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PHY–396 K. Solutions for problem set #6.

Problem **2**(a.1):

Let's start with the classically allowed region. For a bound state of the oscillator with energy $E = E_n = \hbar\omega(n+\frac{1}{2})$, we have

$$\begin{aligned} k(x) &= \frac{\sqrt{2mE - 2mV(x))}}{\hbar} = \frac{1}{\hbar}\sqrt{(2n+1)\hbar\omega m - m^2\omega^2x^2} \\ &= \sqrt{\frac{(2n+1)\omega m}{\hbar}} \times \sqrt{1 - \frac{m\omega x^2}{(2n+1)\hbar}} = \sqrt{\frac{(2n+1)\omega m}{\hbar}} \times \sqrt{1-y^2}. \end{aligned} \tag{S.1}$$

Consequently,

$$k\,dx = \sqrt{\frac{(2n+1)\omega m}{\hbar}}\, 1-y^2 \times \sqrt{\frac{(2n+1)\hbar}{m\omega}}\, dy = (2n+1)\sqrt{1-y^2}\,dy, \tag{S.2}$$

hence

$$\int k(x)\,dx = (2n+1)\int\sqrt{1-y^2}\,dy = \frac{(2n+1)}{2}\big(y\sqrt{1-y^2} - \arccos y\big) + \text{const} \tag{S.3}$$

where the integral evaluates by variable change $y = \cos\phi$, thus

$$\begin{aligned} \int\sqrt{1-y^2}\,dy &= \int \sin\phi \times (-\sin\phi)d\phi = \int \tfrac{1}{2}(\cos(2\phi)-1)d\phi \\ &= \tfrac{1}{4}\sin 2\phi - \tfrac{1}{2}\phi = \tfrac{1}{2}y\sqrt{y} - \tfrac{1}{2}\arccos\phi. \end{aligned} \tag{S.4}$$

Or in terms of definite integrals from the classical turning points x_1 and x_2,

$$\begin{aligned} \int_{x_1}^{x} k(x')\,dx' &= (2n+1)\int_{-1}^{y}\sqrt{1-y'^2}\,dy' = \frac{2n+1}{2}\Big(y\sqrt{1-y^2} + \pi - \arccos y\Big), \\ \int_{x}^{x_2} k(x')\,dx' &= (2n+1)\int_{y}^{+1}\sqrt{1-y'^2}\,dy' = \frac{2n+1}{2}\Big(-y\sqrt{1-y^2} + \arccos y\Big). \end{aligned} \tag{S.5}$$

Therefore, the WKB wave function for the allowed region

$$\Psi^{\rm WKB}(x) \;=\; \frac{\rm const}{\sqrt{k(x)}}\,\sin\left(\frac{\pi}{4}+\int\limits_{x}^{x_2}k(x')dx'\right) \;=\; \pm\frac{\rm const}{\sqrt{k(x)}}\,\sin\left(\frac{\pi}{4}+\int\limits_{x_1}^{x}k(x')dx'\right) \tag{S.6}$$

becomes

for $-1<y<+1$,

$$\begin{aligned}\Psi_n^{\rm WKB}(y) &= \frac{C}{(1-y^2)^{1/4}}\times\sin\Bigl(\tfrac{\pi}{4}\,+\,(n+\tfrac12)\arccos(y)\,-\,(n+\tfrac12)y\sqrt{2-y^2}\Bigr)\\ &= (-1)^n\frac{C}{(1-y^2)^{1/4}}\times\sin\Bigl((n+\tfrac34)\pi\,-\,(n+\tfrac12)\arccos(y)\,+\,(n+\tfrac12)y\sqrt{2-y^2}\Bigr)\end{aligned} \tag{S.7}$$

where C is the overall normalization constant.

