2/3/11

$\vec{J} = \vec{J}_1 + \vec{J}_2$

$\vec{J}_1$ acts on 1 set of degrees of freedom

$\vec{J}_2$ acts on another set of DoF

states $|\alpha; j_1, m_1; j_2 m_2\rangle$

multiplet: fixed α, j_1, j_2

all possible $m_1 = -j_1, \ldots, +j_1$

$m_2 = -j_2, \ldots, +j_2$.

altogether $(2j_1+1) \times (2j_2+1)$ states.

From the PoV of $\vec{J} \equiv \vec{J}^{net} = \vec{J}_1 + \vec{J}_2$

this multiplet is reducible.

→ comprises several multiplets (j).

$$(j_1) \otimes (j_2) = \bigoplus_j (j)$$

$\boxed{Q1}$ $\bigoplus_j$ over what set of j?

$\boxed{Q2}$ $|\alpha, j_1, m_1, j_2, m_2\rangle$ = linear comb.

of states $|\alpha, j, j_2, j_{net}, m_{net}\rangle$

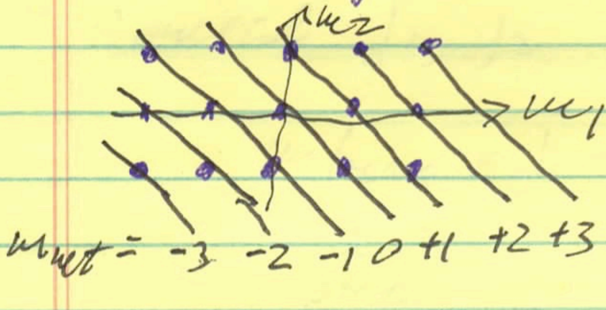
what are the coefficients here?

$$\hat{J}_z^{net} = \hat{J}_z^{(1)} + \hat{J}_z^{(2)}$$

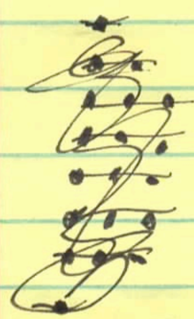
$$\rightarrow \boxed{m_{net} = m_1 + m_2}$$

classify $|m_1, m_2\rangle$ states by m_{net}.

example: $j_1 = 2$, $j_2 = 1$



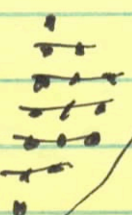
net # states for each m_{net}



1 state for $m_{net} = \pm 3$

2 states for $m_{net} = \pm 2$

3 states for $m_{net} = \pm 1$ or 0.



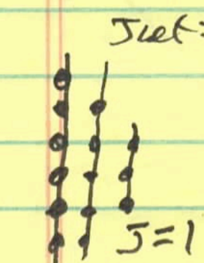
$m_{net} = \pm 3$, $j_{net} \geq 3$

no states with $|m_{net}| > 3$

$\rightarrow$ no $j_{net} > 3$.

$\rightarrow$ 1 multiplet of ($j = 3$)

↳ highest $j_{net} = 3$.



remaining states:

1 with $m_{net} = \pm 2$

2 with $m_{net} = \pm 1$ or 0.

↳ there is 1 multiplet ($j_{net} = 2$)

remaining states of $m_{net} = \pm 1$ or 0

form a ($j_{net} = 1$) multiplet.

3/3

For $j_1 = 2, j_2 = 1$

$j_{net} = 3, 2,$ or 1

$$(2)\otimes(1) = (3)\oplus(2)\oplus(1)$$

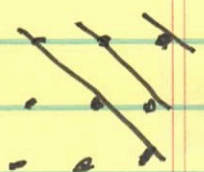
General j_1, j_2.

$|m|_{max} = j_1 + j_2$

1 state for $m = \pm(j_1+j_2)$. each

2 states for $m = \pm(j_1+j_2-1)$ each.

3 states for $m = \pm(j_1+j_2-2)$ each



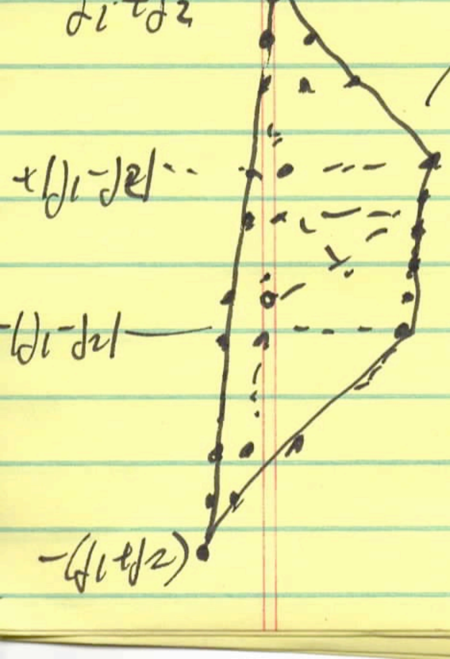
⋮

ν states $m = \pm(|j_1-j_2|)$ each

$(2\min(j_1,j_2)+1)$

for any $|m| < |j_1-j_2|$

$(2\min(j_1,j_2)+1)$ states.



→ on this spindle diagram each column is a multiplet of J_{net}.

→ $J_{net} = j_1+j_2, j_1+j_2-1, j_1+j_2-2, \ldots$

$\ldots, |j_1-j_2|$.

3/4

Adding 2 angular momenta

j_1 & j_2

→ j net runs from $|j_1-j_2|$ to j_1+j_2 by 1.

if j_1 & j_2 both integer, j is integer

if j_1 & j_2 both half-integer, j is integer.

if j_1 integer j_2 half-integer or vice verse, j is half-integer

$\vec{J} = \vec{L} + \vec{S}$ for an electron

l is integer, $s = \frac{1}{2}$

→ j is half-integer.

