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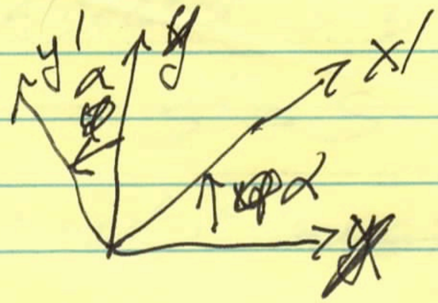
Passive rotation: changing the coordinates



in polar coordinates

$r' = r$

$\varphi' = \varphi + \alpha.$

These are new coordinates of the same particle (or part of some body).

Active rotation: ~~on~~ rotates the body in same old coordinates.

part which used to be @ (r, φ)
moves to $(r, \varphi + \alpha)$.

QM: $|r, \varphi\rangle \to |r, \varphi + \alpha\rangle$

Active rotation & passive rotation have opposite signs.

Change coordinates

$$x_i' = C_{ij} x_j$$

A scaler S stays invariant

$$S' = S$$

A vector must transform like $\vec{x}$

$$V_i \to V_i' = C_{ij} V_j$$

A 2 index tensor T_{ij}

$$T_{ij} \to T_{ij}' = C_{ik} C_{jl} T_{kl}$$

~~n-index ten~~

3-index tensor T_{ijk}

$$T_{ijk} \to T_{ijk}' = C_{il} C_{jm} C_{kn} T_{lmn}$$

$\vdots$

ditto for tensors with any # indices.

$\forall$ 2 vectors A_i & B_j

$\vec{A} \cdot \vec{B}$ is a scalar

$\vec{A} \times \vec{B}$ is a vector

$T_{ij} = A_i B_j$ is a tensor.

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Tensors with 2 or more indices

2 indices $\left\{ \begin{array}{l} \text{symmetric tensor } T_{ij} = T_{ji} \\ \text{antisymmetric tensor } T_{ij} = -T_{ji} \end{array} \right.$

more indices

Totally symmetric $T_{ij \cdots n} = T_{\text{any permutation } ij \cdots n}$

Totally antisymmetric

$$T_{\text{perm. of } ij \cdots n} = (-1)^{\text{parity}} \times T_{ij \cdots n}$$

Mixed permutations symmetries.

in 3d $T_{ij} = -T_{ji}$ is equiv to a vector.

$$V_i = \frac{1}{2} \epsilon_{ijk} T_{jk} \;, \quad T_{jk} = \epsilon_{jki} V_i$$

~~Any n-index tensor is equiv. to a totally symmetric tensor~~

Any n index tensor is equiv to 1 or more totally symmetric tensors.

Rotation group in 3D

Infinitesimal rotation

through angle $\alpha \to 0$

around axis $\vec{n}$

$$\vec{V} \to \vec{V}' = \vec{V} + \alpha\, \vec{n} \times \vec{V} + O(\alpha^2).$$

Finite rotation $R(\alpha, \vec{n})$

$$\vec{V}' = \cos\alpha\, \vec{V} + \sin\alpha\, \vec{n} \times \vec{V} + (1-\cos\alpha)\, \vec{n}\, (\vec{n} \cdot \vec{V})$$

$$V_i' = R_{ij}(\alpha, \vec{n})\, V_j$$

$$R_{ij} = \cos\alpha\, \delta_{ij} + \sin\alpha\, \epsilon_{ikj} n_k + (1-\cos\alpha)\, n_i n_j$$

R_{ij} : $3\times$~~3 matrix~~ orthogonal matrix

$$(R_{ij} V_j) \cdot (R_{ik} W_k) = V_i W_i = \vec{V} \cdot \vec{W}$$

$$R_{ij} R_{ik} = \delta_{jk}$$

$$R^T R = \underline{1} = R R^T$$

$\forall$ orthogonal matrix M, $\det(M) = \pm 1$

but rotation matrix has $\det(R) = +1$

orthogonal matrices of det = +1
form a group $SO(3)$

or $SO(n)$ in n dimensions

$SO(n)$ is isomorphic to
rotation group in n dimensions.

