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Symmetries always form a group

$\forall$ symmetries S_1, S_2

$S_2 S_1$ is a symmetry

$S_2 S_1$ acts as: do S_1 first, then S_2

S^{-1} reverses the action of S

$S^{-1}S = SS^{-1} = 1$: does nothing

associative product $S_3(S_2 S_1) = (S_3 S_2) S_1$

Abelian groups: $S_1 S_2 = S_2 S_1$

Example: translations of space

$T(\bar{a}) \cdot T(\bar{b}) = T(\bar{a}+\bar{b}) = T(\bar{b}) \cdot T(\bar{a})$.

Rotations in 2 dimensions

$R(\alpha) \cdot R(\beta) = R(\alpha+\beta) = R(\beta) R(\alpha)$.

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non-abelian groups

$S_2 S_1 \neq S_1 S_2$ for some S_1, S_2

Example: rotations in 3D

or any dim ≥ 3

$R(90°, x) R(90°, y) \neq R(90°, y) R(90°, x)$

In QM, symmetries are represented by unitary operators $\hat{U}(s)$

$\hat{U}(s_2 s_1) = \hat{U}(s_2)\hat{U}(s_1)$

group product   operator product.

$\hat{U}(1) = \underline{1}$, $\hat{U}(s^{-1}) = \hat{U}^{-1}(s)$

Schrödinger picture

$|\psi\rangle \to |\psi'\rangle = \hat{U}(s)|\psi\rangle$.

$\hat{A} \to \hat{A}' = \hat{A}$, operators stay unchanged.

Heisenberg picture

$|\psi\rangle \to$ same $|\psi'\rangle = |\psi\rangle$.

$\hat{A} \to \hat{A}' = \hat{U}^{-1}\hat{A}\hat{U}$, $\hat{U}^{-1} = \hat{U}^{+}$

Either picture $\langle\psi|\hat{A}|\psi\rangle' = \langle\psi|\hat{U}^{-1}\hat{A}\hat{U}|\psi\rangle$

Dynamical symmetries

$\hat{U}(s)$ commute with the Hamiltonian $\hat{H}$

$\to$ with the time evolution.

Non-abelian dyn. symmetry

$0 = [\hat{U}(S_1), \hat{H}], \; [\hat{U}(S_2), \hat{H}] = 0$

$[\hat{U}(S_1), \hat{U}(S_2)] \neq 0$

$\rightarrow$ $\hat{H}$ must have degenerate spectrum.

Example of abelian dyn. symmetry

Free particle $\hat{H} = \frac{\hat{\vec{p}}^2}{2m}$

Translations $\hat{T}(\vec{a}) = \exp(-i\vec{a}\cdot\hat{\vec{p}}/\hbar)$

commute with $\hat{H}$

Continuous symmetry:

S parametrized by real numbers

$S(a_1, \dots, a_n)$. $\rightarrow \hat{U}(S) = \hat{U}(a_1 \dots a_n)$.

$S(0, \dots, 0) = 1$

For very small $a_1 \dots a_n$, $S \simeq 1$

$\hat{U}(S) \approx \hat{1}$

$\hat{U}(\text{small } a_1 \dots a_n) \approx 1 - \frac{i}{\hbar}\sum_{i=1}^{n} a_i \hat{G}_i + O(a^2)$.

For some Hermitian operators $\hat{G}_i$

$$U^\dagger U = 1 + \sum_{i=1}^{n} \underbrace{\left(-\frac{i}{\hbar} a_i G_i + \frac{i}{\hbar} a_i G_i^\dagger\right)}_{0 \;\Rightarrow\; \hat{G}_i^\dagger = G_i} + O(a^2)$$

$$\boxed{\hat{G}_i = i\hbar \, \frac{\partial U(a_1 \ldots a_n)}{\partial a_i}\Big|_{\text{all } a = 0}}$$

↳ for infinitesimal a_i

$$U = 1 - \frac{i}{\hbar} \sum_i a_i \hat{G}_i + O(a^2)$$

Finite a_i?

Assume $\left[S\left(\frac{a_1}{n}, \frac{a_2}{n}, \ldots \frac{a_n}{n}\right)\right]^n = S(a_1 \ldots a_n)$.

Examples: $T(\text{by } \frac{\vec{a}}{n})$

& $R(\text{by } \frac{\theta}{n})$.

$$\hat{S}(a_1 \ldots a_n) = \lim_{n\to\infty} \left(U\left(\frac{a_1}{n}, \ldots \frac{a_n}{n}\right)\right)^n$$

$$= \lim_{n\to\infty} \left(1 - \frac{i}{\hbar} \sum_i \frac{a_i}{n} \hat{G}_i + O\left(\frac{1}{n^2}\right)\right)^n$$

$$\boxed{U(a_1 \ldots a_n) = \exp\left(-\frac{i}{\hbar} \sum_i a_i \hat{G}_i\right)}$$

$\hat{G}_i$ operators ~~are~~ <u>generate</u>

the symmetries

$\hat{G}_i$ are called the generators

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Examples:

