Cabibbo–Kobayashi–Maskawa Matrix of Flavor Mixing

History:

By 1950s, physicists have noted that the Fermi constant G_F inferred from the β–decays of nuclei is a couple of percent smaller than the G_F inferred from the muon decay. At the same time, a bunch of *strange* particles were discovered in cosmic rays and accelerator labs; these particles were created by the strong interactions but decayed only by the weak interactions, hence the name "strange". Moreover, the effective Fermi constant responsible for the strange particle decays was about $4\frac{1}{2}$ times weaker than the regular G_F responsible for the nuclear β–decays or the pion decays.

Eventually, people noted the similarity of strong interactions between the strange and non-strange hadrons, and in 1961 Murray Gell–Mann and Yuval Ne'eman discovered an approximate $SU(3)$ symmetry between them. Nowadays, this symmetry is called the $SU(3)_{\rm flavor}$ to distinguish from the exact $SU(3)_{\rm color}$ symmetry of QCD. And in 1963, Nicola Cabibbo used the $SU(3)_{\rm flavor}$ symmetry to establish the universality of the weak interactions between all kinds of particles, leptons or hadrons, strange or non-strange. Specifically, in the effective low-energy theory

$$\mathcal{L} \;=\; -2\sqrt{2}G_F \times J^+_\mu J^{\mu-} \tag{1}$$

the Fermi constant is the same for all the weak interactions, while

$$\begin{aligned} J^+_\mu \;&=\; \overline{\Psi}(e)\gamma_\mu\frac{1-\gamma^5}{2}\Psi(\nu_e) \;+\; \overline{\Psi}(\mu)\gamma_\mu\frac{1-\gamma^5}{2}\Psi(\nu_\mu) \;+\; J^+_\mu(\text{hadrons}),\\ J^-_\mu \;&=\; \overline{\Psi}(\nu_e)\gamma_\mu\frac{1-\gamma^5}{2}\Psi(e) \;+\; \overline{\Psi}(\nu_\mu)\gamma_\mu\frac{1-\gamma^5}{2}\Psi(\mu) \;+\; J^-_\mu(\text{hadrons}) \end{aligned} \tag{2}$$

where the hadronic $J^\pm_\mu$ are the $SU(3)$ — or rather the chiral $SU(3)_L$ — symmetry currents corresponding to the mixed generators

$$T^\pm \;=\; \cos\theta\bigl(T^1 \mp iT^2\bigr) \;+\sin\theta\bigl(T^4 \mp iT^5\bigr). \tag{3}$$

In this formula, the T^1 and the T^2 are isospin generators, the T^4 and the T^5 generators mix strange and non-strange hadrons with each other, and $\theta \approx 13^\circ$ is a small mixing angle nowadays called *the Cabibbo angle* θ_c to distinguish from the Weinberg's mixing angle θ_w of the GWS electroweak theory. In terms of the *quark model* — invented by Murray Gell–Mann and George Zweig in 1964 — the hadronic weak currents are

$$J_\mu^\pm(\text{hadrons}) \;=\; \sum_{i,j=u,d,s}^{\text{quarks}} \overline{\Psi}_i\gamma_\mu \frac{1-\gamma^5}{2}\left(\frac{\lambda^\pm}{2}\right)^i_j \Psi^j \tag{4}$$

where

$$\frac{\lambda^+}{2} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ \cos\theta_c & 0 & 0 \\ \sin\theta_c & 0 & 0 \end{pmatrix}, \qquad \frac{\lambda^-}{2} \;=\; \begin{pmatrix} 0 & \cos\theta_c & \sin\theta_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

In the $SU(2)_w \times U(1)_y$ terms, the Cabibbo models means that the quark eigenstates of the electroweak quantum numbers are different from the mass eigenstates: Let u, d, and s denote the quark flavors of definite mass, then the $SU(2)$ doublet comprises $\Psi(u)$ and

$$\Psi(d') \;=\; \cos\theta_c\Psi(d) \;+\; \sin\theta_c\Psi(s) \tag{6}$$

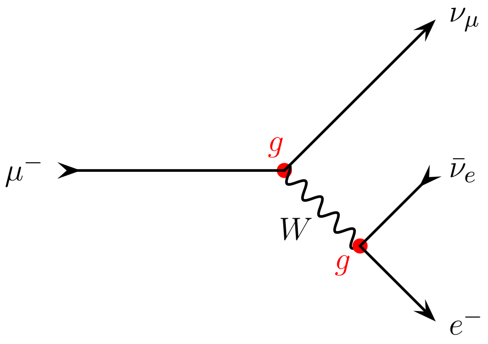
rather than $\Psi(d)$, while the orthogonal combination

$$\Psi(s') \;=\; -\sin\theta_c\Psi(d) \;+\; \cos\theta_c\Psi(s) \tag{7}$$

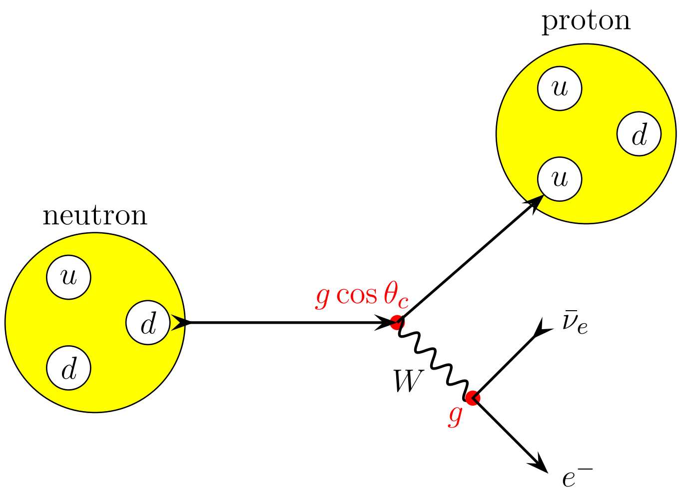
is an $SU(2)$ singlet. Or in chiral terms:

$$\begin{aligned} (u,d')_L &\text{ is an } SU(2)_W \text{ doublet with } Y = +\tfrac{1}{6}, \\ s'_L &\text{ is an } SU(2)_W \text{ singlet with } Y = -\tfrac{1}{3}, \\ u_R &\text{ is an } SU(2)_W \text{ singlet with } Y = +\tfrac{2}{3}, \\ d'_R &\text{ is an } SU(2)_W \text{ singlet with } Y = -\tfrac{1}{3}, \\ s'_R &\text{ is an } SU(2)_W \text{ singlet with } Y = -\tfrac{1}{3}. \end{aligned} \tag{8}$$

