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Representation of SO(3)

matrices $(T^{(\sigma)}_{x,y,z})$ indices (σ labels distinct representations)

$$[T_i^{(\sigma)}, T_j^{(\sigma)}] = i\hbar\, \epsilon_{ijk}\, T_k^{(\sigma)} \qquad i,j,k = x,y,z$$

given $T^{(\sigma)}_{x,y,z}$

$$D^{(\sigma)}(\varphi,\vec{u}) = \text{matrix} \exp\left(-\frac{i\varphi}{\hbar}\vec{T}^{(\sigma)}\cdot\vec{u}\right)$$

↳ ~~repfient~~ represent SO(3) rotations $R(\varphi,\vec{u})$

$$D^{(\sigma)}(R_2R_1) = D^{(\sigma)}(R_2)\cdot D^{(\sigma)}(R_1)$$

$R_2, R_1 \in SO(3)$.

~~similar approach to other cont. groups.~~

~~generators obey~~ $\cancel{[\hat{G}_i,\hat{G}_j] = i\hbar f_{ijk}\hat{G}_k}$

$\cancel{i,j,k = 1,\dots,n}$

~~f_{ijk}: specific to the group in question~~

~~↳ structure constants.~~

→ ~~find $T_i^{(\sigma)}$ which obey~~

$T_i^{(\sigma)}$ represent angular momenta $\hat{J}_i$

$D^{(\sigma)}(\varphi,\vec{u})$ represent finite rotations

$R_g(\varphi,\vec{u})$

so $T_i^{(\sigma)}, D^{(\sigma)}(\varphi,\vec{u})$: matrices $d(\sigma)\times d(\sigma)$

$d(\sigma)$: dimension of repr. (σ)

A multiplet: set of some objects which transform under rotations according to some representation of the rotation group.

For example: states $|\alpha, j, m\rangle$: fixed α, j all m

form a multiplet of $(2j+1)$ states.

under rotations

$$\hat{\mathcal{D}}(\varphi, \vec{n})\,|\alpha, j, m\rangle = \sum_{m'} \underbrace{|\alpha, j, m'\rangle}_{\text{same } \alpha,\text{ same } j}\, \mathcal{D}^{(j)}_{m';m}(\varphi, \vec{n})$$

$\mathcal{D}^{(j)}_{m';m}(\varphi, \vec{n})$ represents $R(\varphi, \vec{n})$

by $(2j+1) \times (2j+1)$ matrices

(α, j) label a multiplet

m labels states within one multiplet.

~~j label: repre~~

j specifies the representation of $SO(3)$

α labels diff. multiplets with same j

a reducible multiplet
can be further block-diagonalized.

an irreducible multiplet cannot.

↳ reducible / irreducible
representations.

For $SO(3)$, the complete list
of finite irreducible
representations is given
by repr(j) $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

For each j, represented by
$(2j+1) \times (2j+1)$ matrices.

In A quantum system with a
rotational symmetry
states form multiplets of $SO(3)$
Each (α, j) multiplet has
$(2j+1)$ states $|\alpha, j, m\rangle$.

but which (α, j) ~~appear in the [illegible]~~
are present depends on the system.

Spinless particle in a central potential $V(r)$

$$\hat{H} = \frac{\vec{p}^2}{2m} + V(\hat{r}).$$

$$\vec{J} = \vec{L} = \hat{\vec{x}} \times \hat{\vec{p}}$$

$\llcorner$ orbital angular momentum.

single valued $\psi(x, y, z)$

$$\psi(r, \theta, \varphi + 2\pi) = \psi(r, \theta, \varphi).$$

& $\forall\ |\alpha, j, m\rangle,\ \psi \sim e^{im\varphi}$

$$L_z = -i\hbar \frac{\partial}{\partial\varphi}$$

single-valued ψ requires integer m

$\rightarrow$ integer $j = \ell = 0, 1, 2, \ldots$

For the orbital angular momentum ℓ must be integer.

For state $|\alpha, \ell, m\rangle$

$$\psi(r, \theta, \varphi) = \psi_\alpha(r) \times Y_{\ell, m}(\theta, \varphi)$$

$\uparrow$ spherical harmonic

$$\hat{H} = \frac{\hat{p}_r^2}{2m} + \frac{\vec{L}^2}{2mr^2} + V(r)$$

$\psi_\alpha(r)$ obeys

$$\frac{1}{r}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right) r\,\psi_\alpha(r)$$
$$= E_{\alpha,\ell}\,\psi_\alpha(r)$$

Energies $E_{\alpha,\ell}$ depend on $\alpha \equiv n_{\text{radial}}$ and ℓ but not on m.

$\rightarrow$ each energy level has $(2\ell+1)$-fold degeneracy

For ~~actual~~ almost all $V(r)$

there is no further degeneracy

$E_{\alpha,\ell} \neq E_{\alpha'\ell'}$ unless both $\ell=\ell'$, $\alpha=\alpha'$.

2 exceptions:

Harmonic $V(r) = \frac{\omega^2 m}{2} r^2$

Coulomb $V(r) = -\frac{Ze^2}{r}$

due to extra symmetries beyond $so(3)$.

