E6/1

classical bit: descrete choice
0 or 1

Qubit: chose a state ~~s~~ in a
2dim. Hilbert space

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

N classical bits: descrete choice
of 2^N states

N qubits: Hilbert space of dim $= 2^N$

basis: $|0 \text{ or } 1, 0 \text{ or } 1, 0 \text{ or } 1, \ldots N \text{ times}\rangle$

Entangled ~~bits~~ qubits

2 qubit state $\frac{4}{5}|0,0\rangle + \frac{3i}{5}|1,1\rangle$

each qubit has 64% probability 1
36% probability 0
100% probability of 2 qubits being same.

No Quantum Xerox theorem.

cannot copy quantum information

no linear operator $\hat{C}|\psi\rangle \to |\psi,\psi\rangle$

or rather $\hat{C}|\psi, I\rangle \to |\psi,\psi\rangle$
for any $|\psi\rangle$ and some prepared $|I\rangle$

single qubit example.

identify $|0\rangle$ & $|1\rangle$ as a states $|z-\rangle$ & $|z+\rangle$ of a spin-$\frac{1}{2}$ atom.

Take 2 atoms $\hat{C}|z-, I\rangle = |z-, z-\rangle$

$\hat{C}|z+, I\rangle = |z+, z+\rangle$.

$|x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$.

by linearity $\hat{C}|x+, I\rangle = \frac{1}{\sqrt{2}}|z-, z-\rangle + \frac{1}{\sqrt{2}}|z+, z+\rangle$.

$|x+, x+\rangle = \frac{1}{2}(|z+, z+\rangle + |z+, z-\rangle + |z-, z+\rangle + |z-, z-\rangle)$.

$\neq \hat{C}|x+, I\rangle$

Can copy a classical bit

$|z+\rangle = |1\rangle$, $|z-\rangle = |0\rangle$

but not the qubit.

Quantum computer: all gates are unitary operations on a few qubits @ a time

No information loss.

Information loss $\rightarrow$ entropy gain

$\rightarrow$ heat

Information loss

2 particles collide and never meet again.

$\mathcal{H}(2\text{ particles}) = \mathcal{H}(1^{st}) \otimes \mathcal{H}(2^{nd})$.

before collision $|in\rangle = |in_1\rangle \otimes |in_2\rangle$

for ex $\Psi_{in}(\vec{x}_1, \vec{x}_2) = \psi_{in_1}(\vec{x}_1) * \psi_{in_2}(\vec{x}_2)$.

After collision $|fin\rangle$ does not factorize into $|fin_1\rangle \otimes |fin_2\rangle$

instead, you get some entangled state $|fin\rangle$.

$\{|\alpha\rangle\}$ basis of $\mathcal{H}(1^{st})$

$\{|i\rangle\}$ basis of $\mathcal{H}(2^{nd})$

$\{|\alpha, i\rangle\}$ basis of $\mathcal{H}(2\text{ particles})$.

$$|fin\rangle = \sum_{\alpha, i} C_{\alpha i} |\alpha, i\rangle.$$

Suppose we have no access to 2^{nd} particle & want to measure some property of the 1^{st} particle.

→ operator $\hat{A}$ in $\mathcal{H}(1^{st})$

in $\mathcal{H}(2\text{ particles})$ it becomes $\hat{A} \otimes \mathbb{1}$

$$\langle \alpha, i|\hat{A}|\beta, j\rangle = \langle\alpha|\hat{A}|\beta\rangle \times \delta_{ij}$$

$$|\psi_{tw}\rangle = \sum_{\alpha,i} C_{\alpha i}|\alpha, i\rangle$$

$$\langle\psi_{tw}|\hat{A}|\psi_{tw}\rangle = \sum_{\alpha i}\sum_{\beta j} C^*_{\alpha i} C_{\beta j}\,\langle\alpha|\hat{A}|\beta\rangle \times \delta_{ij}$$

$$= \sum_{\alpha\beta} \cancel{\beta_j}\, \rho_{\beta\alpha}\langle\alpha|\hat{A}|\beta\rangle$$

$$\cancel{\rho}\ \boxed{\rho_{\beta\alpha} = \sum_i C_{\beta i} C^*_{\alpha i}}$$

└ density matrix

$\rho_{\beta\alpha}$ is all we need to measure some property of the 1st particle w/o access to the 2nd particle.

$$\boxed{\langle\psi_{tw}|\hat{A}|\psi_{tw}\rangle = \sum_{\alpha\beta} \rho_{\beta\alpha}\langle\alpha|\hat{A}|\beta\rangle}$$

density operator $\hat{\rho} = \sum_{\beta\alpha} |\beta\rangle\rho_{\beta\alpha}\langle\alpha|.$

$$\langle\hat{A}\rangle = \sum_{\beta\alpha}\langle\beta|\hat{\rho}|\alpha\rangle\langle\alpha|\hat{A}|\beta\rangle$$

$$= \sum_{\beta}\langle\beta|\hat{\rho}\hat{A}|\beta\rangle = Tr(\hat{\rho}\hat{A})$$

$$\cancel{\langle\hat{A}\rangle}\ \boxed{\langle\hat{A}\rangle_\rho = tr(\hat{\rho}\hat{A})}$$

Tr (matrix) = Σ diagonal elements.

