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General Harmonic Oscullator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{q}^2$$

$$\hat{p}^\dagger = \hat{p}, \quad \hat{q}^\dagger = \hat{q}, \quad [\hat{q}, \hat{p}] = i\hbar$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega m}}(\omega m\hat{q} + i\hat{p})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega m}}(\omega m\hat{q} - i\hat{p}).$$

$$\hat{a}\hat{a}^\dagger \neq \hat{a}^\dagger\hat{a}$$

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2\hbar\omega m}\left(-i\omega m[\hat{q}, \hat{p}] + i\omega m[\hat{p}, \hat{q}]\right)$$

$$= \frac{1}{2\hbar\omega m}\left(-i\omega m \times i\hbar + i\omega m \times (-i\hbar) = 2\omega m\hbar\right)$$

$$= 1$$

$$\boxed{[\hat{a}, \hat{a}^\dagger] = 1}$$

$$\hat{n} = \hat{a}^\dagger\hat{a} = \hat{n}^\dagger$$

$$\hat{a}\hat{a}^\dagger = a^\dagger a + [a, a^\dagger] = \cancel{\hat{a}\hat{a}^\dagger + 1 = \hat{n} + 1}$$

$$= \hat{a}^\dagger\hat{a} + 1 = \hat{n} + 1$$

Theorem: $\forall\, \hat{a}, \hat{a}^\dagger$ such that $[\hat{a}, \hat{a}^\dagger] = 1$

$\hat{n} = \hat{a}^\dagger \hat{a}$ has spectrum composed of non-negative integers

$$n = 0, 1, 2, 3, \ldots$$

$$\hat{a} + \hat{a}^\dagger = \frac{2m\omega}{\sqrt{2\hbar\omega m}} \hat{q}$$

$$\hat{a} - \hat{a}^\dagger = \frac{2i}{\sqrt{2\hbar\omega m}} \hat{p}$$

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega m}} (\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = \sqrt{\tfrac{1}{2}\hbar\omega m} (i\hat{a}^\dagger - i\hat{a}).$$

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{\omega^2 m}{2}\hat{q}^2$$

$$\frac{1}{2m}\hat{p}^2 = \frac{\hbar\omega m}{2} \times \frac{1}{2m}(-i\hat{a} + i\hat{a}^\dagger)^2$$

$$= \frac{\hbar\omega}{4}\left(-\hat{a}^2 - (\hat{a}^\dagger)^2 + \{\hat{a}, \hat{a}^\dagger\}\right).$$

$$\frac{\omega^2 m}{2}\hat{q}^2 = \frac{\omega^2 m}{2} \times \frac{\hbar}{2\omega m}(\hat{a} + \hat{a}^\dagger)^2$$

$$= \frac{\hbar\omega}{4}\left(+\hat{a}^2 + (\hat{a}^\dagger)^2 + \{\hat{a}, \hat{a}^\dagger\}\right)$$

$$\hat{H} = \frac{\hbar\omega}{4} \times 2\{\hat{a}, \hat{a}^\dagger\}.$$

$$\hat{H} = \frac{\hbar\omega}{2}\Big(\{\hat{a}, \hat{a}^{\dagger}\} = \underbrace{\hat{a}\hat{a}^{\dagger}}_{\hat{n}+1} + \underbrace{\hat{a}^{\dagger}\hat{a}}_{\hat{n}}\Big)$$

$$= \frac{\hbar\omega}{2}(2\hat{n}+1).$$

$$\boxed{\hat{H} = \hbar\omega\left(\hat{n}+\tfrac{1}{2}\right)}$$

in the eigenbasis of $\hat{n} = \hat{a}^{\dagger}\hat{a}$

$$\hat{n}|n\rangle = n|n\rangle \quad , n = 0, 1, 2, \dots$$

$$\hat{H}|n\rangle = E_n|n\rangle, \; E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

Ground state $|n=0\rangle$, $E_0 = \frac{1}{2}\hbar\omega$.

zero-point energy

$$E_n = E_0 + \underbrace{n \times \hbar\omega}$$

n quanta,

~~ach~~ each quantum having

$\varepsilon = \hbar\omega$.