Problem **2**(a.2):

Next, the classically forbidden region on the right side, $x>x_2$ or $y>+1$. In this region

$$\kappa \;=\; ik \;=\; \sqrt{\frac{(2n+1)\omega m}{\hbar}}\times\sqrt{y^2-1}, \tag{S.8}$$

$$\kappa(x)\,dx \;=\; (2n+1)\sqrt{y^2-1}\,dy, \tag{S.9}$$

hence

$$\begin{aligned}\int\kappa(x)\,dx &= (2n+1)\int\sqrt{y^2-1}\,dy\\ &= \frac{2n+1}{2}\Bigl(y\sqrt{y^2-1}\,-\,\operatorname{arcosh}(y)\Bigr)\;+\;\text{const},\end{aligned} \tag{S.10}$$

where the integral evaluates by variable change $y=\cosh\tau$, thus

$$\begin{aligned}\int\sqrt{y^2-1}\,dy &= \int\sinh\tau\times\sinh\tau\,d\tau \;=\; \int\tfrac12(\cosh(2\tau)-1)d\tau\\ &= \tfrac14\sinh(2\tau)\,-\,\tfrac12\tau \;=\; \tfrac12 y\sqrt{y^2-1}\,-\,\tfrac12\operatorname{arcosh} y.\end{aligned} \tag{S.11}$$

As a definite integral from the turning point x_2, the integral (S.10) becomes

$$\int\limits_{x_2}^{x}\kappa(x')\,dx' \;=\; (2n+1)\int\limits_{1}^{y}\sqrt{y'^2-1}\,dy' \;=\; (n+\tfrac12)\times\Bigl(y\sqrt{y^2-1}\,-\,\operatorname{arcosh}(y)\Bigr), \tag{S.12}$$

so the WKB wave function which shrinks for $x \to +\infty$

$$\Psi^{\rm WKB} = \frac{\rm const}{\sqrt{\kappa(x)}} \times \exp\left(-\int\limits_{x_2}^{x} \kappa(x')dx'\right) \tag{S.13}$$

becomes

$$\text{for } y > +1, \quad \Psi_n^{\rm WKB}(y) = \frac{C}{2(1-y^2)^{1/4}} \times \exp\Bigl(-(n+\tfrac{1}{2}) \times \bigl(y\sqrt{y^2-1} - \operatorname{arcosh}(y)\bigr)\Bigr). \tag{S.14}$$

Note: by the Airy function matching rules, the overall coefficient here is $C/2$ for the same C as in eq. (S.7) for the classically allowed region.

Finally, consider the other classically forbidden region to the left side of the oscillator, $x < x_1$ or $y < -1$. By the $x \to -x$ symmetry of the harmonic oscillator, the WKB wave-function in that region looks exactly like the wave-function (S.14) for the right forbidden region, except for $y \to -y$ and maybe the overall sign. In light of the second eq. (S.7) and the Airy function matching rules,

$$\text{for } y < -1, \quad \Psi_n^{\rm WKB}(y) = (-1)^n \frac{C}{2(1-y^2)^{1/4}} \times \exp\Bigl(-(n+\tfrac{1}{2}) \times \bigl(|y|\sqrt{y^2-1} - \operatorname{arcosh}(y)\bigr)\Bigr). \tag{S.15}$$

Problem **2**(b):

Let

$$x = \sqrt{\frac{(2n+1)\hbar}{\omega m}} \times y, \qquad k = \sqrt{\frac{(2n+1)\omega m}{\hbar}} \times q, \tag{S.16}$$

then

$$\frac{\omega m x^2}{2\hbar} = (2n+1)\,\frac{y^2}{2}, \quad kx = (2n+1)qy, \quad \frac{\hbar k^2}{4\omega m} = (2n+1)\,\frac{q^2}{4}, \tag{S.17}$$

and therefore

$$\exp\left(+\frac{\omega m x^2}{2\hbar}\right) \times \exp\left(ikx - \frac{\hbar k^2}{4\omega m}\right) = \exp\left((2n+1) \times \left(\frac{y^2}{2} + iqy - \frac{q^2}{4}\right)\right). \tag{S.18}$$

In the context of the integral (3), this means

$$\Psi_n^{\rm true} \;=\; {\rm const}\times\int\limits_{-\infty}^{+\infty} dq\, q^n \times \exp\left((2n+1)\times\left(\frac{y^2}{2}\,+\,iqy\,-\,\frac{q^2}{4}\right)\right), \qquad \text{(S.19)}$$

which looks almost like (4) except for the n-dependent pre-exponential factor q^n. But with a bit of algebra, we may move most of this factor inside the exponents; indeed,

$$q^n \;=\; \frac{q^{(n\frac{1}{2})}}{\sqrt{q}} \;=\; \frac{1}{\sqrt{q}}\times\exp\bigl((n+\tfrac{1}{2})\log q\bigr), \qquad \text{(S.20)}$$

hence

$$\Psi_n^{\rm true} \;=\; {\rm const}\times\int\limits_{-\infty}^{+\infty} \frac{dq}{\sqrt{q}} \times \exp\left((2n+1)\times\left(\frac{\log q}{2}\,+\,\frac{y^2}{2}\,+\,iqy\,-\,\frac{q^2}{4}\right)\right). \qquad \text{(S.21)}$$

