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If $V(x) > E$ on an infinite region of space.

→ out 2 solutions of the schrödinger eqn

1 dies down } for $x \to \infty$
1 blows up }

only the dieing-down solution is physical

⇒ when $V(\pm\infty) > E$ @ both ends

→ each end selects a solution

for generic E, they mismatch

→ no good solutions.

⇒ E spectrum is discrete.

↳ E's for which same solution dies down @ both $x \to \pm\infty$

Physically such solutions are bound states.

For any bound state $\langle\psi|\psi\rangle = \int |\psi|^2 dx < \infty$

→ normalizable $|\psi\rangle$.

→ well defined $\langle x \rangle$

$\langle p \rangle = m \frac{d}{dt} \langle x \rangle = 0$ (stationary $|\psi\rangle$).

un-bound states

$\rightarrow E > V(\pm\infty)$ @ at ~~the~~ least 1 end

$\rightarrow$ ~~o~~scillating $\psi(x)$

both solutions are ok.

but un-normalizable $\langle\psi|\psi\rangle = \infty$

$\rightarrow$ continuous spectrum of E.

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in an un-bound state

$\langle x\rangle = \dfrac{\langle\psi|\hat{x}|\psi\rangle}{\langle\psi|\psi\rangle}$ is ill-defined.

$\Rightarrow \langle p\rangle \neq 0$ is ok

Asymptotic region $x \to +\infty$ or $-\infty$

$V(x) \simeq$ const $\quad E > V$.

$$\psi(x) = A e^{+ikx} + B e^{-ikx}$$

$$k = \frac{1}{\hbar}\sqrt{2m(E-V)}$$

Meaning of $\psi = e^{+ikx}$ or $\psi = e^{-ikx}$?

$\rightarrow$ need wave packets.

$$\psi(x) = \int dE\, \psi_E(x) \times F(E)$$

where $F(E)$ has a narrow peak @ E_0.

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$$F(E) = \exp(-(E-E_0)^2/2\epsilon^2)$$

$$\epsilon \ll E-V$$

$$\psi_E(x) = e^{\pm i k(E) x}$$

For small ϵ, $k(E) = k_0 + \frac{E-E_0}{\hbar u}$

u = velocity $= \frac{\hbar k_0}{m}$

$$\psi(x) = \int dE\, e^{\pm i k_0 x}\, e^{\pm i (E-E_0) x/\hbar u}\, e^{-(E-E_0)^2/2\epsilon^2}$$

$$\psi(x,t) = \int dE\, e^{\pm i k_0 x}\, e^{-i\omega_0 t}$$
$$\times e^{\pm i (E-E_0) x/\hbar u}\, e^{-i(E-E_0)t/\hbar}$$
$$\times e^{-(E-E_0)/2\epsilon^2}.$$

$$= e^{\pm i k_0 x - i\omega_0 t} \int dE \exp\left(\frac{i(E-E_0)}{u\hbar}(\pm x - ut) - \frac{1}{2\epsilon^2}(E-E_0)^2\right)$$

$$= e^{\pm i k_0 x - i\omega_0 t} \times \sqrt{2\pi\epsilon^2}\, \exp\left(-\frac{\epsilon^2}{2\hbar^2 u^2}(\pm x - ut)^2\right).$$

→ Gaussian wave packet moving ~~left or right @~~ right or left @ speed $= u$.

e^{+ikx}: moves right

e^{-ikx} moves left.

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$$\psi_E(x) = A e^{+ikx} + B e^{-ikx}$$

$$\Downarrow$$

$$\psi(x,t) = A e^{+ik_0x - i\omega_0 t} \times \exp\left(-\frac{\epsilon^2(x-ut)^2}{\hbar^2 u^2}\right)$$

$$+ B e^{-ik_0x - i\omega_0 t} \times \exp\left(-\frac{\epsilon^2(x+ut)^2}{2\hbar^2 u^2}\right).$$

: 2 wave packets, ϵ

1 moves right, Amplitude $= A$

1 moves left, Amplitude $= B$.

---

2 asymptotic regions $x \to -\infty$ & $x \to +\infty$.

For $x \to -\infty$ $\psi_E(x) = A_1 e^{+ik_1x} + B_1 e^{-ik_1x}$

For $x \to +\infty$ $\psi_E(x) = A_2 e^{+ik_2x} + B_2 e^{-ik_2x}$

2 linear relations between

A_1, A_2, B_1, B_2

@ each end, wave packets

centered @ $x = \pm ut$.