Spin

A single particle has

a fixed spin $j = s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$\rightarrow$ a single multiplet of $(2j+1)$ st spin states.

For ex, an electron, a proton, a neutron have $s = \frac{1}{2}$

$\rightarrow$ 2 spin states.

in the basis of $|s = \frac{1}{2}, m_s = \pm\frac{1}{2}\rangle$

① $\hat{\vec{S}} = \frac{\hbar}{2} \cdot \vec{\sigma}$

↑ Pauli matrices.

② 2×2 matrix $\mathcal{D}^{1/2}(\varphi, \vec{n}) =$

$$= Q(\varphi, \vec{n}) = \exp\left(-\frac{i\varphi}{2}\, \vec{n} \cdot \vec{\sigma}\right)$$

$$= \cos\frac{\varphi}{2} - i \sin\frac{\varphi}{2}\,(\vec{n} \cdot \vec{\sigma}).$$

For Rotation by 2π around any axis $\vec{n}$

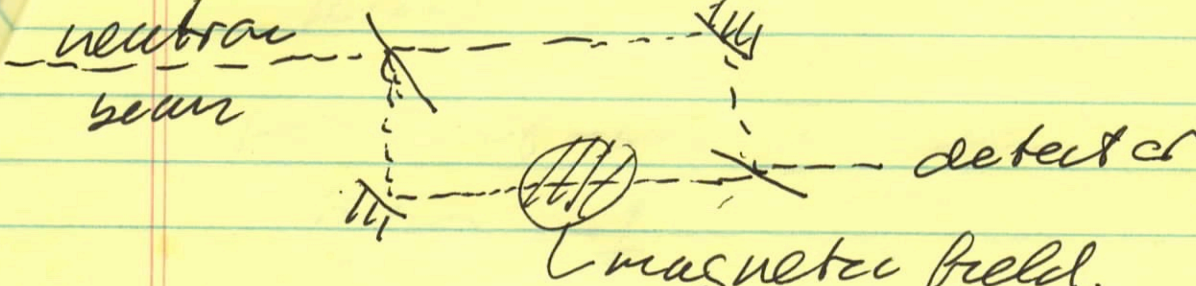
$\boxed{\cancel{Q(2\pi, \vec{n}) = -1}}$

&

$\boxed{Q(2\pi, \vec{n}) = -1}$

$\boxed{Q(4\pi, \vec{n}) = +1}$

Experiment



mag field rotates the spin through

angle $\varphi = \Omega_{\text{Larmor}} \times$ time of flight through B.

we ~~can~~ can measure φ.

interference

constructive for $\varphi = 0, 4\pi, 8\pi, \ldots$

destructive for $\varphi = 2\pi, 6\pi, \ldots$

More generally, any j, any $m = -j, \ldots, +j$

$$\hat{R}(2\pi, z)\,|j,m\rangle = e^{-2\pi i J_z/\hbar}\,|j,m\rangle$$
$$= e^{-2\pi i m}\,|j,m\rangle$$

$$e^{-2\pi i m} = (-1)^{2m} = (-1)^{2j}$$

$\forall$ axes $\hat{R}(2\pi, \vec{n})\,|j,m\rangle = (-1)^{2j}\,|j,m\rangle$

$$\hat{R}(4\pi, \vec{n})\,|j,m\rangle = +|j,m\rangle.$$

For integer spin

$\hat{R}(2\pi) = 1$

For half-integer spin

$\hat{R}(2\pi) = -1$

half-integer representations of $SO(3)$ are double-valued function of R_{ij}: $x_i' = R_{ij} x_j$.

~~R_{ij}~~ R_{ij} does not distinguish between $\underset{=1}{R(2\pi)}$ and $R(0) = 1$

$\rightarrow$ ~~See~~ Spin(3) group: double cover of $SO(3)$

spin $= \frac{1}{2}$ $\quad R \rightarrow Q = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\, \vec{n}\cdot\vec{\sigma}$

$Q \in SU(2)$: unitary 2x2 matrix $\det(Q) = 1$

Cale Cayley-Klein $\quad Q^{\dagger}\sigma_i Q = R_{ij}\sigma_j$

$Q \rightarrow R$ $\quad$ ~~$Q^{\dagger}\sigma_i Q = R_{ij}\sigma_j$~~

2-to-1 map $\quad Q$ & $-Q$ define same R_{ij}.

identify $\text{spin}(3) \cong SU(2)$.

$\mathcal{D}^{(j)}_{m', m}(R)$: single-valued
for both integer & half-integer j

$n > 3$ dimensions;

$Spin(n)$: double cover of $SO(n)$

$n = 4, 5, 6$ Spin(3) is related
to other groups

$Spin(4) = SU(2) \times SU(2)$
$Spin(6) = SU(4)$.

$n > 6$: Spin(n) ~~independent~~
$\Rightarrow$ distinct group
no relation to SU(?)
or Sp groups.

$SO(2) = U(1)$: unimodal or complex
number.

$Spin(2)$ is infinite cover of $SO(2)$.

but $\hat{R}(4\pi) \neq 1$

$d(r)$; dimension of repr. (r)