Generic eigenstate $\hat{n}|n\rangle = n|n\rangle$.

$$|\psi_1\rangle = \hat{a}|n\rangle, \ |\psi_2\rangle = \hat{a}^{+}|n\rangle.$$

$$[\hat{n},\hat{a}] = [\hat{a}^{+}\hat{a},\hat{a}] = [\hat{a}^{+},\hat{a}]\hat{a} + \hat{a}^{+}[\hat{a},\hat{a}]$$

$$= (-1)\hat{a} + \hat{a}^{+}(0) \ = -\hat{a}$$

$$\hat{n}\hat{a} - \hat{a}\hat{n} = -\hat{a}$$

$$\hat{n}\hat{a} = \cancel{(\hat{n}-1)}\hat{a}(\hat{n}-1).$$

$$[\hat{n},\hat{a}^{+}] = [\hat{a}^{+}a,a^{+}] = [a^{+}a^{+}]a + a^{+}[a\,a^{+}]$$

$$= 0 + \hat{a}^{+}\cdot 1 = +\hat{a}^{+}$$

$$\hat{n}\hat{a}^{+} = \hat{a}^{+}\hat{n} + ([\hat{n},a^{+}] = \hat{a}^{+})$$

$$= \hat{a}^{+}(\hat{n}+1).$$

$$\hat{n}\hat{a} = \hat{a}(\hat{n}-1), \ \hat{n}\hat{a}^{+} = \hat{a}^{+}(\hat{n}+1).$$

$$|\psi_1\rangle = \hat{a}|n\rangle, \ |\psi_2\rangle = \hat{a}^{+}|n\rangle.$$

$$\hat{n}|\psi_1\rangle = \hat{n}\hat{a}|n\rangle = \hat{a}(\hat{n}-1)|n\rangle$$

$$= \hat{a}((n-1)|n\rangle) = (n-1)\hat{a}|n\rangle$$

$$\hat{n}|\psi_1\rangle = (n-1)|\psi_1\rangle.$$

$\rightarrow |\psi_1\rangle = \hat{a}|n\rangle$ is an eigenstate

eigenvalue $= n-1$

$\hat{n}|\psi_2\rangle = \hat{n}\hat{a}^\dagger|n\rangle = \hat{a}^\dagger(\hat{n}+1)|n\rangle$

$= (n+1)\hat{a}^\dagger|n\rangle = (n+1)|\psi_2\rangle$

$\rightarrow |\psi_2\rangle = \hat{a}^\dagger|n\rangle$ is an eigenstate of $\hat{n}$

eigenvalue $= n+1$.

$\hat{a}$ & $\hat{a}^\dagger$ turn eigenstates of $\hat{n}$ into other eigenstates

$\hat{a}$ lowers n by 1

$\hat{a}^\dagger$ raises n by 1

$\hat{a}^\dagger$: raising operator
creation operator

$\hat{a}$: lowering operator
annihilation operator

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start with a general eigenstate $|n_0\rangle$, $\hat{n}|n_0\rangle = n_0|n_0\rangle$.

$\rightarrow$ ~~by~~ apply $\hat{a}$ & $\hat{a}^\dagger$ operators to build an infinite chain

$$\cdots |n_0-2\rangle \underset{\hat{a}}{\overset{\hat{a}^\dagger}{\rightleftarrows}} |n_0-1\rangle \underset{\hat{a}}{\overset{\hat{a}^\dagger}{\rightleftarrows}} |n_0\rangle \underset{\hat{a}}{\overset{\hat{a}^\dagger}{\rightleftarrows}} |n_0+1\rangle \underset{\hat{a}}{\overset{\hat{a}^\dagger}{\rightleftarrows}} |n_0+2\rangle \cdots$$

$\hookrightarrow$ get all $|n_0 \pm$ any integer$\rangle$ states.

chain stops when $\hat{a}|\text{previous}\rangle = 0$
or $\hat{a}^+|\text{previous}\rangle = 0$.

