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Identical Particles

Exact permutation symmetry

$(\vec{x}_1, s_1) \leftrightarrow (\vec{x}_2, s_2)$

~~N~~ N particles: all N! permutations are exact symmetries

discrete symmetry group S_N

⇒ Quantum states ∈ multiplets of S_N

non-singlet multiplets: trouble

~~wave~~ ~~wavefn~~ multi-component $\psi(\vec{x}_1 \cdots \vec{x}_N)$

permuting 2 particles in another galaxy affects the wave function here.

↳ need singlet representations of S_N.

2 choices: totally symmetric

$\psi(\text{permutation of } x_1 \cdots x_N) = \psi(x_1 \cdots x_N)$.

totally antisymmetric

$\psi(\text{permutation of } x_1 - x_N) = (-1)^{\text{parity}} \times \psi(x_1 \cdots x_N)$.

→ 2 particle types: bosons & fermions.

Spin-statistics Theorem

Particles of integer spin are bosons

Particles of half-integer spin are fermions

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hand-waving argument

take 2 particles in same spin state & permute

for 2 particles permutation is equiv. to a 180° rotation

$$\hat{R}(\alpha, \hat{z}) = e^{i\alpha J_z/\hbar}$$

net $J_z = S_{1z} + S_{2z}$

same spin state m

net $S_z = 2\hbar m$

$$e^{i\pi \cdot 2m} = (-1)^{2m} = (-1)^{2s}$$

permutation carries + sign for integer s

− sign for half-integer s

$d \geq 3$ dimensions

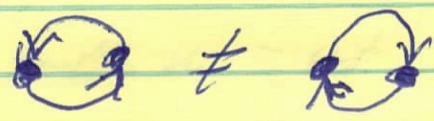
If $\hat{R}(2\pi) = +1 \rightarrow$ bosons

$\tilde{R}(2\pi) = -1 \rightarrow$ fermions

$d = 2$ dimensions

Particles coming from 3d
$\rightarrow$ usual 3d rules.

~~For~~ quasi-particles in 2d.
may have fractional spin &
fractional statistics.

Exch in 2d 

$$\psi(x_2, x_1) = e^{\pm i\theta}\psi(x_1, x_2)$$

sign of $\pm i\theta$ depends
on the manner of permutation.

$\theta = 0$: bosons
$\theta = \pi$: fermions
$\theta \neq 0, \pi$: anyons.

Multiple anyons : singlet ~~unitary~~ representation
of the braid group B_n instead of S_n

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For anyons

$$J_z = \hbar \left(\pm \frac{\theta}{2\pi} + \text{integer} \right)$$

~~Nex~~ Degeneracy in a central potential.

$$\psi(\vec{x}) = \psi_{n_r}(r) \times Y_{\ell m}(\theta, \varphi)$$

$\downarrow$ $|n_r, \ell, m\rangle$.

$E(n_r, \ell)$, degenerate wrt ~~m~~. $(2\ell+1)$-fold

usually $E(n_r', \ell') \neq E(n_r, \ell)$
unless both $n_r = n_r'$, $\ell = \ell'$

2 exceptions:

Harmonic $V = \frac{\omega^2 m}{2} r^2$

Coulomb $V = -\frac{Ze^2}{r}$

~~Harmo~~

$$\hat{H} = \frac{\bar{P}^2}{2m} + \frac{m\omega^2}{2}\bar{r}^2$$

$$\rightarrow \sum_{i=x,y,z}\left(\frac{p_i^2}{2m} + \frac{m\omega^2}{2}x_i^2\right)$$

$\rightarrow$ 3 indep. Harmonic osc., same ω.

$$\hat{H} = \hbar\omega\left(\hat{a}_x^{+}\hat{a}_x + \hat{a}_y^{+}a_y + \hat{a}_z^{+}a_z + \frac{3}{2}\right)$$

$$= \hbar\omega\left(\hat{n}_{tot} + \frac{3}{2}\right)$$

$\hat{Q}_{ij} = \hat{a}_i^{+}a_j$, any $i,j = x,y,z$.

~~All~~ All ~~9~~ $\hat{Q}_{ij}$ commute with $\hat{\vec{L}}$

$\rightarrow$ with $\hat{H}$.

$\rightarrow$ more conserved quantities than just $\hat{\vec{L}}$ & $\hat{H}$

Together, Q_{ij} generate ~~a~~ the $U(3)$ symmetry group: unitary 3×3 matrices.

$$\hat{U}\hat{a}_\alpha^{+}U^{+} = \sum_\beta \underbrace{U_{\alpha\beta}}_{\text{unitary } 3\times3 \text{ matrix}}\hat{a}_\beta^{+}$$

$$\hat{U}\hat{a}_\alpha U^{+} = \sum_\beta U_{\alpha\beta}^{*}\hat{a}_\beta$$

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any unitary matrix $U = e^{i\varphi}\tilde{U}$ for $\det\tilde{U} = 1$ for some overall phase φ

$\Rightarrow U(3) = U(1) \times SU(3)$.

6) SU(3) Multiplets realized on the 3-oscillator system

totally-symmetric tensors ν indices

$\leftrightarrow$ states of ν quanta.

in terms n_r & l

$$\nu = \cancel{n_r+2}\, l + 2n_r.$$

$\nu = 0$ level ; 1 state $(l=0)$
$\nu = 1$ level 3 states $(l=1)$
$\nu = 2$ level 6 states $(l=2)$ or $(l=0)$
$\nu = 3$ level 10 states $(l=3)$ or $(l=1)$
$\nu = 4$ " — " 15 " $(l=4)$ or $(l=2)$ or $(l=0)$

level ν : $\frac{1}{2}(\nu+1)(\nu+2)$ states.

$l = \nu$, or $l = \nu - 2$, or $\nu - 4$, ----

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simultaneous rotations
of 2 particles or 2 sets of degrees
of freedom like $\vec{x}$ & spin state.

or 2 particles.

$$\hat{R}(\theta,\vec{n}) = \hat{R}_1(\theta,\vec{n}) \otimes \hat{R}_2(\theta,\vec{n}).$$

generator: $\hat{\vec{J}}^{net} = \hat{\vec{J}}^{(1)} + \hat{\vec{J}}^{(2)}$

$$\left\{\begin{array}{l} [\hat{J}_i^{(1)}, \hat{J}_j^{(1)}] = i\hbar\,\epsilon_{ijk}\,\hat{J}_k^{(1)} \\ [\hat{J}_i^{(2)}, \hat{J}_j^{(2)}] = i\hbar\,\epsilon_{ijk}\,\hat{J}_k^{(2)} \\ [\hat{J}_i^{(1)}, \hat{J}_j^{(2)}] = 0 \end{array}\right.$$

bec they act on different degrees of freedom

$$\rightarrow [\hat{J}_i^{net}, \hat{J}_j^{net}] = i\hbar\,\epsilon_{ijk}\,\hat{J}_k^{net}.$$

multiplet: (j_1) of 1^{st} $\times$ (j_2) of 2^{nd}.
$\rightarrow$ $(2j_1+1)\times(2j_2+1)$ states.
$\hookrightarrow$ reducible multiplet of J^{net}

$\hookrightarrow$ decomposes into several irred. multiplets with diff. j^{net}.

$$(j_1) \times (j_2) = \bigoplus_{j^{tot}} (j)^{tot}$$

j^{tot} runs from $|j_1 - j_2|$ to $j_1 + j_2)$
by 1