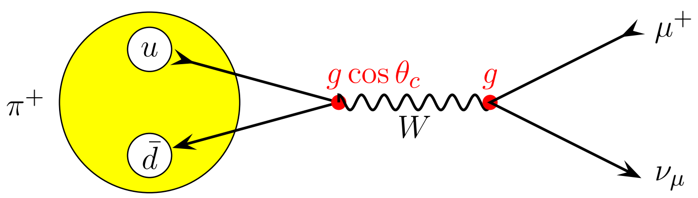
The Cabibbo theory explains the relative strengths of weak interactions involving the leptons, the non-strange hadrons, and the strange hadrons. For examples, compare muon decay $\mu^- \to \nu_\mu + e^- + \bar{\nu}_e$, neutron β-decay $n^0 \to p^+ + e^- + \bar{\nu}_e$, pion decay $\pi^+ \to \mu^+ + \bar{\nu}_\mu$, and kaon decay $K^+ \to \mu^+ + \bar{\nu}_\mu$:



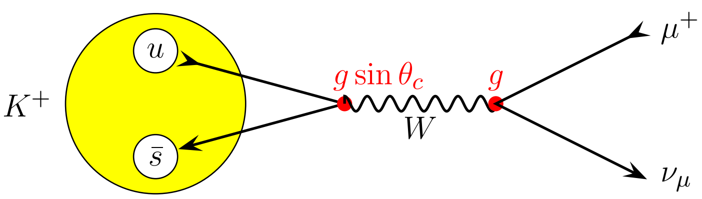
$$\mathcal{M} \propto G_F\,, \tag{9}$$



$$\mathcal{M} \propto G_F \times \cos\theta_c\,, \tag{10}$$

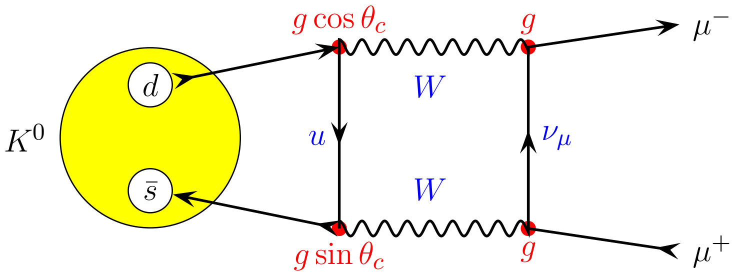


$$\mathcal{M} \propto G_F \times \cos\theta_c$$



$$\mathcal{M} \propto G_F \times \sin\theta_c$$

Unfortunately for the Cabibbo model, it predicts unrealistically high decay rates of neutral kaons to $\mu^+\mu^-$ pairs. By late 1960s, the branching ratio for this decay mode was known to be less then $2{\cdot}10^{-7}$ (the current particle data book gives $B(K_0 \to \mu^+\mu^-) \approx 7{\cdot}10^{-9}$), but the Cabibbo model allows this decay at the one-loop level:



(11)

which yields amplitude

$$\mathcal{M}(K^0 \to \mu^+\mu^-) \;\propto\; \frac{\alpha_2}{\pi} \times G_F \times \sin\theta_c \cos\theta_c \tag{12}$$

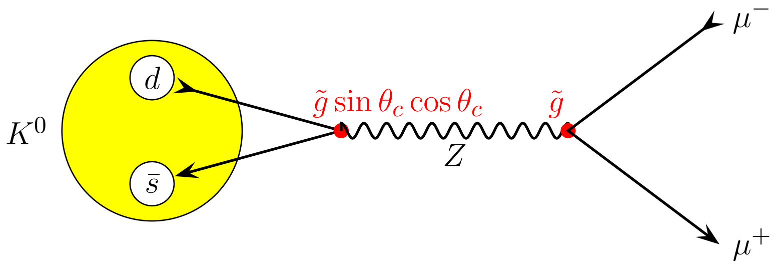
and hence branching ratio

$$B(K^0 \to \mu^+\mu^-) \;\sim\; \frac{\alpha_2}{\pi} \;\sim\; 10^{-2}. \tag{13}$$

Worse, in the $SU(2)\times U(1)$ Glashow model, mixing of the d' and s' quarks with different $SU(2)$ quantum numbers leads to the *flavor changing terms in the neutral current*

$$\begin{aligned}
J_Z^\mu \;=\; & J_Z^\mu(\text{leptons}) \;+\; \overline{\Psi}(u)\gamma^\mu\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta_w\right)\Psi(u) \\
& + \overline{\Psi}(d')\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta_w\right)\Psi(d') \;+\; \overline{\Psi}(s')\gamma^\mu\left(0 + \frac{1}{3}\sin^2\theta_w\right)\Psi(s') \\
=\; & J_Z^\mu(\text{leptons}) \;+\; \overline{\Psi}(u)\gamma^\mu\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta_w\right)\Psi(u) \\
& + \overline{\Psi}(d)\gamma^\mu\left(-\cos^2\theta_c\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta_w\right)\Psi(d) \\
& + \overline{\Psi}(s)\gamma^\mu\left(-\sin^2\theta_c\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta_w\right)\Psi(s) \\
& + \overline{\Psi}(s)\gamma^\mu\left(-\sin\theta_c\cos\theta_c\frac{1-\gamma^5}{4}\right)\Psi(d) \;+\; \overline{\Psi}(d)\gamma^\mu\left(-\sin\theta_c\cos\theta_c\frac{1-\gamma^5}{4}\right)\Psi(s).
\end{aligned} \tag{14}$$

Consequently, there is a tree diagram for the $K^0 \to \mu^+\mu^-$ decay:



$$\mathcal{M} \;\propto\; G_F \times \sin\theta_c \cos\theta_c \tag{15}$$

and hence $O(1)$ branching ratio.