This formula indeed has the desired form (4) for

$$f(q) \;=\; \frac{1}{\sqrt{q}} \qquad \text{(S.22)}$$

and

$$g(q,y) \;=\; \frac{\log q}{2}\,-\,\frac{q^2}{4}\,+\,iyq\,+\,\frac{y^2}{2}\,. \qquad \text{(S.23)}$$

Problem **2**(c):

Following the saddle point method explained in my notes, we treat the integral (4) as an integral over a complex contour Γ which happens to lie along the real axis. In the large n limit this integral is dominated by the saddle points where $g'(q)=(\partial g/\partial q)=0$, and if these saddle points happen to be complex rather than real, then the integration contour Γ should be deformed to Γ' which goes through these saddle points.

For $g(q)$ as in eq. (S.23), $g'(q)$ is

$$g'(q) = \frac{\partial g}{\partial q} = \frac{1}{2q} - \frac{q}{2} + iyq, \tag{S.24}$$

so its zeros are the roots of the quadratic polynomial

$$q^2 - 2iyq - 1 = 0, \tag{S.25}$$

thus

$$q_{1,2} = iy \pm \sqrt{-y^2+1}\,. \tag{S.26}$$

Note: for y in the allowed region $-1 < y < +1$, the two saddle points (S.26) are unimodular complex numbers related by reflection off the imaginary axis,

$$|q_1| = |q_2| = 1, \quad q_2 = -q_1^*. \tag{S.27}$$

But for y in a forbidden region $y > +1$ or $y < -1$, the two saddle points

$$q_{1,2} = iy \pm i\sqrt{y^2-1} \tag{S.28}$$

are imaginary but not unimodular and have different magnitudes. And that's why the two saddle points are co-dominant when y is in the allowed region, but there is only one dominant saddle point for y in a forbidden region.

Problem **2**(c,1) — the allowed region:

For y in the allowed region

$$\begin{aligned}\log(q_{1,2}) &= \log(iy \pm \sqrt{1-y^2}) = i\arg(iy \pm \sqrt{1-y^2}) \\ &= i \times \begin{cases} \arcsin y & \text{for } +, \\ \pi - \arcsin y & \text{for } -, \end{cases} \\ &= \frac{i\pi}{2} \mp i\arccos y. \end{aligned} \tag{S.29}$$

At the same time,

$$y^2 + 2iyq_{1,2} - \tfrac{1}{2}q_{1,2}^2 \;=\; y^2 + iq_{1,2}y - \tfrac{1}{2} \;=\; y^2 + iy(iy\pm\sqrt{1-y^2}) - \tfrac{1}{2} \;=\; \pm iy\sqrt{1-y^2} - \tfrac{1}{2}, \tag{S.30}$$

hence

$$g(q_{1,2}) \;=\; \frac{i\pi}{4} \;-\; \frac{1}{4} \;\pm\; \frac{i}{2}\big(y\sqrt{1-y^2} \;-\; \arccos y\big). \tag{S.31}$$

Note: at both saddle points g has the same real part, so their contributions to the integral (4) are of similar magnitudes. Therefore, we should deform the integration contour Γ to Γ' which goes through both of these points.

