region is to the left of x_0 . On the forbidden side $x < x_0$, $z > 0$ we have:

$$\kappa(x) = \frac{\sqrt{2mF(x_0 - x)}}{\hbar} = \sqrt[3]{\frac{2mF}{\hbar^2}} \times z^{1/2}, \quad (2.27)$$

$$\int_x^{x_0} \kappa(x') dk' = \frac{2}{3}(x_0 - x)^{3/2} \frac{\sqrt{2mF}}{\hbar} = \frac{2}{3} z^{3/2}, \quad (2.28)$$

hence the two WKB wave functions (2.26) become

$$\begin{aligned} \Psi_1(x) &= \frac{C_1}{2\sqrt{\kappa(x)}} \exp\left(-\int_x^{x_0} \kappa(x') dk'\right) \rightarrow \frac{C_1}{2} \sqrt[6]{\frac{\hbar^2}{2mF}} \times z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right), \\ &\langle\langle \text{note factors of 2 in the denominator} \rangle\rangle \\ \Psi_2(x) &= \frac{C_2}{\sqrt{\kappa(x)}} \exp\left(+\int_x^{x_0} \kappa(x') dk'\right) \rightarrow C_2 \sqrt[6]{\frac{\hbar^2}{2mF}} \times z^{-1/4} \exp\left(+\frac{2}{3} z^{3/2}\right), \end{aligned} \quad (2.29)$$

in perfect agreement with the $z \rightarrow +\infty$ limits of the two Airy functions (2.19) and (2.17). Specifically,

$$\text{for } z \rightarrow +\infty, \quad \Psi_1(x) = C_1 \sqrt{\pi} \sqrt[6]{\frac{\hbar^2}{2mF}} \times Ai(z), \quad \Psi_2(x) = C_2 \sqrt{\pi} \sqrt[6]{\frac{\hbar^2}{2mF}} \times Bi(z). \quad (2.30)$$

Likewise, in the allowed side $x > x_0$, $z < 0$ we have:

$$k(x) = \frac{\sqrt{2mF(x - x_0)}}{\hbar} = \sqrt[3]{\frac{2mF}{\hbar^2}} \times |z|^{1/2}, \quad (2.31)$$

$$\int_{x_0}^x k(x') dx' = \frac{2}{3}(x - x_0)^{3/2} \frac{\sqrt{2mF}}{\hbar} = \frac{2}{3} |z|^{3/2}, \quad (2.32)$$

hence the two WKB wave functions (2.25), — or rather, their real linear combinations

$$\begin{aligned} \Psi_1(x) &= \frac{C_1}{\sqrt{k(x)}} \times \sin\left(\frac{\pi}{4} + \int_{x_0}^x k(x') dx'\right), \\ \Psi_2(x) &= \frac{C_2}{\sqrt{k(x)}} \times \cos\left(\frac{\pi}{4} + \int_{x_0}^x k(x') dx'\right), \end{aligned} \quad (2.33)$$

become

$$\begin{aligned}\Psi_1(x) &\rightarrow C_1 \sqrt[6]{\frac{\hbar^2}{2mF}} \times |z|^{-1/4} \sin\left(\frac{\pi}{4} + \frac{2}{3}|z|^{3/2}\right), \\ \Psi_2(x) &\rightarrow C_2 \sqrt[6]{\frac{\hbar^2}{2mF}} \times |z|^{-1/4} \cos\left(\frac{\pi}{4} + \frac{2}{3}|z|^{3/2}\right),\end{aligned}\tag{2.34}$$

in perfect agreement with the $z \rightarrow -\infty$ limits of the Airy functions (2.19) and (2.17). Specifically,

$$\text{for } z \rightarrow -\infty, \quad \Psi_1(x) = C_1 \sqrt{\pi} \sqrt[6]{\frac{\hbar^2}{2mF}} \times Ai(z), \quad \Psi_2(x) = C_2 \sqrt{\pi} \sqrt[6]{\frac{\hbar^2}{2mF}} \times Bi(z).\tag{2.35}$$

Moreover, comparing eqs. (2.30) and (2.35) to each other, we immediately learn how to match the WKB solutions on the allowed and forbidden sides of the classical turning point x_0 :

$$\begin{aligned}\text{forbidden side } x < x_0 &\quad \longleftrightarrow \quad \text{allowed side } x > x_0 \\ \Psi_1(x) = \frac{C_1}{2\sqrt{\kappa(x)}} \exp\left(-\int_x^{x_0} \kappa(x') dx'\right) &\quad \longleftrightarrow \quad \Psi_1(x) = \frac{C_1}{\sqrt{k(x)}} \sin\left(\frac{\pi}{4} + \int_{x_0}^x k(x') dx'\right), \\ \Psi_2(x) = \frac{C_2}{\sqrt{\kappa(x)}} \exp\left(+\int_x^{x_0} \kappa(x') dx'\right) &\quad \longleftrightarrow \quad \Psi_2(x) = \frac{C_2}{\sqrt{k(x)}} \cos\left(\frac{\pi}{4} + \int_{x_0}^x k(x') dx'\right).\end{aligned}\tag{2.36}$$