To avoid both of these problems, Sheldon Glashow, John Iliopoulos, and Luciano Maiani proposed in 1970 that the s' quark (or rather the LH s' quark) should be a member of an $SU(2)_W$ doublet $(c, s')_L$ rather than a singlet. Consequently, there should be a fourth quark flavor c — which they called *charm* — whose mass they estimated as $m_c = 1$–2 GeV; this flavor — or rather any hadrons containing this quark flavor — were not known in 1970, but was soon discovered in 1974 by Sam Ting's group at BNAL and Burton Richter's group at SLAC. Altogether, Glashow, Iliopoulos, and Maiani — collectively called GIM — had two similar families of quarks:

$$\begin{aligned} (u, d')_L \text{ and } (c, s') \ \text{ are } SU(2)_W \text{ doublets } \ \text{with } Y &= +\tfrac{1}{6}, \\ u_R \text{ and } c_R \ \text{ are } SU(2)_W \text{ singlets } \ \text{with } Y &= +\tfrac{2}{3}, \\ d'_R \text{ and } s'_R \ \text{ are } SU(2)_W \text{ singlets } \ \text{with } Y &= -\tfrac{1}{3}, \end{aligned} \tag{16}$$

just like the two similar families of leptons known back then,

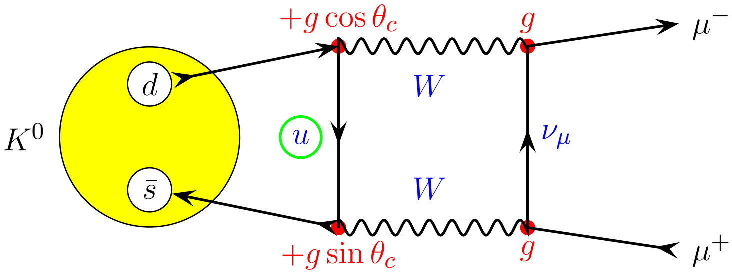
$$\begin{aligned} (\nu_e, e^-)_L \text{ and } (\nu_\mu, \mu^-) \ \text{ are } SU(2)_W \text{ doublets } \ \text{with } Y &= -\tfrac{1}{2}, \\ e^-_R \text{ and } \mu^-_R \ \text{ are } SU(2)_W \text{ singlets } \ \text{with } Y &= -1. \end{aligned} \tag{17}$$

Consequently, despite the mass eigenstates d and s being mixtures of the d' and s' electroweak eigenstates, the neutral current remains diagonal in both (d', s') and (d, s) bases,

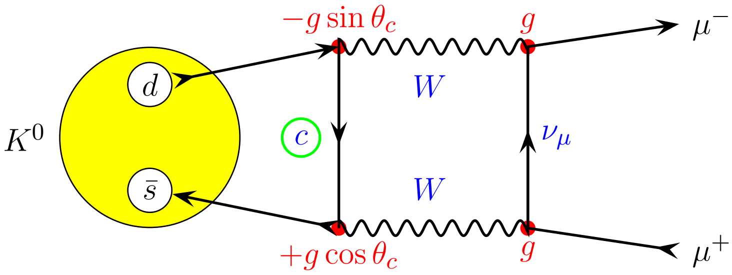
$$\begin{aligned} J^\mu_Z(d, s) &= \sum_{q'=d',s'} \overline{\Psi}(q')\gamma^\mu \left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta_w\right)\Psi(q') \\ &= \sum_{q=d,s} \overline{\Psi}(q)\gamma^\mu \left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta_w\right)\Psi(q) \end{aligned} \tag{18}$$

so there are no flavor-changing neutral currents. This immediately eliminates the tree-level $K^0 \to \mu^+\mu^-$ decay.

As to the one-loop diagram (11) for this decay, adding the c quark to the theory adds another diagram with a c quark in the propagator instead of u. Altogether,



$$\mathcal{M} \;=\; +\cos\theta_c\sin\theta_c\times\text{common},$$



$$\mathcal{M} \;=\; -\sin\theta_c\cos\theta_c\times\text{common}, \tag{19}$$

so the net decay amplitude cancels out. This cancellation is knows as the *GIM mechanism* after its authors.

The GIM mechanism can be easily generalized to electroweak models with more than two families of quarks and leptons, as long as each family has exactly the same $SU(2)_W\times U(1)_Y$ quantum numbers. Moreover, in 1973 Makoto Kobayashi and Toshihide Maskawa found that the flavor mixing in the 3–flavor model can break the CP symmetry of the theory. The CP violation was experimentally discovered in 1964 by James Cronin and Val Fitch, and there were great may theories trying to explain it, but the Kobayashi–Maskawa theory had the advantage of not introducing any new interactions, just a new family of particles subject to the well-known strong and electroweak interactions. Lo and behold, this third family was soon experimentally discovered: the τ lepton in 1975, the *bottom* b quark in 1977, the *top* t quark in 1995, and finally the τ-type neutrino ν_τ in 2000. Also, up until 2001 the CP violation was seen only in the neutral kaons, but then it was also seen in the neutral B-mesons, and all the data was consistent with the Kobayashi–Maskawa theory. So in 2008, Makoto Kobayashi and Toshihide Maskawa got Nobel prises for their theory, — or rather shared $\frac{1}{2}$ of the Nobel prise, the other $\frac{1}{2}$ going to Yoichiro Nambu