Moreover, Γ' should cross each saddle point as a mountain highway rather than a mountain goat trail: from a lower value of $\operatorname{Re} g(q)$ to a maximum and back to a lower value. In terms of unit tangent vector

$$\eta \;=\; \frac{dq}{|dq|} \tag{S.32}$$

at a saddle point q_1 or q_2, we need

$$\operatorname{Re}(\eta^2 g'') \;<\; 0. \tag{S.33}$$

For $g(q)$ as in eq. (S.24), the second derivative $g''(q)$ is

$$g''(q) \;=\; \frac{\partial^2 g}{\partial q^2} \;=\; \frac{-1}{2q^2} \;-\; \frac{1}{2}\,, \tag{S.34}$$

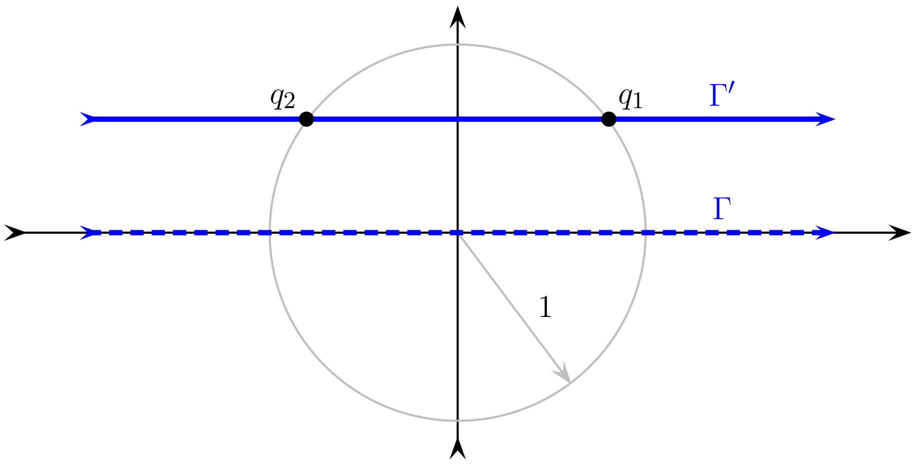
so let's find its values at the two saddle points $Q_{1,2}$:

$$\frac{1}{q_{1,2}^2} \;=\; \frac{(q_{1,2}^*)^2}{|q_{1,2}|^4} \;=\; \frac{(-iy \pm \sqrt{1-y^2})^2}{1} \;=\; (1-y^2) \;-\; y^2 \;\mp\; 2iy\sqrt{1-y^2}, \tag{S.35}$$

hence

$$g''(q_{1,2}) \;=\; -\tfrac{1}{2} \;+\; y^2 \;\pm\; iy\sqrt{1-y^2} \;-\; \tfrac{1}{2} \;=\; -(1-y^2) \;\pm\; iy\sqrt{1-y^2}\,. \qquad \text{(S.36)}$$

Note negative real part of g'' at both fixed points, so a contour Γ' crossing each points in the direction $\eta = 1$ would work as a proper mountain pass. For example



As explained in my notes on the saddle points (eq. (1.118) on page 6), the contribution of each saddle point to the integral

$$I \;=\; \int\limits_{\Gamma\,\text{or}\,\Gamma'} dq\, f(q) \times \exp\bigl((2n+1)\times g(q,y)\bigr) \qquad \text{(S.37)}$$

in the large n limit becomes

$$I_{1,2} \;=\; \sqrt{\frac{2\pi}{2n+1}} \times \exp\bigl((2n+1)\times g(q_{1,2},y)\bigr) \times \frac{\eta f(q_{1,2})}{\sqrt{-\eta^2 g''(q_{1,2},y)}} \;\times \bigl(1 \,+\, O(1/n)\bigr). \quad \text{(S.38)}$$

In our case $\eta = 1$, $f(q) = 1/\sqrt{q}$, so

$$\frac{\eta f(q_{1,2})}{\sqrt{-\eta^2 g''(q_{1,2},y)}} \;=\; \frac{1}{-g''(q_{1,2},y) \times q_{1,2}}$$

where

$$-g''(q_{1,2}) \times q_{1,2} \;=\; \bigl[(1-y^2) \mp iy\sqrt{1-y^2}\bigr] \times (iy \pm \sqrt{1-y^2}) \;=\; \pm\sqrt{1-y^2}\,, \qquad \text{(S.39)}$$

hence

$$\frac{\eta f(q_{1,2})}{\sqrt{-\eta^2 g''(q_{1,2},y)}}\ =\ \frac{\sqrt{\pm 1}}{(1-y^2)^{1/4}} \tag{S.40}$$

