E3/1

2 boson Hilbert space

$$\psi(\vec{x}_1, \vec{x}_2) = \psi(\vec{x}_2, \vec{x}_1)$$

N boson wave functions

$$\psi(\vec{x}_1, \cdots \vec{x}_N) = \psi\begin{pmatrix}\text{any permutation} \\ \text{of } \vec{x}_1, \cdots, \vec{x}_N\end{pmatrix}$$

$$\hat{\psi}(\vec{x}) = \sum_{\vec{p}} L^{-3/2} e^{-i\vec{p}\cdot\vec{x}/\hbar} \hat{a}_p$$

$$\hat{\psi}^{+}(x) = \sum_{\vec{p}} L^{-3/2} e^{+i\vec{p}\cdot\vec{x}/\hbar} \hat{a}_p^{+}$$

$\hat{\psi}^{+}$ creates an extra atom @ point $\vec{x}$

$\hat{\psi}$ destroys an atom @ $\vec{x}$.

Field-Particle Duality

Exactly same Quantum theory
but 2 different classical
(or semi-classical) limits

α) classical Fields
β) classical Particles.

QED $\quad$ α) classical EM fields
$\quad\quad\quad$ β) Photons.

α) classical atoms
$\downarrow$
Quantum theory
$\downarrow$
β) classical $\Phi(x)$ & $\Phi^*(x)$ fields.

$\exists$ non zero $\Phi(x)$ in a
Bose-Einstein condensate.

$\Phi(x)$ describe the hydrodynamics
of the superfluid component.

Bosonic atoms like ^{4}He at zero temperature

Naively: all atoms in same state $|\vec{p}=0\rangle$

$$|\text{naive}\rangle = |n_0 = N, \text{ other } n_{\vec{p}} = 0\rangle$$

Trouble: long-distance quantum correlations

$$G(x,y) = \langle\text{naive}|\hat{\Psi}^\dagger(x)\hat{\Psi}(y)|\text{naive}\rangle$$
$$- \langle\text{naive}|\hat{\Psi}^\dagger(x)|\text{naive}\rangle * \langle\text{naive}|\hat{\Psi}(y)|\text{naive}\rangle$$

$$= \frac{N}{\text{volume}} \neq 0 \text{ even for } |x-y| \to \infty$$

$\to$ bad.

Simplest solution: coherent state

$$|\text{coherent}\rangle = e^{-\bar{N}/2}\exp(\sqrt{\bar{N}}\,\hat{a}_0^\dagger)|\text{vac}\rangle.$$

$\hookrightarrow \langle N\rangle = \bar{N}$

$\Delta N = \sqrt{\bar{N}} \ll \langle N\rangle$

$$\hat{\Psi}(x)|\text{coherent}\rangle = \sqrt{\bar{n}}\,|\text{coherent}\rangle$$

$$\bar{n} = \frac{\bar{N}}{\text{volume}}$$

Landau - Ginzburg Model

$V_2(\vec{x}-\vec{y}) = \lambda\, \delta^{(3)}(\vec{x}-\vec{y})$

good for BEC of heavy atoms

starting approx for LiHe2

$$\hat{H} - \mu\hat{N} = \int d^3x \left(\frac{\hbar^2}{2M} \nabla\Psi^\dagger \cdot \nabla\Psi + \frac{\lambda}{2}\Psi^\dagger\Psi^\dagger\Psi\Psi - \mu\Psi^\dagger\Psi \right)$$

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coherent states $\rightarrow$ classical fields

~~$\Phi = \langle \hat{\Psi} \rangle$ and $\Phi^* = \langle$~~

$\Phi(x) = \langle \hat{\Psi}(x) \rangle \quad , \Phi^*(x) = \langle \hat{\Psi}^\dagger(x) \rangle .$

Free Energy $F = E - \mu N = \int d^3\vec{x} \left(\frac{\hbar^2}{2M} |\nabla\Phi|^2 + \frac{\lambda}{2}|\Phi|^4 - \mu|\Phi|^2 \right)$

@ Temperature $= 0$, minimize F.

$\hookrightarrow \nabla\Phi(x) \equiv 0 \rightarrow \Phi(x) = \text{const}.$

$\hookrightarrow$ minimize $\frac{\lambda}{2}|\Phi|^4 - \mu|\Phi|^2 = \frac{\lambda}{2}\left(|\Phi|^2 - \frac{\mu}{\lambda}\right)^2 - \frac{\mu^2}{2\lambda}$

For $\mu < 0$, minimum @ $\Phi(x) \equiv 0$
vacuum

For $\mu > 0$, minimum @ $|\Phi(x)|^2 = \frac{\mu}{\lambda}$

$\rightarrow \Phi(x) \equiv \sqrt{\frac{\mu}{\lambda}}\, e^{i\,\text{phase}}$

any const. phase.

undergraduate stat mech

$\rightarrow$ free atoms, $\lambda = 0$

$\hookrightarrow$ BEC requires $\mu < 0$ @ $T > 0$

$\mu = 0$ @ $T = 0$.

real life: interaction $\lambda > 0$ requires

$\mu > 0$ @ $T = 0$

$\mu = \lambda \underset{\text{avg.}}{\bar{n}}$ density

Moving BEC for $\Phi(x) \neq$ const.

$\hat{N} = \int d^3\vec{x}\, \hat{n} \qquad \hat{n}(\vec{x}) = \hat{\Psi}^\dagger(x)\hat{\Psi}(x)$

$\rightarrow$ classical $n(x) = |\Phi(x)|^2$

is the BEC density.

[In liq He, $|\Phi|^2$ = superfluid density].

$\vec{P}_{net} = \int d^3x\, \hat{\Psi}^\dagger(-i\hbar)\vec{\nabla}\hat{\Psi}$

$\rightarrow \int d^3\vec{x}\, \Phi^*(-i\hbar)\nabla\Phi = \frac{-i\hbar}{2}\int d^3\vec{x}(\Phi^*\vec{\nabla}\Phi - \Phi\nabla\Phi^*)$

$= \hbar \int d^3x\, \mathrm{Im}(\Phi^*\nabla\Phi)$

$\rightarrow$ momentum density

$\frac{\vec{P}}{\mathrm{vol}} = \hbar\, \mathrm{Im}(\Phi^*\vec{\nabla}\Phi) = \hbar |\Phi|^2 \vec{\nabla}\,\mathrm{phase}(\Phi(\vec{x}))$

$$\frac{\vec{P}}{Vol.} = \hbar |\Phi|^2 \, \nabla \text{phase}(\Phi(\vec{x})).$$

$$\frac{\vec{P}}{Vol} = \rho \vec{v} = m n \vec{v}$$

$\vec{v}$: flow velocity

$$m n \vec{v} = \hbar \cdot (|\Phi|^2 = n) \cdot \nabla \text{phase}(\Phi)$$

$$\boxed{\vec{v} = \frac{\hbar}{m} \vec{\nabla} \text{phase}(\Phi)}$$

$|\Phi(x)|^2 \to$ superfluid density

$\nabla \text{phase}(\Phi) \to \frac{m}{\hbar}$ velocity