Present Day

At present (2022), the Standard Model appears to have 3 families of quarks and leptons and no other fermions. So let's work out the 3-family flavor mixing in some detail. Let the un-primed quark flavors u, d, s, c, b, t be the eigenstates of the quark mass matrix. In terms of these flavors, the 3 $SU(2)_W$ doublets comprise

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L, \tag{20}$$

for

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \times \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{21}$$

where

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{22}$$

is a 3×3 unitary matrix called the CKM matrix after Cabibbo, Kobayashi, and Maskawa. The CKM matrix has 9 parameters — 3 real angles and 6 complex phases — but we may eliminate 5 of the phases by adjusting the relative phases of the basic flavors,

$$\Psi(u) \to e^{i\theta_u}\Psi(u), \quad \Psi(c) \to e^{i\theta_c}\Psi(c), \quad \Psi(t) \to e^{i\theta_t}\Psi(t), \tag{23}$$

and likewise

$$\Psi(d') \to e^{i\theta_u}\Psi(d'), \quad \Psi(s') \to e^{i\theta_c}\Psi(s'), \quad \Psi(b') \to e^{i\theta_t}\Psi(b'), \tag{24}$$

but

$$\Psi(d) \to e^{i\theta_d}\Psi(d), \quad \Psi(s) \to e^{i\theta_s}\Psi(s), \quad \Psi(b) \to e^{i\theta_b}\Psi(b), \tag{25}$$

hence

$$V_{f'f} \to \exp(i\theta_{f'} - i\theta_f) V_{f'f}\,. \tag{26}$$

The standard parametrization of the remaining 4 parameters of the CKM matrix is

$$\mathrm{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \times \begin{pmatrix} \cos\theta_{13} & 0 & e^{-i\delta}\sin\theta_{13} \\ 0 & 1 & 0 \\ -e^{+i\delta}\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \times \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

According to the 2022 particle data book, the best fit to all the current experimental data is

$$\begin{aligned} \theta_{12} &= 13.00^\circ \pm 0.04^\circ, \\ \theta_{23} &= 2.40^\circ \pm 0.05^\circ, \\ \theta_{13} &= 0.211^\circ \pm 0.006^\circ, \\ \delta &= 65.5^\circ \pm 1.5^\circ. \end{aligned} \tag{28}$$

Since the CKM matrix mixes flavors with exactly the same T^3 and Y quantum numbers — and hence exactly the same couplings

$$\tilde{g} \times \left(T^3 - Q\sin^2\theta_w\right) \tag{29}$$

to the Z^0 vector particle, — changing the quark flavor basis to the mass eigenstates does not result in any flavor-changing terms in the neutral weak current:

$$\begin{aligned} J_Z^\mu(\text{quarks}) &= \sum_{q=u,c,t} \overline{\Psi}^q\gamma^\mu\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta_w\right)\Psi^q \\ &\quad + \sum_{q'=d',s',b'} \overline{\Psi}^{q'}\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta_w\right)\Psi^{q'} \\ &= \sum_{q=u,c,t} \overline{\Psi}^q\gamma^\mu\left(+\frac{1-\gamma^5}{4} - \frac{2}{3}\sin^2\theta_w\right)\Psi^q \\ &\quad + \sum_{q=d,s,b} \overline{\Psi}^q\gamma^\mu\left(-\frac{1-\gamma^5}{4} + \frac{1}{3}\sin^2\theta_w\right)\Psi^q, \end{aligned} \tag{30}$$

exactly as in eq. (48) of my previous set of notes.

On the other hand, the charged weak currents of the quarks do get all kinds of off-diagonal terms in the basis of mass eigenstates. Indeed, in the electroweak basis

$$J^-_\mu(\text{quarks}) \;=\; \sum_{U=u,c,t} \overline{\Psi}(U)\gamma_\mu \frac{1-\gamma^5}{2} \Psi(\text{the } SU(2) \text{ partner of } U) \tag{31}$$

where

$$\Psi(\text{the } SU(2) \text{ partner of } U) \;=\; \sum_{D=d,s,b} V_{UD} \times \Psi(D), \tag{32}$$

thus

$$J^-_\mu(\text{quarks}) \;=\; \sum_{U=u,c,t} \sum_{D=d,s,b} V_{UD} \times \overline{\Psi}(U)\gamma_\mu \frac{1-\gamma^2}{2} \Psi(D). \tag{33}$$

Likewise,

$$J^+_\mu(\text{quarks}) \;=\; \sum_{U=u,c,t} \overline{\Psi}(\text{the } SU(2) \text{ partner of } U)\gamma_\mu\gamma_\mu \frac{1-\gamma^2}{2} \Psi(U) \tag{34}$$

where

$$\overline{\Psi}(\text{the } SU(2) \text{ partner of } U) \;=\; \sum_{D=d,s,b} V^*_{UD} \times \overline{\Psi}(D), \tag{35}$$

thus

$$J^+_\mu(\text{quarks}) \;=\; \sum_{U=u,c,t} \sum_{D=d,s,b} V^*_{UD} \times \overline{\Psi}(D)\gamma_\mu \frac{1-\gamma^2}{2} \Psi(U). \tag{36}$$

The off-diagonal terms in the CKM matrix are very important, as they allow the strange and the b-flavored hadrons to decay into lighter particles. Indeed, a weak decay of a b-flavored particle involves an electroweak transition from the b quark to a charge $+\frac{2}{3}$ quark u, c, or t and a virtual W^- (which then splits into a pair of light quarks or leptons). But the top quark is heavier than the bottom quark, so the only allowed decays are $b \to c$ and $b \to u$, which involve the off-diagonal CKM matrix elements V_{ub} and V_{cb}, thus

$$\Gamma(b \to \text{anything}) \;\propto\; |V_{ub}|^2 \;+\; |V_{cb}|^2. \tag{37}$$

Without these off-diagonal matrix elements, the lightest b-flavored mesons and baryons would be stable.