which we may rewrite as

$$\frac{e^{-i\pi/4}\times e^{\pm i\pi/4}}{(1-y^2)^{1/4}}\,. \tag{S.41}$$

At the same time,

$$\exp\bigl((2n{+}1)\times g(q_{1,2},y)\bigr)\ =\ e^{-(2n+1)/4}\,e^{i\pi(2n+1)/4}\times\exp\Bigl(\pm i(n{+}\tfrac12)\times\bigl(y\sqrt{1-y^2}-\arccos y\bigr)\Bigr), \tag{S.42}$$

so altogether

$$I_{1,2}\ =\ D\times\frac{e^{\pm i\pi/4}}{(1-y^2)^{1/4}}\times\exp\Bigl(\pm i(n+\tfrac12)\times\bigl(y\sqrt{1-y^2}\ -\ \arccos y\bigr)\Bigr)\times\bigl(1\,+\,O(1/n)\bigr) \tag{S.43}$$

where

$$D\ =\ \sqrt{\frac{2\pi}{2n+1}}\times e^{-(2n+1)/4}\,e^{i\pi n/2} \tag{S.44}$$

is an overall constant factor.

Finally, combining the two saddle points' contributions, we have

$$\begin{aligned} I\ &\mathop{\longrightarrow}_{n\to\infty}\ I_1\ +\ I_2\\ &=\ \frac{D}{(1-y^2)^{1/4}}\left(\begin{array}{c}\exp\Bigl(+i(n+\frac12)\times\bigl(y\sqrt{1-y^2}\ -\ \arccos y\bigr)\ +\ i\frac{\pi}{4}\Bigr)\\ +\ \exp\Bigl(-i(n+\frac12)\times\bigl(y\sqrt{1-y^2}\ -\ \arccos y\bigr)\ -\ i\frac{\pi}{4}\Bigr)\end{array}\right)\\ &=\ \frac{2D}{(1-y^2)^{1/4}}\times\cos\Bigl((n+\tfrac12)\times\bigl(y\sqrt{1-y^2}\ -\ \arccos y\bigr)\ +\ \tfrac{\pi}{4}\Bigr)\\ &=\ \frac{2D}{(1-y^2)^{1/4}}\times\sin\Bigl(\tfrac{\pi}{4}\ +\ (n+\tfrac12)\times\bigl(\arccos y\ -\ y\sqrt{1-y^2}\bigr)\Bigr). \end{aligned} \tag{S.45}$$

In light of eq. (S.21), this means that

$$\begin{aligned} \Psi_n^{\rm true}(y)\ &=\ {\rm const}\times I(y)\\ &\mathop{\longrightarrow}_{n\to\infty}\ \frac{C}{(1-y^2)^{1/4}}\times\sin\Bigl(\tfrac{\pi}{4}\ +\ (n+\tfrac12)\times\bigl(\arccos y\ -\ y\sqrt{1-y^2}\bigr)\Bigr) \\ &=\ \Psi_n^{WKB}(y)\quad\langle\!\langle\ \textit{cf.}\ \text{eq. (S.7)}\ \rangle\!\rangle, \end{aligned} \tag{S.46}$$

provided we identify $C = 2D \times$ pre-integral constant factor from eq. (S.21).

Problem **2**(c,1) — the forbidden regions:

Thanks to the $y \to -y$ symmetry between the two forbidden regions, let's focus on the right forbidden region $y > +1$. For such y, the two saddle points of $g(q)$ function lie on the imaginary axis rather than on the unit circle, and they have different magnitudes

$$q_1 \;=\; i(y + \sqrt{y^2 - 1}) \quad \text{and} \quad q_2 \;=\; i(y - \sqrt{y^2 - 1}). \tag{S.47}$$

For these points,

$$\log q_{1,2} \;=\; \frac{i\pi}{2} \;+\; \log(y \pm \sqrt{y^2 - 1}) \;=\; \frac{i\pi}{2} \;\pm\; \operatorname{arcosh} y, \tag{S.48}$$

$$\begin{aligned} y^2 \;+\; 2iyq_{1,2} \;-\; \tfrac{1}{2}q_{1,2}^2 \;&=\; y^2 \;+\; iyq_{1,2} \;-\; \tfrac{1}{2} \;=\; y^2 \;-\; \tfrac{1}{2} \;+\; iy(iy \pm i\sqrt{y^2 - 1}) \\ &=\; -\tfrac{1}{2} \;\mp\; y\sqrt{y^2 - 1}\,, \end{aligned} \tag{S.49}$$

hence

$$g(q_{1,2}) \;=\; \frac{i\pi}{4} \;-\; \frac{1}{4} \;\mp\; \frac{1}{2}\big(y\sqrt{y^2 - 1} \;-\; \operatorname{arcosh} y\big). \tag{S.50}$$