$\hat{n} = \hat{a}^+\hat{a} \geq 0$

means $\langle\psi|\hat{n}|\psi\rangle \geq 0 \; \forall |\psi\rangle$.

$\langle\psi|\hat{n} = \hat{a}^+\hat{a}|\psi\rangle = (\hat{a}|\psi\rangle)^+(\hat{a}|\psi\rangle)$

$= \text{norm}^2(\hat{a}|\psi\rangle) \geq 0$.

$\rightarrow$ all eigenvalues $n \geq 0$.

$\rightarrow$ downward chain

$|n_0\rangle \rightarrow |n_0-1\rangle \rightarrow |n_0-2\rangle \cdots$

has to stop somewhere.

$\rightarrow \hat{a}|\text{last}\rangle = 0$.

~~$\langle\text{last}|\hat{a}^+\hat{a}|\text{last}\rangle = \text{norm}^2(\hat{a}|\text{last}\rangle) = 0$.~~

$\rightarrow \hat{a}^+\hat{a}|\text{last}\rangle = 0 \quad \hat{n}|\text{last}\rangle = 0$

$\Rightarrow |\text{last}\rangle = |n=0\rangle$.

$|\text{last}\rangle = (\text{coeff}) \cdot \hat{a}\hat{a}\cdots\hat{a}|n_0\rangle$.

has $n = n_0 - \text{integer}$.

$\overset{\parallel}{0}$ $\Rightarrow n_0$ must be integer.

All eigenvalues of $\hat{n}$ must be integers ≥ 0.

All $n = 0, 1, 2, \ldots$ are present in the spectrum of $\hat{n}$.

start with any eigenstate $|n_0\rangle$ & act with several $\hat{a}^\dagger$

$(\text{coeff})(\hat{a}^\dagger)^k |n_0\rangle = |n_0+k\rangle \neq 0.$

$\forall |\psi\rangle$, $\hat{a}^\dagger|\psi\rangle$ has norm^2 =

$= \langle\psi|\hat{a}\hat{a}^\dagger|\psi\rangle = \langle\psi|\hat{n}+1|\psi\rangle \geq \langle\psi|\psi\rangle \neq 0.$

$\rightarrow \hat{a}^\dagger|\psi\rangle \neq 0.$

$\rightarrow |n_0+k\rangle = (\text{coeff})(\hat{a}^\dagger)^k|n_0\rangle$ exists $\forall k = 1, 2, 3, \ldots$

$\rightarrow$ all n_0+k exist.

all $n_0 - k \geq 0$ exist.

$\rightarrow \exists$ eigenstates $|n\rangle$ $\forall$ integer $n \geq 0$.

Degeneracies.

start with eigenvalue n_0

~~that~~ build a basis $\{|n_0, \alpha\rangle\}$.

$$\langle n_0, \alpha | n_0, \beta\rangle = \delta_{\alpha\beta}.$$

Let $|n_0+1, \alpha\rangle = \frac{1}{\sqrt{n_0+1}} \hat{a}^\dagger |n_0, \alpha\rangle$.

eigenstate, eigenvalue $= n_0+1$

$\langle n_0+1, \alpha | n_0+1, \beta\rangle$

$$= \cancel{\frac{1}{\sqrt{n_0}} \frac{1}{n_0+1} \langle n_0 \alpha | \hat{n} | n_0 \hat{\beta}\rangle}$$

$=$

$$= \frac{1}{n_0+1} \langle n_0, \alpha | \underbrace{\hat{a}\hat{a}^\dagger}_{\hat{n}_0+1} | n_0, \beta\rangle$$

$$= \frac{n_0+1}{n_0+1} \langle n_0 \alpha | n_0 \beta\rangle = \delta_{\alpha\beta}.$$

$\rightarrow$ a basis for n_0+1 eigenstates

completeness: Let $\hat{n} |\psi\rangle = (n_0+1)|\psi\rangle$.