Likewise, the only allowed weak decay channel of the s quark is to a virtual W^- and a u quark, because the other two charge $+\frac{2}{3}$ flavors are heavier than s. Consequently,

$$\Gamma(s \to \text{anything}) \ \propto \ |V_{us}|^2, \tag{38}$$

and without this off-diagonal matrix element, the strange quark — and hence some strange mesons and baryons — would be stable.

I wish I could explore the phenomenological consequences of the CKM matrix in some detail in this class, but alas the time is too short and the semester is almost finished. Instead, let me simply recommend you take the Particle Phenomenology class Professor Can Kilic is going to teach in Fall 2023.

Yukawa Couplings and the Origin of the CKM Matrix.

In the un-broken $SU(2) \times U(1)$ theory the quarks are massless and we cannot tell which quark is u, which is c, *etc.*, *etc.*; we cannot even tell which left-handed Weyl field pairs up with which right-handed Weyl field into a Dirac spinor. We can use the $SU(2)$ symmetry to form doublets, but we are free to choose any basis we like for the 3 doublets — let's call them Q_α for $\alpha = 1,2,3$ — and we are free to change this basis by a unitary field re-definition,

$$\psi^i_L(Q_\alpha) \ \to \ \psi^i_L(Q'_\alpha) \ = \sum_\beta (\mathcal{U}^Q)_{\alpha,\beta} \times \psi^i_L(Q_\beta), \tag{39}$$

where $\mathcal{U}^Q$ is a unitary 3×3 matrix. Similarly, we may use any basis D_α for the 3 right-handed quarks of charge $-\frac{1}{3}$, any basis U_α for the 3 right-handed quarks of charge $+\frac{2}{3}$, and we are free to change these two bases by unitary transforms

$$\begin{aligned} \psi_R(U_\alpha) \ \to \ \psi_R(U'_\alpha) \ &= \sum_\beta (\mathcal{U}^U)_{\alpha,\beta} \times \psi_R(U_\beta), \\ \psi_R(D_\alpha) \ \to \ \psi_R(D'_\alpha) \ &= \sum_\beta (\mathcal{U}^D)_{\alpha,\beta} \times \psi_R(D_\beta), \end{aligned} \tag{40}$$

where $\mathcal{U}^U$ and $\mathcal{U}^D$ are two independent unitary 3×3 matrices. However, we cannot mix the U_α with the D_α because of their different $U(1)$ hypercharges.

Likewise, we are free to use any basis L_α for the 3 doublets of left-handed leptons, any basis E_α for the 3 right-handed charged leptons, and we are free to change these bases by unitary transforms,

$$\begin{aligned}
\psi_L^i(L_\alpha) \;\to\; \psi_L^i(L'_\alpha) &= \sum_\beta (\mathcal{U}^L)_{\alpha,\beta} \times \psi_L^i(L_\beta), \\
\psi_R(E_\alpha) \;\to\; \psi_R(E'_\alpha) &= \sum_\beta (\mathcal{U}^E)_{\alpha,\beta} \times \psi_R(E_\beta).
\end{aligned} \tag{41}$$

(I'll take care of the neutrinos in my notes on neutrino masses.)

The Yukawa couplings involve one Higgs field H^i or H_i^* and two fermion fields, — one left-handed, one right-handed — and for each choice of their $SU(2)\times U(1)$ quantum numbers, there are three ψ_L fields and three ψ_R fields. Consequently, there is a big lot of the Yukawa terms in the Lagrangian, namely

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} = &-\sum_{\alpha,\beta} Y^U_{\alpha\beta} \times \psi_R^\dagger(U_\alpha)\psi_L^i(Q_\beta) \times \epsilon_{ij}H^j \;-\; \sum_{\alpha,\beta} Y^D_{\alpha\beta} \times \psi_R^\dagger(D_\alpha)\psi_L^i(Q_\beta) \times H_i^* \\
&-\sum_{\alpha,\beta} Y^E_{\alpha\beta} \times \psi_R^\dagger(E_\alpha)\psi_L^i(L_\beta) \times H_i^* \;+\; \text{Hermitian conjugates},
\end{aligned} \tag{42}$$

where the $Y^U_{\alpha,\beta}$, the $Y^D_{\alpha,\beta}$, and the $Y^E_{\alpha,\beta}$ comprise three 3×3 complex matrices of the Yukawa coupling constants. And when the Higgs develops symmetry-breaking VEV, these matrices of Yukawa couplings give rise to the complex 3×3 *mass matrices*

$$M^U_{\alpha,\beta} = \frac{v}{\sqrt{2}} \times Y^U_{\alpha,\beta}\,, \qquad M^U_{\alpha,\beta} = \frac{v}{\sqrt{2}} \times Y^D_{\alpha,\beta}\,, \qquad M^E_{\alpha,\beta} = \frac{v}{\sqrt{2}} \times Y^E_{\alpha,\beta}\,, \tag{43}$$

$$\mathcal{L}_{\text{mass}} = -\sum_{\alpha,\beta} M^U_{\alpha\beta} \times \psi_R^\dagger(U_\alpha)\psi_L^1(Q_\beta) \;-\; \sum_{\alpha,\beta} M^D_{\alpha\beta} \times \psi_R^\dagger(D_\alpha)\psi_L^2(Q_\beta) \tag{44}$$

$$-\sum_{\alpha,\beta} M^E_{\alpha\beta} \times \psi_R^\dagger(E_\alpha)\psi_L^2(L_\beta) \;+\; \text{Hermitian conjugates.} \tag{45}$$

To get the physical masses of quarks and leptons, we need to diagonalize these mass matrices via suitable unitary transforms (39)–(41). In matrix notations, these transforms lead to

$$(Y^U)' = \mathcal{U}^U \times Y^U \times (\mathcal{U}^Q)^\dagger, \quad (Y^D)' = \mathcal{U}^D \times Y^D \times (\mathcal{U}^Q)^\dagger, \quad (Y^E)' = \mathcal{U}^E \times Y^E \times (\mathcal{U}^L)^\dagger, \tag{46}$$

and consequently

$$(M^U)' = \mathcal{U}^U \times M^U \times (\mathcal{U}^Q)^\dagger, \quad (M^D)' = \mathcal{U}^D \times M^D \times (\mathcal{U}^Q)^\dagger, \quad (M^E)' = \mathcal{U}^E \times M^E \times (\mathcal{U}^L)^\dagger. \tag{47}$$