---

very early $t \to -\infty$

$\to$ $x = +ut$ is @ left side only

$x = -ut$ is @ right side only.

[5] For early times $t \to -\infty$.

Left side $A_1 e^{+ik_1 x} \exp\left(-\frac{\sigma^2(x-u_1 t)}{2t_0^2 u_1^2}\right)$

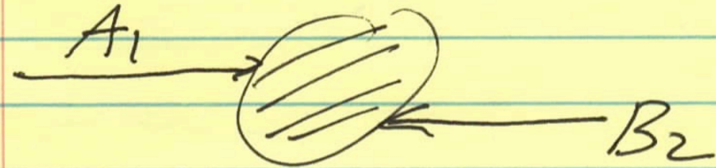
only.

$\Rightarrow$ incident wave packet coming from the left.

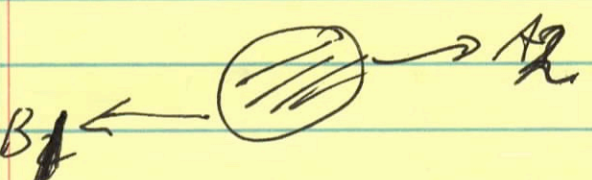
Right side, $B_2 e^{-ik_2 x} \exp\left(-\frac{\sigma^2}{2t_0^2 u_2^2}(x+u_2 t)\right)$

only

incident wave packet comes from the right.



incoming @ early times



outgoing @ late times

Typical situation: incoming particles from side only

say from the left.

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A_1: incident amplitude

A_2: transmitted amplitude

B_1: reflected amplitude

$B_2 = 0$.

$$\psi = \begin{cases} A_1 e^{+ik_1 x} + B_1 e^{-ik_1 x} & x \to -\infty \\ A_2 e^{+ik_2 x} + B_2 e^{-ik_2 x} & x \to +\infty \end{cases}$$

2 linear relations between

A_1, B_1, A_2, B_2.

Add $B_2 = 0$ eq-n

$\to$ solve for $\frac{A_2}{A_1}$ and $\frac{B_1}{A_1}$

$$R = \left|\frac{B_1}{A_1}\right|^2 : \text{reflection probability}$$

$$T = \left|\frac{A_2}{A_1}\right|^2 \times \left(\frac{u_2}{u_1} = \frac{k_2}{k_1}\right) : \text{transmission probability.}$$

---

probability $\sim \int dx |\psi|^2$

for $\psi = c \exp\left(-\frac{\epsilon^2 (x \pm ut)^2}{2t_0^2 u^2}\right)$

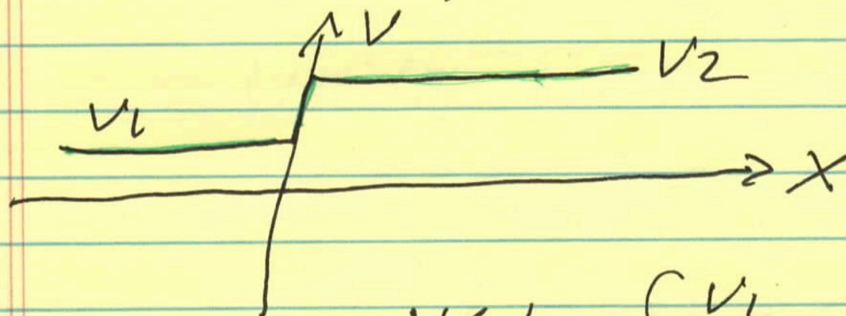
$\int dx |\psi|^2 \to |c|^2 \sqrt{2\pi} \frac{t_0 u}{\epsilon}$

probability $\sim |c|^2 \times u$.

net probability = 1

$\rightarrow$ must have $R+T=1$.

Example: potential step



$$V(x) = \begin{cases} V_1 & x<0 \\ V_2 & x>0. \end{cases}$$

Take $E > V_1$, $E > V_2$.

For $x<0$ $\quad \psi_1(x) = A_1 e^{+ik_1 x} + B_1 e^{-ik_1 x}$

$$k_1^2 = \frac{2m(E-V_1)}{\hbar^2}.$$

For $x>0$ $\quad \psi_2(x) = A_2 e^{+ik_2 x} + B_2 e^{-ik_2 x}$

$$k_2^2 = \frac{2m(E-V_2)}{\hbar^2}$$

Matching conditions @ $x=0$.