On the last line here the expression inside $(\cdots)$ is always positive (for $y > +1$), thus

$$\operatorname{Re} g(q_1) \;<\; \operatorname{Re} g(q_2), \tag{S.51}$$

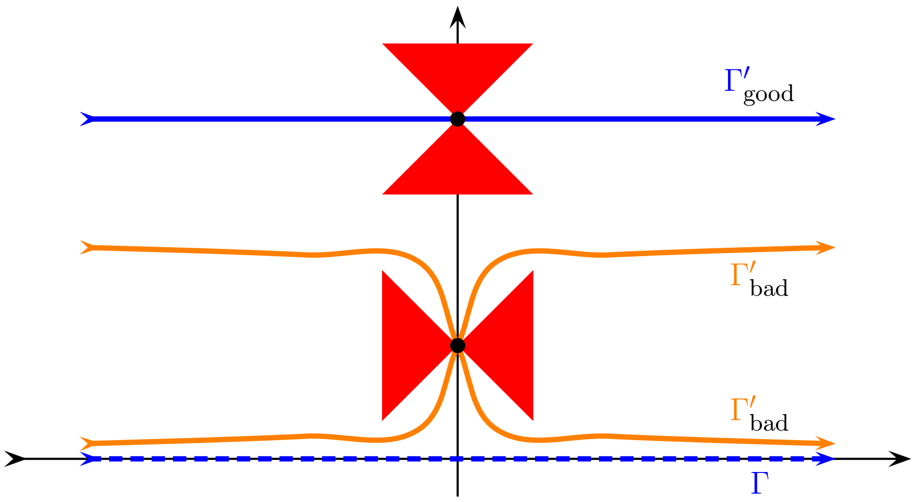
which *naively* suggests that the dominant contribution to the integral (4) comes from the q_2 point rather than q_1 (or both points).

However, we shall see in the moment that the mountain-pass like directions of crossing each saddle point are different: the q_1 point should be crossed horizontally — parallel to the original direction of the integration contour Γ along the real axis, — while the q_2 point should be crossed vertically, along the imaginary axis. By the $\operatorname{Re} q \to -\operatorname{Re} q$ symmetry of the integral, this means that the q_2 saddle point does not contribute to the integral; instead, the leading contribution comes from the other saddle point q_1.

In general, mountain-pass-like directions η of crossing a saddle point have negative real parts of $\eta^2 \times g''$, so let's calculate the second derivative $g''(q_{1,2})$ at the two fixed points:

$$\begin{aligned} g''(q_{1,2}) &= \frac{-1}{2q_{1,2}^2} - \frac{1}{2} = \frac{+1}{2(y \pm \sqrt{y^2-1})^2} - \frac{1}{2} \\ &= \frac{(y \mp \sqrt{y^2-1})^2}{2} - \frac{1}{2} = y^2 - 1 \mp y\sqrt{y^2-1} \\ &= \mp\sqrt{y^2-1}(y \mp \sqrt{y^2-1}). \end{aligned} \tag{S.52}$$

Thus, g'' has real values at both fixed points but $g''(x_1)$ is negative while $g''(x_2)$ is positive, so the mountain-pass-like directions at q_1 are horizontal (or close to horizontal) while the mountain-pass-like directions at q_2 are vertical (or close to vertical). Graphically,



where the black circles are the saddle points — q_1 above the q_2, — the red triangles indicates the bad directions of approaching them — the hills instead of the valleys, — the dashed blue line is the original integration contour along the real axis, the solid blue line is the good shifted contour going through the q_1 saddle point, and the solid orange lines are the bad shifted contours going through the q_2 point. By symmetry, the two orange contours should be exactly equivalent, so we may just as well average the respective integrals, but since they cross the q_2 point in opposite directions, its contribution cancels out from the average of the two contours. And that's why the integral is dominated (in the large n limit) by the q_1 saddle point instead of q_2.