$tr(\hat{A}) = \sum_i \langle i|\hat{A}|i\rangle$ in any orthonormal basis.

same trace in any basis.

~~Eg.~~ if $\hat{A}$ has a basis of eigenstates

then $tr(\hat{A}) = \Sigma$ eigenvalues.

$tr(\hat{A}\hat{B}) = tr(\hat{B}\hat{A})$ if both are convergent.

Properties of the density operator $\hat{\rho}$.

1) $\hat{\rho}$ is Hermitian $\hat{\rho}^{+} = \hat{\rho}$

$\leftrightarrow \langle\beta|\hat{\rho}|\alpha\rangle = \sum_i C_{\beta i} C_{\alpha i}^{*} = \langle\alpha|\rho|\beta\rangle^{*}$

2) $tr(\hat{\rho}) = 1$

$tr(\hat{\rho}) = \sum_\beta \rho_{\beta\beta} = \sum_{i,\beta} C_{\beta i} C_{\beta i}^{*} = \langle\Psi_{in}|\Psi_{in}\rangle = 1$

3) $\hat{\rho} \geq 0 \leftrightarrow$ all eigenvalues ≥ 0

$\updownarrow$ $\langle\psi|\hat{\rho}|\psi\rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}(1^{st})$.

$\langle\psi|\hat{\rho}|\psi\rangle = \sum_{\alpha\beta} \langle\psi|\beta\rangle \langle\alpha|\psi\rangle \sum_i C_{\beta i} C_{\alpha i}^{*}$

$= \sum_i \left| \sum_\beta \langle\psi|\beta\rangle C_{\beta i} \right|^2 \geq 0.$

Physically $\langle\psi|\hat{\rho}|\psi\rangle$ is the probability of the 1^{st} particle being in state $|\psi\rangle$ assuming $\langle\psi|\psi\rangle = 1$.

Prob (1st in $|\psi\rangle$) or $= \sum_i$ Prob (1st in $|\psi\rangle$, 2nd in $|i\rangle$).

$$= \sum_i \cancel{\Phi} |\langle \psi, i | \Psi \rangle|^2$$

$$= \sum_i \left| \sum_\beta \cancel{\Phi} \langle \psi | \beta \rangle \underbrace{\langle \beta, i | \Psi \rangle}_{C_{\beta i}} \right|^2$$

$$= \langle \psi | \hat{\rho} | \psi \rangle.$$

$\hat{\rho}$: eigenvalues ≥ 0, add up $tr(\hat{\rho}) = 1$

$\Rightarrow$ 0 $\leq$ eigenvalues ≤ 1

$\rightarrow$ probabilities betw 0 and 1

add up to 1.

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what if $|\psi\rangle$ un-normalizable

like $|\vec{x}\rangle$ or $|\vec{p}\rangle$?

$\rightarrow$ Probability density $\langle \vec{x} | \hat{\rho} | \vec{x} \rangle \times d^3\vec{x}$

$\langle \vec{p} | \hat{\rho} | \vec{p} \rangle \times \frac{d^3\vec{p}}{(2\pi\hbar)^3}$

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eigenvalues & eigenstates of $\hat{\rho}$

$$\hat{\rho} = \sum_a |a\rangle w_a \langle a|$$

$$w_a \geq 0, \quad \sum w_a = 1.$$

Physically, 1st particle is in state $|a\rangle$ with probability w_a.

Suppose the $|\Psi\rangle$ state of 2 particles is <u>not</u> entangled.

$|\Psi\rangle = |f_1\rangle \otimes |f_2\rangle$.

~~$\rho_{\beta\alpha}$~~ $C_{\alpha i} = \langle \alpha | f_1 \rangle \times \langle i | f_2 \rangle$

$\rho_{\beta\alpha} = \sum_i C_{\beta i} C^*_{\alpha i}$

$= \sum_i \langle \beta | f_1 \rangle \langle f_1 | \alpha \rangle \times \underbrace{\sum_i |\langle i | f_2 \rangle|^2}_{1}$

~~ρ_{β}~~ $\hat{\rho}_{\beta\alpha} = \langle \beta | f_1 \rangle \langle f_1 | \alpha \rangle$

$\hat{\rho} = |f_1\rangle \langle f_1|$.

1 eigenvalue $= 1$ for $|\alpha\rangle = |f_1\rangle$
other eigenvalues $= 0$.

→ particle in state $|f_1\rangle$ with probability 100%

→ such states are called pure ~~state~~ states

Pooh: ~~states wi $\hat{\rho}$~~
when $\hat{\rho}$ has several non-zero eigen-values → mixed state.

Mixed states follow from entanglement with un-observable part of the quantum system