1) Momenta $\hat{p}_x, \hat{p}_y, \hat{p}_z$

generate the translation group

$$\hat{T}(\vec{a}) = \exp\left(-\frac{i}{\hbar}\vec{a}\cdot\hat{\vec{p}}\right).$$

2) Angular momenta

$\hat{J}_x, \hat{J}_y, \hat{J}_z$ generate

the 3d rotation group

R (angle α, axis $\vec{n}$)

└ $\vec{n}$: unit vector.

$$\hat{R}(\alpha, \vec{n}) = \exp\left(-\frac{i}{\hbar}\alpha\vec{n}\cdot\hat{\vec{J}}\right).$$

$$= \exp\left(-\frac{i}{\hbar}\left(\alpha n_x\cdot\hat{J}_x + \alpha n_y\cdot\hat{J}_y + \alpha n_z\cdot\hat{J}_z\right)\right).$$

Toy model: Rotations in 2d

in (x,y) plane, $\psi(x,y)$.

→ rotation group in 2d is abelian

$$\hat{R}(\varphi_1)\hat{R}(\varphi_2) = \hat{R}(\varphi_2)\hat{R}(\varphi_1).$$

Let $\hat{J} = +i\hbar\frac{\partial}{\partial\varphi}\hat{R}(\varphi)\Big|_{\varphi=0}$.

$$\to \hat{R}(\varphi) = \exp\left(-\frac{i}{\hbar}\varphi\hat{J}\right).$$

coordinate basis

$$\psi(x,y) = \langle x,y|\psi\rangle$$

polar coordinates (r,φ).

$$\psi(r,\varphi) = \langle r,\varphi|\psi\rangle.$$

Active rotations

$$\hat{R}(\alpha)|r,\varphi\rangle = |r,\varphi+\alpha\rangle.$$

$$\cancel{\psi'(r,\varphi) = \langle\psi|\hat{R}|r,\varphi\rangle = \langle\psi'|r,\varphi+\alpha\rangle}$$

$$\psi'(r,\varphi+\alpha) = \psi(r,\varphi)$$

$$\psi'(r,\varphi) = \psi(r,\varphi-\alpha).$$

$$\hat{R}(\alpha)\psi(r,\varphi) = \psi(r,\varphi-\alpha).$$

$$\frac{\partial}{\partial\alpha}\hat{R}(\alpha)\psi(r,\varphi)\Big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\psi(r,\varphi-\alpha)\Big|_{\alpha=0}$$

$$= -\frac{\partial}{\partial\varphi}\psi(r,\varphi).$$

$$\boxed{\hat{J} = +i\hbar\frac{\partial}{\partial\alpha}\hat{R}(\alpha)\Big|_{\alpha=0}.}$$

$$\hat{J}\psi(r,\varphi) = -i\hbar\frac{\partial}{\partial\varphi}\psi(r,\varphi)$$

General continuous symmetry

$$\hat{G}_i = +i\hbar \frac{\partial}{\partial a_i} \hat{U}(a_1 \ldots a_n)\Big|_{\text{all } a_i = 0}$$

Back to 2d rotations

$$\hat{J} = +i\hbar \frac{\partial}{\partial \varphi} R(\varphi)\Big|_{\varphi=0}$$

in coordinate basis

$$\hat{J}\psi(r,\varphi) = -i\hbar \frac{\partial}{\partial \varphi}\psi(r,\varphi).$$

cartesian (x,y)

$$\frac{\partial}{\partial \varphi} \to x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\hat{J}\psi = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi(x,y)$$

$$= (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)\psi(x,y)$$

when there are no degrees of freedom except (x,y)

$$\hat{J}_{2d} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \hat{L}_z$$

More generally

$$\hat{J}_{2d} = \hat{L}_z + \underbrace{\hat{S}_{2d}}_{\text{2d spin}}$$

spectrum:

$L_z|\psi\rangle = \hbar m|\psi\rangle$ for some number m.

$\hat{R}(\alpha)|\psi\rangle = \exp(-im\alpha \cancel{B_z})|\psi\rangle$

$\to \psi(r, \varphi-\alpha) = e^{-im\alpha}\psi(r, \varphi)$

$\to \psi(r, \varphi) = \psi_r(r) \times e^{im\varphi}$.

single valued $\psi(r, \varphi+2\pi) = \psi(r, \varphi)$

$\to$ integer $\underline{\underline{m}}$

$\to$ spectrum of $\hat{L}_{2d} \equiv \hat{L}_z$

spans ~~ts~~ ($\hbar$ × integer m)

Spin: cf $S_{2d} = \hat{S}_z$ from 3d.

then eigenvalues are $\hbar \times m$

~~for integer or half-integer m.~~

$\to$ spectrum of $J_{2d} = \hat{J}_z$ from 3d

is $\hbar \times m_J$ for integer or half-integer m_J

$2m_J$ ~~or must be integer.~~

For any species of a 3d particle

m_s is either always integer

or ~~all~~ always half integer.

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whereas for purely 2d quasi-particles

S_{2d} does <u>not</u> come from 3d S_z

→ m_s can be any fraction

↳ ~~such~~ ~~any~~ $\boxed{\text{anyons}}$