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infinitesimal rotations commute
to 1st order in ~~the~~ angle

$R_1(\alpha_1, n_1)$, $R(\alpha_2 n_2)$

$V \to V' = R_1 V \to V'' = R_2 R_1 V.$

$\vec{V}' = \vec{V} + \alpha_1 \vec{n}_1 \times \vec{V} + O(\alpha_1^2)$

$\vec{V}'' = \vec{V}' + \alpha_2 \vec{n}_2 \times \vec{V}' + O(\alpha_2^2)$

$= \vec{V} + (\alpha_1 \vec{n}_1 + \alpha_2 \vec{n}_2) \times \vec{V} + O(\alpha_1^2, \alpha_1\alpha_2, \alpha_2^2)$

To 1st order

$(\alpha \vec{n})_{net} = \alpha_1 \vec{n}_1 + \alpha_2 \vec{n}_2$

symmetric in $(1, 2)$

$R_1 R_2 = R_2 R_1 + O(\alpha^2)$,

$\to$ infinitesimal rotation angles
add up as vectors $\alpha\vec{n}$

QM: For infinitesimal α

$$\hat{R}(\alpha,\vec{n}) = 1 - \frac{i}{\hbar}\alpha\vec{n}\cdot\hat{\vec{J}} + O(\alpha^2)$$

$\hat{J}_x, \hat{J}_y, \hat{J}_z$: generators.

For finite α

$$\hat{R}(\alpha,\vec{n}) = \exp\left(-\frac{i}{\hbar}\alpha\vec{n}\cdot\hat{\vec{J}}\right)$$

Schrödinger picture

rotation $|\psi\rangle \to \hat{R}|\psi\rangle$.

$$\hat{A} \to \hat{A}.$$

Heisenberg picture

$$|\psi\rangle \to |\psi\rangle$$

$$\hat{A} \to \hat{R}^\dagger\hat{A}\hat{R}$$

scalar operator $\hat{S} \to \hat{R}^\dagger\hat{S}\hat{R} = \hat{S}$

vector operator $\hat{\vec{V}}_i \to \hat{R}^\dagger\hat{V}_i\hat{R} = R_{ij}V_j$

Beyond 1st order in α

rotations do not commute

$$R(\alpha_1\vec{n}_1)R(\alpha_2\vec{n}_2) \neq R(\alpha_2\vec{n}_2)R(\alpha_1\vec{n}_1)$$

unless $\vec{n}_1 = \vec{n}_2$

$$\vec{V} \xrightarrow{R_1} \vec{V}_1 \xrightarrow{R_2} V_3$$

$$\downarrow R_2$$

$$V_2 \xrightarrow{R_1} V_4 \qquad V_3 \neq V_4.$$

To 2nd order in α_1, α_2

$$\vec{V}_1 = \vec{V} + \alpha_1 \vec{u}_1 \times \vec{V} + \frac{\alpha_1^2}{2}(\vec{u}_1(\vec{u}_1 \cdot \vec{V}) - \vec{V}) + O(\alpha_1^3).$$

$$\vec{V}_3 = \vec{V}_1 + \alpha_2 \vec{u}_2 \times \vec{V}_1 + \frac{\alpha_2^2}{2}(\vec{u}_2 (\vec{u}_2 \cdot \vec{V}_1) - \vec{V}_1)$$

$$= \vec{V} + \alpha_1 \vec{u}_1 \times \vec{V} + \frac{\alpha_1^2}{2}(u_1(u_1 \cdot V) - \vec{V})$$

$$+ \alpha_2 \vec{u}_2 \times \vec{V} + \alpha_2 \vec{u}_2 \times (\alpha_1 \vec{u}_1 \times \vec{V})$$

$$+ \frac{\alpha_2^2}{2}(\vec{u}_2 (u_2 \cdot \vec{V}) - \vec{V})$$

$$+ O(\text{cubic in } \alpha_1, \alpha_2).$$

$$\vec{V}_3 = \vec{V} + (\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2) \times \vec{V}$$

$$+ \frac{\alpha_2^2}{2}(\vec{u}_2(\vec{u}_2 \cdot \vec{V}) - \vec{V}) + \frac{\alpha_1^2}{2}(\vec{u}_1(\vec{u}_1 \cdot \vec{V}) - \vec{V})$$

$$+ \alpha_1 \alpha_2 \vec{u}_2 \times (\vec{u}_1 \times \vec{V}) + \text{cubic}$$

different

$$\vec{V}_4 = \vec{V} + (\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2) \times \vec{V}$$