The bottom line is,

$$I \;=\; \int_{\Gamma} dq\, f(q) \times \exp((2n+1)g(q,y)) \;=\; \int_{\Gamma'_{\rm good}} dq\, f(q) \times \exp((2n+1)g(q,y)), \tag{S.53}$$

which in the large n limit becomes

$$I \;\underset{n\to\infty}{\longrightarrow}\; \sqrt{\frac{2\pi}{2n+1}} \times \exp\bigl((2n+1) \times g(q_1,y)\bigr) \;\times\; \frac{\eta f(q_1)}{\sqrt{-\eta^2 g''(q_1,y)}}\,. \tag{S.54}$$

Specifically, the integration contour is horizontal, $\eta = 1$,

$$-g''(q_1) \;=\; \sqrt{y^2-1} \times (y - \sqrt{y^2-1}) \;=\; \frac{\sqrt{y^2-1}}{y+\sqrt{y^2-1}} \;=\; \frac{\sqrt{y^2}-1}{|q_1|} \tag{S.55}$$

while

$$f(q_1) \;=\; \frac{1}{\sqrt{q_1}} \;=\; \frac{e^{-i\pi/4}}{|q_1|^{1/2}}, \tag{S.56}$$

hence

$$\frac{\eta f(q_1)}{\sqrt{-\eta^2 g''(q_1,y)}} \;=\; \frac{e^{-i\pi/4}}{(1-y^2)^{1/4}}\,. \tag{S.57}$$

At the same time

$$\exp\bigl((2n+1) \times g(q_1,y)\bigr) \;=\; e^{-(2n+1)/4)}\, e^{i\pi(2n+1)/4} \times \exp\Bigl(-(n+\tfrac{1}{2}) \times \bigl(y\sqrt{y^2-1} - \operatorname{arcosh} y\bigr)\Bigr), \tag{S.58}$$

so altogether

$$I \;\underset{n\to\infty}{\longrightarrow}\; \frac{D}{(1-y^2)^{1/4}} \times \exp\Bigl(-(n+\tfrac{1}{2}) \times \bigl(y\sqrt{y^2-1} \;-\; \operatorname{arcosh} y\bigr)\Bigr) \tag{S.59}$$

where

$$D \;=\; \sqrt{\frac{2\pi}{2n+1}} \times e^{-(2n+1)/4}\, e^{i\pi n/2}, \tag{S.60}$$

exactly as in eq. (S.44) for the allowed region of y. Therefore, in the forbidden region $y > +1$

$$\begin{aligned}\Psi_n^{\rm true}(y) &= \text{const} \times I(y) \\ &\underset{n\to\infty}{\longrightarrow} \frac{C}{(1-y^2)^{1/4}} \times \exp\Big(-(n+\tfrac{1}{2}) \times \big(y\sqrt{y^2-1} - \operatorname{arcosh} y\big)\Big) \\ &= \Psi_n^{WKB}(y) \quad \langle\!\langle\, \textit{cf.}\text{ eq. (S.14)} \,\rangle\!\rangle,\end{aligned} \tag{S.61}$$

and even the overall coefficient C is the same as in the allowed region. Thus,

$$\text{for } n \to \infty, \quad \Psi_n^{\rm true}(y) \longrightarrow \Psi_n^{WKB}(y) \quad \text{for all regions of } y, \text{ allowed or forbidden}, \tag{S.62}$$

quod erat demonstrandum.

Problem **3**(a):
For the effective potential (7) and negative energies $E < 0$, we have

$$V_{\rm eff}(r) - E = \frac{\mathbf{L}^2}{2mr^2} - \frac{Ze^2}{r} - E = \frac{-E}{r^2}\left(r^2 - \frac{Ze^2}{(-E)} \times r + \frac{\mathbf{L}^2}{2m(-E)}\right). \tag{S.63}$$

At the radial turning points r_1 and r_2, — which correspond to the pericenter and the apocenter of an elliptic orbit, — the quadratic polynomial inside $(\cdots)$ must vanish, so the turning points are the roots

$$r_{1,2} = \frac{Ze^2}{2(-E)} \mp \sqrt{\left(\frac{Ze^2}{2(-E)}\right) - \frac{\mathbf{L}^2}{2m(-E)}}. \tag{S.64}$$