$j = l + \frac{1}{2}$ or $j = l - \frac{1}{2}$

only for $l \neq 0$.

states with definite j_{net}, m_{net} are linear combinations of states with definite m_1 & m_2.

And vice verse.

$$|\alpha, j_1, j_2, j_{net}, m_{net}\rangle = \sum_{\substack{m_1+m_2 \\ m_1, m_2 = m_{net}}} |\alpha, j_1, m_1, j_2, m_2\rangle \times C(j_1, m_1, j_2, m_2 | j_{net}, m_{net})$$

Clebbsch-Gordan coefficients.

$$C(j_1 m_1 j_2 m_2 | j_{net} m_{net}) = C(j_1 m_1 j_2 m_2 | j_1 j_2 j_{net} m_{net})$$

$$= \langle j_1, m_1, j_2, m_2 | j_1, j_2, j_{net}, m_{net}\rangle.$$

$$|\alpha, j_1, m_1, j_2 m_2\rangle = \sum_{j_{net}} |\alpha j_1 j_2 j_{net}\ m_{net} = m_1 + m_2\rangle \times$$

$$\times C(j_1 j_2 j_{net}, m_{net} | j_1, m_1, j_2, m_2)$$

P8
23/6

3 sets of degrees of freedom

$\vec{J}_{net} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3$

$\hookrightarrow$ multiplet $(j_1) \otimes (j_2) \otimes (j_3)$

~~fixed~~ $|\alpha j_1 m_1 j_2 m_2 j_3 m_3\rangle$

fixed α, j_1, j_2, j_3

all possible m_1, m_2, m_3.

Stage 1: Add $\vec{J}_1 + \vec{J}_2 = \vec{J}_{12}$

$$\hookrightarrow (j_1) \otimes (j_2) = \bigoplus_{j_{12} = |j_1 - j_2|}^{j_1 + j_2} (j_{12})$$

Then for each j_{12}, add $\vec{J}_{12} + \vec{J}_3 = \vec{J}_{net}$

$$(j_1) \otimes (j_2) \otimes (j_3) = \bigoplus (j_{net})$$

but: ~~see~~ You may end up with 2 or more different multiplets of same (j_{net}).

Example: 3 spins, $s_1 = s_2 = s_3 = \frac{1}{2}$

$\rightarrow 2^3 = 8$ states altogether

$\cancel{(\tfrac{1}{2})\otimes(\tfrac{1}{2}) = (0)\oplus(1)}$

$$(j_1=\tfrac{1}{2})\otimes(j_2=\tfrac{1}{2}) = (j_{12}=0)\oplus(j_{12}=1).$$

$$(j_1=\tfrac{1}{2})\otimes(j_2=\tfrac{1}{2})\otimes(j_3=\tfrac{1}{2}) = \left[(j_{12}=0)\oplus(j_{12}=1)\right]\otimes(j_3=\tfrac{1}{2})$$

$$= (j_{12}=0)\otimes(j_3=\tfrac{1}{2}) \oplus (j_{12}=1)\otimes(j_3=\tfrac{1}{2})$$

$$= (j_{net}=\tfrac{1}{2}) \oplus (j_{net}=\tfrac{1}{2}) \oplus (j_{net}=\tfrac{3}{2}).$$

states $2 + 2 + 4 = 8$

2 different doublets.

$|s_1 m_1, s_2 m_2, s_3 m_3\rangle \rightarrow$ lin. combinations

of $|s_1 s_2 s_3, s_{net}, m_{net}, \xi\rangle$

ξ distinguishes between

2 different doublets of

$s_{net} = \frac{1}{2}$, $m_{net} = \pm\frac{1}{2}$.

3/8

Back to $\vec{J} = \vec{J}_1 + \vec{J}_2$, 2 similar particles

Suppose $j_1 = j_2$

$|n_1 j_1 m_1; n_2 j_2 m_2\rangle$

$n_1 = n_2$ as well as $j_1 = j_2$

Permutation symmetry

$$\hat{P}\,|n_1 j_1 m_1; n_2 j_2 m_2\rangle \rightarrow |n_2 j_2 m_2, n_1 j_1 m_1\rangle$$

$$\hat{P}\,|n_1 = n_2; j_1 = j_2; J_{net}, M_{net}\rangle$$

$$= \pm\,|n_1 = n_2; j_1 = j_2; J_{net}, M_{net}\rangle$$

$$\boxed{\text{sign} = (-1)^{j_1 + j_2 - J_{net}}}$$

state $|n_1 = n_2, j_1 = j_2, J_{net}, M_{net}\rangle$

is symmetric $\quad j = j_1 = j_2$

for $J_{net} = 2j, 2j-2, 2j-4, \ldots,$

down to 1 or 0.

antisymmetric for

$J_{net} = 2j-1, 2j-3, 2j-5, \ldots$

down to 0 or 1

23/9

~~Tstate~~

Titanium atom

outer electrons : 2 in 3d state.

each has $\ell = 2$, $s = \frac{1}{2}$.

Add up 2 spins $S_{net} = 0$ or 1

$S = 1$ is symmetric in 2 spins

$S = 0$ is antisymmetric.

Add up 2 orbital angular momenta

$L_{net} = 4, 3, 2, 1, 0$

$L_{net} = 4, 2, 0$ are symmetric in spatial wave functions

$L_{net} = 3$ or 1 are antisymmetric.

By Fermi statistics

$\hat{P}_{net} = P_{spin} \times P_{space} = -1$

$S = 1$ comes with $L = 3$ or 1

$S = 0$ comes with $L = 4, 2, 0$