$$+ \frac{\alpha_1^2}{2}(\cdots) + \frac{\alpha_2^2}{2}(\cdots)$$

$$+ \alpha_1 \alpha_2 \vec{u}_1 \times (\vec{u}_2 \times \vec{V})$$

$$\vec{V}_4 - \vec{V}_3 = \alpha_1\alpha_2(\vec{u}_2 \times (\vec{u}_1 \times \vec{V}) - \vec{u}_1 \times (\vec{u}_2 \times \vec{V}))$$

$$= \alpha_1\alpha_2 (\vec{u}_1 \times \vec{u}_2) \times \vec{V}.$$

$$\vec{V}_4 = R_1 \cdot R_2 \cdot \vec{V}$$

$$\vec{V}_3 = R_2 \cdot R_1 \cdot \vec{V}$$

$$(R_1R_2 - R_2R_1) \cdot \vec{V} = \alpha_1\alpha_2 (u_1 \times u_2) \times \vec{V}$$

+ cubic

$$= R(\alpha_1\alpha_2(\vec{u}_1 \times \vec{u}_2)) \cdot \vec{V} - \vec{V}.$$

+ quartic

$$\boxed{[R_1, R_2] = R(\alpha_1\alpha_2 \vec{u}_1 \times \vec{u}_2) - 1,} + O(\text{cubic}).$$

operators $\hat{R}(\alpha, \vec{u})$ obey the same algebraic relations as matrices $R(\alpha, \vec{u})$.

$$[\hat{R}(\alpha_1, \vec{u}_1), \hat{R}_2(\alpha_2 \hat{u}_2)] = \hat{R}(\alpha_1\alpha_2 \vec{u}_1 \times \vec{u}_2) - 1$$

+ ~~O(...)~~

+ ~~cub~~ $O(\text{cubic})$.

$$\hat{R}(\alpha_1 \vec{n}_1) = 1 - \frac{i}{\hbar}\alpha_1 \vec{n}_1 \cdot \vec{J} - \frac{\alpha_1^2}{2\hbar^2}(\vec{n}_1 \cdot \vec{J})^2 + O(\alpha_1^3)$$

$$\hat{R}(\alpha_2 \vec{n}_2) = 1 - \frac{i\alpha_2}{\hbar}\vec{n}_2 \cdot \vec{J} - \frac{\alpha_2^2}{2\hbar^2}(\vec{n}_2 \cdot \vec{J})^2 + O(\alpha_2^3).$$

$$[\hat{R}(\alpha_1 n_1), R(\alpha_2 \hat{n}_2)] = \frac{-\alpha_1\alpha_2}{\hbar^2}[\vec{n}_1 \cdot \vec{J}, n_2 \cdot \vec{J}] + O(\alpha_1^2\alpha_2, \alpha_1\alpha_2^2).$$

$$R(\alpha_1\alpha_2 \vec{n}_1 \times \vec{n}_2) - 1$$

$$= \frac{-i\alpha_1\alpha_2}{\hbar}(\vec{n}_1 \times \vec{n}_2) \cdot \vec{\hat{J}} + \cdots$$

$$[(\vec{n}_1 \cdot \vec{\hat{J}}), (\vec{n}_2 \cdot \vec{\hat{J}})] = i\hbar (n_1 \times n_2) \cdot \vec{\hat{J}}$$

$$\begin{array}{ccc} \| & & \| \\ n_{1i} n_{2j} [\hat{J}_i, \hat{J}_j] & & i\hbar \epsilon_{ijk} n_{1i} n_{2j} \hat{J}_k \end{array}$$

$$\boxed{[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon_{ijk} \hat{J}_k}$$

Any angular momentum operator generating rotations of <u>any</u> quantum degrees of freedom must obey these commutation relations in 2d, For ~~particle~~ 1 particle without spin, $\psi(x,y)$

$\exists$ SO(2) rotations are generated

by $\hat{J} = \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$

↳ in 3D, for a single spinless particle

$\hat{J}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \hat{L}_z$

$\hat{J}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x$

$\hat{J}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \hat{L}_y$

$\vec{\hat{J}} = \vec{\hat{L}} = \vec{\hat{x}} \times \vec{\hat{p}}$

For a particle with spin

$\vec{J} = \vec{L} + \vec{S}$

For several particles

$\vec{J} = \sum_{\text{particles}} (\vec{L} + \vec{S})_{\text{particle}}$

in all cases $[\hat{J}_i, \hat{J}_j] = i\hbar\,\epsilon_{ijk}\,\hat{J}_k$