Now, any complex matrix M can be written as a product $M = W_1 D W_2$ where W_1 and W_2 are unitary matrices while D is diagonal, real, and non-negative.⋆ Consequently, using appropriate unitary matrices $\mathcal{U}^E$ and $\mathcal{U}^Q$ we can make the charged leptons' mass matrix diagonal and real

$$M^E \to (M^E)' = \mathcal{U}^E \times M^E \times (\mathcal{U}^L)^\dagger = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{48}$$

Note that it is in the transformed bases — where the $(M^E)'$ matrix is diagonal — that the LH and the RH Weyl fields combine into Dirac fields of the physical electron, muon, and tau fields

$$\begin{aligned} \Psi_e &= \begin{pmatrix} \psi_L^2(L_1') = \mathcal{U}^L_{1\beta}\psi_L^2(L_\beta) \\ \psi_R(E_1') = \mathcal{U}^E_{1\beta}\psi_R(E_\beta) \end{pmatrix}, \\ \Psi_\mu &= \begin{pmatrix} \psi_L^2(L_2') = \mathcal{U}^L_{2\beta}\psi_L^2(L_\beta) \\ \psi_R(E_2') = \mathcal{U}^E_{2\beta}\psi_R(E_\beta) \end{pmatrix}, \\ \Psi_\tau &= \begin{pmatrix} \psi_L^2(L_3') = \mathcal{U}^L_{3\beta}\psi_L^2(L_\beta) \\ \psi_R(E_3') = \mathcal{U}^E_{3\beta}\psi_R(E_\beta) \end{pmatrix}. \end{aligned} \tag{49}$$

Likewise, using the $\mathcal{U}^U$ and the $\mathcal{U}^Q$ unitary matrices we may diagonalize the mass matrix

⋆ To prove, start with a polar decomposition $M = UH$ where U is unitary and $H = \sqrt{M^\dagger M}$ is hermitian and positive semi-definite. Then diagonalize the hermitian matrix H, *i.e.*, write it as $H = W^\dagger D W$ for some unitary matrix W. Consequently, $M = UW^\dagger DW = W_1 D W_2$ for $W_2 = W$ and $W_1 = UW^\dagger$.

for the charge $+\frac{2}{3}$ quarks,

$$\begin{aligned}
M^U \ \rightarrow\ \left(M^U\right)' \ &=\ \mathcal{U}^U \times M^U \times \left(\mathcal{U}^Q\right)^\dagger \ =\ \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \\
\Psi_u \ &=\ \begin{pmatrix} \psi^1_L(Q'_1) = \mathcal{U}^Q{}_{1\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_1) = \mathcal{U}^U_{1\beta}\psi_R(U_\beta) \end{pmatrix}, \\
\Psi_c \ &=\ \begin{pmatrix} \psi^1_L(Q'_2) = \mathcal{U}^Q{}_{2\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_2) = \mathcal{U}^U_{2\beta}\psi_R(U_\beta) \end{pmatrix}, \\
\Psi_t \ &=\ \begin{pmatrix} \psi^1_L(Q'_3) = \mathcal{U}^Q{}_{3\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_3) = \mathcal{U}^U_{3\beta}\psi_R(U_\beta) \end{pmatrix},
\end{aligned} \tag{50}$$

and similarly for the charge $-\frac{1}{3}$ quarks,

$$\begin{aligned}
M^D \ \rightarrow\ \left(M^D\right)' \ &=\ \mathcal{U}^D \times M^D \times \left(\widetilde{\mathcal{U}}^Q\right)^\dagger \ =\ \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \\
\Psi_d \ &=\ \begin{pmatrix} \psi^2_L(Q'_1) = \widetilde{\mathcal{U}}^Q{}_{1\beta}\psi^2_L(Q_\beta) \\ \psi_R(U'_1) = \mathcal{U}^D_{1\beta}\psi_R(D_\beta) \end{pmatrix}, \\
\Psi_s \ &=\ \begin{pmatrix} \psi^2_L(Q'_2) = \widetilde{\mathcal{U}}^Q{}_{2\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_2) = \mathcal{U}^D_{2\beta}\psi_R(D_\beta) \end{pmatrix}, \\
\Psi_b \ &=\ \begin{pmatrix} \psi^2_L(Q'_3) = \widetilde{\mathcal{U}}^Q{}_{3\beta}\psi^1_L(Q_\beta) \\ \psi_R(U'_3) = \mathcal{U}^D_{3\beta}\psi_R(D_\beta) \end{pmatrix},
\end{aligned} \tag{51}$$

However, *it takes different unitary matrices $\mathcal{U}^Q \neq \widetilde{\mathcal{U}}^Q$ to diagonalize the up-type and down-type quark mass matrices, and that's what messes up the $SU(2)$ doublet structure!* Indeed, in terms of the upper components $\psi^1_L(Q_\alpha)$ of the original doublets, the left-handed u, c, t quarks of definite mass are linear combinations

$$\begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} = \mathcal{U}^Q \times \begin{pmatrix} \psi^1_L(Q_1) \\ \psi^1_L(Q_2) \\ \psi^1_L(Q_3) \end{pmatrix}, \tag{52}$$

so their $SU(2)$ partners are *similar* linear combinations of the lower components $\psi^2_L(Q_\alpha)$ of the original doublets,

$$\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = \mathcal{U}^Q \times \begin{pmatrix} \psi^2_L(Q_1) \\ \psi^2_L(Q_2) \\ \psi^2_L(Q_3) \end{pmatrix}, \tag{53}$$

for the same $\mathcal{U}^Q$ matrix as the up-type quarks. On the other hand, the d, s, b quarks defined as mass eigenstates obtain from different linear combinations

$$\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} = \widetilde{\mathcal{U}}^Q \times \begin{pmatrix} \psi^2_L(Q_1) \\ \psi^2_L(Q_2) \\ \psi^2_L(Q_3) \end{pmatrix}. \tag{54}$$