$\hat{n}(\hat{a}|\psi\rangle) = n_0 \hat{a}|\psi\rangle$.

$\rightarrow \hat{a}|\psi\rangle = \sum_\alpha c_\alpha |n_0, \alpha\rangle$ for

$$c_\alpha = \langle n_0, \alpha | \hat{a} |\psi\rangle.$$

$\sum_\alpha c_\alpha |n_0+1, \alpha\rangle$

$$= \frac{1}{\sqrt{n_0+1}} \sum_\alpha \underbrace{\hat{a}^\dagger |n_0, \alpha\rangle \cdot \langle n_0, \alpha|}_{\text{complete basis}} \hat{a} |\psi\rangle$$

$$= \frac{1}{\sqrt{n_0+1}} \underbrace{\hat{a}^\dagger \hat{a}}_{\hat{n}} |\psi\rangle = \frac{1}{\sqrt{n_0+1}} (n_0+1) |\psi\rangle.$$

any eigenstate with eigenvalue n_0+1

$$|\psi\rangle = \sum_\alpha \frac{c_\alpha}{\sqrt{n_0+1}} |n_0+1, \alpha\rangle.$$

given a complete $\perp$ basis $|n_0, \alpha\rangle$
for $n = n_0$

$\frac{1}{\sqrt{n_0+1}} \hat{a}^\dagger |n_0, \alpha\rangle$ make a complete $\perp$ basis for $n = n_0+1$.

likewise $\frac{1}{\sqrt{n_0}} \hat{a} |n_0, \alpha\rangle = |n_0-1, \alpha\rangle$

make a complete $\perp$ basis for $n = n_0-1$.

→ extend to all n

→ $\{|n, \alpha\rangle\}$

use same set of α $\forall n$.

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~~same~~ All energy levels

$$E_n = \hbar\omega(n+\tfrac{1}{2})$$

have exactly same degeneracy.

If the ground state is non-degenerate then all states $|n\rangle$ are non-degenerate.

Suppose there no degrees of freedom besides $\hat{q}$ & $\hat{p}$:

$\rightarrow \{|q\rangle, q \in \mathbb{R}\}$ is a complete basis.

For ex: 1 dim vector, no spin.

$\psi(q)$, $\hat{q}\psi(q) = q \times \psi(q)$

$$\hat{p}\psi(q) = -i\hbar \frac{d\psi}{dq}$$

$$\hat{a}\psi = \frac{1}{\sqrt{2\hbar\omega m}}\left(m\omega q\psi + \hbar\frac{d\psi}{dq}\right).$$

ground state $\hat{n}|0\rangle = 0.$

$\rightarrow \hat{a}|0\rangle = 0.$

$$\rightarrow (\cancel{\text{const}})\left(m\omega q + \hbar\frac{d}{dq}\right)\psi_0(q) = 0.$$

$$\rightarrow \boxed{\psi_0(q) = (\text{const}) \times \exp\left(-\frac{m\omega}{2\hbar}q^2\right)}$$

→ unique ground state

→ all states $|n\rangle$ are non degenerate.

$\hat{q}$, $\hat{p}$, + other degrees of freedom

→ eigenstates are $|n, \alpha\rangle$

α describes the other d of DOF

→ same set of α @ each $n = 0, 1, 2, \ldots$

$\hat{a}^{+}|n, \alpha\rangle = \sqrt{n+1}\,|n+1, \alpha\rangle$

same α

$\hat{a}\,|n, \alpha\rangle = \sqrt{n}\,|n-1, \alpha\rangle$

$E(n, \alpha) = \hbar\omega\left(n + \frac{1}{2}\right)$

$+ E_\alpha$ (other degrees of freedom)

$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\omega^2 m}{2}\hat{q}^2 + \hat{H}_\alpha(\text{other DOF}).$