Rule 1: $\psi(x)$ is <u>always</u> continuous.

Rule 2: $\psi'(x)$ is continuous where $V(x)$ is finite

$= \int \psi^* \cdot \frac{-\hbar^2}{2m} \psi'' \, dx$

$\int |\psi'|^2 dx$

worse than $\int |\psi|^2 dx$

→ finite $\psi'(x)$ @ any discontinuity of $V(x)$.

→ continuous $\psi(x)$.

$$\psi''(x) = \frac{2m(V-E)}{\hbar^2}\psi(x)$$

If $V(x)$: jumps but stay finite.

→ $\psi''(x)$ jumps but stays finite

→ $\psi'(x)$ is continuous.

But for an infinite jump of $V(x)$

$\psi'(x)$ becomes discontinuous.

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$$H_{kin} = \int \psi^* \cdot \frac{-\hbar^2}{2m} \psi'' \, dx$$

$$= +\frac{\hbar^2}{2m} \int |\psi'|^2 dx.$$

→ $\int |\psi'|^2 dx$ should not diverse worse than $\int |\psi|^2 dx$

→ finite $\psi'(x)$ @ any discontinuity of $V(x)$.

→ continuous $\psi(x)$.

$$\psi''(x) = \frac{2m(V-E)}{\hbar^2} \psi(x)$$

If $V(x)$: jumps but stay finite.

→ $\psi''(x)$ jumps but stays finite

→ $\psi'(x)$ is continuous.

But for an ~~a~~ infinite jump of $V(x)$

$\psi'(x)$ becomes discontinuous.

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For $x < 0$ $\psi_1(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$

For $x > 0$ $\psi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}$

$\psi_1(0) = \psi_2(0)$

$$A_1 + B_1 = A_2 + B_2$$

$\psi_1'(0) = \psi_2'(0)$ $ik_1(A_1 - B_1) = ik_2(A_2 - B_2)$

$\rightarrow$ 2 linear relations between A_1, B_1, A_2, B_2

Let $B_2 = 0$ (incoming from left side only)

$$\begin{array}{l|l} k_1 & A_1 + B_1 = A_2 \\ \pm & k_1 A_1 - k_1 B_1 = k_2 A_2 \end{array}$$

$$2k_1 A_1 = (k_1 + k_2) A_2$$

$$2k_1 B_1 = (k_1 - k_2) A_2$$

$$\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2}$$

$$\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2}$$

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Reflection coefficient

$$R = \left|\frac{B_1}{A_1}\right|^2 = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

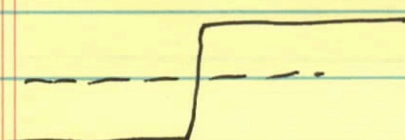
Transmission coefficient

$$T = \frac{k_2}{k_1} \times \left|\frac{A_2}{A_1}\right|^2 = \frac{k_2}{k_1} \times \frac{4k_1^2}{(k_1+k_2)^2}$$

$$T = \frac{4k_1k_2}{(k_1+k_2)^2}$$

$$(k_1-k_2)^2 + 4k_1k_2 = (k_1+k_2)^2$$

$$\boxed{T+R=1}$$



$V_1 < E < V_2$.

$x<0 \quad \psi_1(x) = A_1 e^{+ikx} + B_1 e^{-ikx}$

$x>0 \quad \psi_2(x) = A_2 e^{+\kappa x} + B_2 e^{-\kappa x}$

↑ forbidden $\Rightarrow A_2 = 0$

@ $x=0 \quad \psi_1 = \psi_2 \rightarrow A_1 + B_1 = B_2$

$\psi_1' = \psi_2' \rightarrow ikA_1 - ikA_2 = -\kappa B_2$.

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$$k_1 A_1 + k_1 B_1 = k_1 B_2$$

$$k_1 A_1 - k_1 B_1 = -i\varkappa_2 B_2$$

$$\frac{B_2}{A_1} = \frac{2k_1}{k_1 - i\varkappa_2}$$

$$\frac{B_1}{A_1} = \frac{k_1 + i\varkappa_2}{k_1 - i\varkappa_2}$$

$$R = \left| \frac{k_1 + i\varkappa_2}{k_1 - i\varkappa_2} \right|^2 = 1$$

no transmission, full reflection.