Comparing the sets of down-type quark fields, we immediately see that

$$\begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix} = \mathcal{U}^Q \times \widetilde{\mathcal{U}}^{Q\dagger} \times \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}, \tag{55}$$

which gives us the Cabibbo–Kobayashi–Maskawa matrix

$$V_{\text{CKM}} = \mathcal{U}^Q \times \widetilde{\mathcal{U}}^{Q\dagger}. \tag{56}$$

CP Violation

Missing Sections

Since the CP violation was first discovered by Cronin and Fitch in the neutral K-meson sector, I need to tell you a few things about these mesons. the $K^0 \leftrightarrow \overline{K}^0$ mixing and kaon oscillations, the K-long and K-short and their decays, and what the CP symmetry — and its violation — has to do with all of that. Unfortunately, I do not have time to write all that, so I am going to teach this subject using the document camera and the class notes of Professor Mark Thomson from the Cambridge University (pages 408–428).

CKM Matrix and the CP Violation

All the experimentally measured CP-violating effects — not only in the neutral kaons but also in the b-flavored and c-flavored heavy mesons — can be explained by the imaginary part $\mathrm{Im}(V_{U,D})$ of the CKM matrix. In general, the relation between the CP violation and the CKM matrix is quite complicated and involves loop diagrams: At the tree level, there is no CP violation. For example, the CP violation in the neutral kaons stem from the one-loop GIM diagrams, which I shall explain later in this section. But before I get there, let me show you how the imaginary part of the CKM matrix violates the CP symmetry of the electroweak Lagrangian.

First of all, remember that quarks and leptons form chiral multiplets of the the electroweak $SU(2)_W \times U(1)_Y$ gauge theory, so the weak interactions have no semblance of the parity symmetry **P** or the charge conjugation symmetry **C**. In particular, the charged currents involve only the left-chirality Weyl spinors, which in particle terms mean left-helicity quarks and leptons but right-helicity anti-quarks or anti-leptons. However, the chirality is perfectly consistent with the combined **CP** symmetry, which does not mix the ψ_L and the ψ_R fields; instead it acts as

$$\mathbf{CP}: \quad \psi_L(\mathbf{x},t) \ \to \pm\sigma_2\psi_L^*(-\mathbf{x},t), \quad \psi_R(\mathbf{x},t) \ \to \pm\sigma_2\psi_R^*(-\mathbf{x},t). \tag{57}$$

Since the neutral weak current does not care about the CKM matrix, let me focus on the charged currents. Under **CP**, the charged vector fields $W_\mu^\pm(x)$ transform as

$$\mathbf{CP}: \quad W_0^\pm(\mathbf{x},t) \ \to \ -W_0^\mp(-\mathbf{x},+t), \quad \mathbf{W}^\pm(\mathbf{x},t) \ \to \ +\mathbf{W}i^\mp(-\mathbf{x},+t), \tag{58}$$

where the exchange $W^+ \leftrightarrow W^-$ is due to charge conjugation while different signs for the 3-scalar and the 3-vector components are due to reflection $\mathbf{x} \to -\mathbf{x}$ of the space coordinates. Consequently, in a **CP** symmetric theory we would need a similar relation for the charged currents,

$$\mathbf{CP}: \quad J_0^\pm(\mathbf{x},t) \ \to \ -J_0^\mp(-\mathbf{x},+t), \quad \mathbf{J}^\pm(\mathbf{x},t) \ \to \ +\mathbf{J}^\mp(-\mathbf{x},+t), \tag{59}$$

In terms of fermions, the charged weak currents are sums of left-handed current terms of general form

$$j_L^\mu \;=\; \psi_L^{1\dagger}\bar\sigma^\mu\psi_L^2 \;=\; \overline{\Psi}^1\gamma^\mu\frac{1-\gamma^5}{2}\Psi^2 \tag{60}$$

— where $\overline{\Psi}^1$ and Ψ^2 run over appropriate fermion species, — so let's work out how such terms transform under **CP**. Assuming the Weyl fermions ψ_L^1 and ψ_L^2 have same intrinsic CP signs as members of the same $SU(2)$ doublet, we have

$$\begin{aligned}\mathbf{CP}:\ \psi_L^{1\dagger}\bar\sigma^\mu\psi_L^2 \;&\to\; +(\psi_L^1)^\top\sigma_2\times\bar\sigma^\mu\times\sigma_2(\psi_L^2)^*\\ &=\; +(\psi_L^1)^\top\times\left(\sigma_2\bar\sigma^\mu\sigma_2\;=\;(\sigma^\mu)^\top\right)\times(\psi_L^2)^*\\ &=\; -\psi_L^{2\dagger}\sigma^\mu\psi_L^1\\ &=\; \psi_L^{2\dagger}\bar\sigma^\mu\psi_L^1\times\begin{cases}+1 & \text{for } \mu=1,2,3,\\ -1 & \text{for } \mu=0.\end{cases}\end{aligned} \tag{61}$$

The μ dependence of the overall sign here — which comes from comparing $-\sigma^\mu$ to $+\bar\sigma^\mu$ — is in perfect agreement with eq. (59). In Dirac notations, eq (61) amounts to

$$\mathbf{CP}:\ \overline{\Psi}^1\gamma^\mu\frac{1-\gamma^5}{2}\Psi^2\;\to\;\overline{\Psi}^2\gamma^\mu\frac{1-\gamma^5}{2}\Psi^1\times\begin{cases}+1 & \text{for } \mu=1,2,3,\\ -1 & \text{for } \mu=0.\end{cases} \tag{62}$$