Problem **3**(b):
Given the roots of a quadratic polynomial $ax^2 + bx + c$, the polynomial itself can be written as $a(x - x_1)(x - x_2)$. For the quadratic equation (S.63), this means

$$V_{\rm eff}(r) - E = \frac{-E}{r^2}(r - r_1)(r - r_2). \tag{S.65}$$

Consequently, between the classical turning radii r_1 and r_2, the electron has classical radial momentum[2]

$$p_r^2 \;=\; 2m\times(E-V_{\rm eff}(r)) \;=\; +2mE\times(r_1-r)(r_2-r)-r^2 \;=\; 2m(-E)\times\frac{(r-r_1)(r_2-r)}{r^2}\,, \tag{S.66}$$

hence

$$p_r \;=\; \pm\sqrt{2m(-E)}\times\frac{\sqrt{(r-r_1)(r_2-r)}}{r}\,. \tag{S.67}$$

Therefore, the bounce action of the radial motion over a complete period is

$$\begin{aligned} S \;&=\; \oint p_r\,dr \;=\; 2\int\limits_{r_1}^{r_2}\sqrt{2m(-E)}\times\frac{\sqrt{(r-r_1)(r_2-r)}}{r}\,dr \\ &\langle\!\langle \text{ using eq. (8) } \rangle\!\rangle \\ &=\; 2\sqrt{2m(-E)}\times\frac{\pi}{2}\Bigl(r_1\,+\,r_2\,-\,2\sqrt{r_1r_2}\Bigr) \end{aligned} \tag{S.68}$$

where

$$r_1\,+\,r_2 \;=\; \frac{Ze^2}{(-E)}\,, \tag{S.69}$$

$$r_1\times r_2 \;=\; \frac{\mathbf{L}^2}{2m(-E)}\,, \tag{S.70}$$

hence

$$\begin{aligned} S \;&=\; \pi\sqrt{2m(-E)}\times\left(\frac{Ze^2}{(-E)}\;-\;2\sqrt{\frac{\mathbf{L}^2}{2m(-E)}}\right) \\ &=\; 2\pi\left(Ze^2\sqrt{\frac{m}{2(-E)}}\;-\;\sqrt{\mathbf{L}^2}\right). \end{aligned} \tag{S.71}$$

Problem **3**(c):

By the (corrected) Bohr–Sommerfeld quantization rule, the bounce action (S.71) should be a half-integer multiple of the Planck constant as in eq. (6) Thus, for the hydrogen-like atom at hand, we need

$$2\pi\left(Ze^2\sqrt{\frac{m}{2(-E)}}\;-\;\sqrt{\mathbf{L}^2}\right) \;=\; 2\pi\hbar(n_r+\tfrac12),\quad n_r=0,1,2,3,\ldots, \tag{S.72}$$

and hence

$$Ze^2\sqrt{\frac{m}{2(-E)}} \;=\; \hbar(n_r+\tfrac{1}{2}) \;+\; \sqrt{\mathbf{L}^2}\,. \tag{S.73}$$

Next, we approximate

$$\mathbf{L}^2 \;=\; \hbar^2\ell(\ell+1) \quad \text{for } \ell=0,1,2,\ldots \quad \text{as}\mathbf{L}^2 \;=\; \hbar^2(\ell+\tfrac{1}{2})^2, \tag{9}$$

so eq. (S.73) becomes

$$Ze^2\sqrt{\frac{m}{2(-E)}} \;=\; \hbar(n_r+\tfrac{1}{2}) \;+\; \hbar(\ell+\tfrac{1}{2}) \;=\; \hbar(n_r+\ell+1) \;=\hbar N \tag{S.74}$$

where

$$N \;=\; n_r \;+\; \ell \;+\; 1 \;=\; 1,2,3,\ldots \tag{S.75}$$

is the principal quantum number of a hydrogen-like atom. Finally, solving eq. (S.74) for the bond state energy E, we get

$$-E \;=\; \frac{mZ^2e^4}{2N^2}\,, \tag{S.76}$$

which is precisely the right formula you should remember from your undergraduate QM class.