Besides the μ–dependent sign, the **CP** exchanges the two fermionic species $\Psi^1\leftrightarrow\Psi^2$ involved in the current j_L^μ. For the leptonic charged weak currents

$$\begin{aligned}J_\mu^+(\text{leptons}) \;&=\; \overline{\Psi}^e\gamma_\mu\frac{1-\gamma^5}{2}\Psi^{\nu_e}\;+\;\overline{\Psi}^\mu\gamma_\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\mu}\;+\;\overline{\Psi}^\tau\gamma_\mu\frac{1-\gamma^5}{2}\Psi^{\nu_\tau},\\ J_\mu^-(\text{leptons}) \;&=\; \overline{\Psi}^{\nu_e}\gamma_\mu\frac{1-\gamma^5}{2}\Psi^{e}\;+\;\overline{\Psi}^{\nu_\mu}\gamma_\mu\frac{1-\gamma^5}{2}\Psi^{\mu}\;+\;\overline{\Psi}^{\nu_\tau}\gamma_\mu\frac{1-\gamma^5}{2}\Psi^{\tau},\end{aligned} \tag{63}$$

the CP action (62) leads to $J_\mu^+\leftrightarrow J_\mu^-$, exactly as in eq. (59); indeed,

$$J_\mu^-\;\supset\;\overline{\Psi}^e\gamma_\mu\frac{1-\gamma^5}{2}\Psi^{\nu_e}\;\longleftrightarrow\;\overline{\Psi}^{\nu_e}\gamma_\mu\frac{1-\gamma^5}{2}\Psi^e\;\subset J_\mu^- \tag{64}$$

and likewise for the muonic and tauonic terms in the charged currents (63). Consequently, the interactions

$$\mathcal{L}\;\supset=\;-\frac{g_2}{\sqrt2}\times\left(W_\mu^+J^{\mu-}_{\text{leptonic}}+W_\mu^-J^{\mu+}_{\text{leptonic}}\right) \tag{65}$$

of the leptons with the vector fields $W_\mu^\pm$ are invariant under **CP**.

But the CP transformation rules of the quarks' charged currents

$$J^{-\mu}(\text{quarks}) \;=\; \sum_{U=u,c,t}\;\sum_{D=d,s,b} V_{U,D}\times\overline{\Psi}^U\gamma^\mu\frac{1-\gamma^5}{2}\Psi^D, \tag{33}$$

$$J^{+\mu}(\text{quarks}) \;=\; \sum_{U=u,c,t}\;\sum_{D=d,s,b} V^*_{U,D}\times\overline{\Psi}^D\gamma^\mu\frac{1-\gamma^5}{2}\Psi^U, \tag{36}$$

are more complicated due to CKM matrix elements $V_{U,D}$. Specifically,

$$\begin{aligned}
&\mathbf{CP}:\; J^{-\mu}(\text{quarks}) \;\to\; \pm(\mu)\times\sum_{U=u,c,t}\;\sum_{D=d,s,b} V_{U,D}\times\overline{\Psi}^D\gamma^\mu\frac{1-\gamma^5}{2}\Psi^U,\\
&\qquad\text{which is almost like}\quad \pm(\mu)\times J^{+\mu}(\text{quarks}),\;\text{except for } V_{U,D} \text{ instead of } V^*_{U,D};\\
&\mathbf{CP}:\; J^{+\mu}(\text{quarks}) \;\to\; \pm(\mu)\times\sum_{U=u,c,t}\;\sum_{D=d,s,b} V^*_{U,D}\times\overline{\Psi}^U\gamma^\mu\frac{1-\gamma^5}{2}\Psi^D,\\
&\qquad\text{which is almost like}\quad \pm(\mu)\times J^{-\mu}(\text{quarks}),\;\text{except for } V^*_{U,D} \text{ instead of } V_{U,D};
\end{aligned} \tag{66}$$

so the net effect of **CP** on the interactions

$$\mathcal{L} \;\supset =\; -\frac{g_2}{\sqrt{2}}\times\Big(W^+_\mu J^{\mu-}_{\text{quark}} + W^-_\mu J^{\mu+}_{\text{quark}}\Big) \tag{67}$$

of the $W^\pm_\mu$ with the quarks is equivalent to complex conjugating the CKM matrix,

$$\mathbf{CP}:\quad V_{U,D} \;\leftrightarrow\; V^*_{U,D}\,. \tag{68}$$

Thus, the weak interactions of quarks (and hence hadrons) are **CP** symmetric if and only if the CKM matrix is real.

Caveat: The specific action of the **CP** symmetry can be modified by changing the phase conventions of the particle and antiparticle states and the corresponding fields. For example, if we change the phase of a Dirac spinor field

$$\Psi(x) \;\to\; \Psi'(x) \;=\; e^{i\theta}\Psi(x), \tag{69}$$

then the **CP** action on that field

$$\mathbf{CP}:\Psi(\mathbf{x},t)\ \to\ \gamma^0\gamma^2\Psi^*(-\mathbf{x},t) \tag{70}$$

becomes

$$\mathbf{CP}:\Psi'(\mathbf{x},t)\ \to\ \gamma^0\gamma^2\Psi'^*(-\mathbf{x},t), \tag{71}$$

which in terms of the original $\Psi(x)$ field becomes

$$\mathbf{CP}:\Psi(\mathbf{x},t)\ \to\ e^{-2i\theta}\gamma^0\gamma^2\Psi^*(-\mathbf{x},t), \tag{72}$$

with an extra phase factor $e^{-2i\theta}$.

In the context of quarks in the GWS theory, redefinitions of the quark fields must keep the quark mass matrices M^U and M^D real and diagonal. Thus, we must preserve the pairings of the LH and RH Weyl spinors into Dirac spinors, but we may multiply each such Dirac spinor by a separate phase factor:

$$\Psi^u\to e^{i\theta_u}\Psi^u,\quad \Psi^c\to e^{i\theta_c}\Psi^c,\quad \Psi^t\to e^{i\theta_t}\Psi^t;\quad \Psi^d\to e^{i\theta_d}\Psi^d,\quad \Psi^s\to e^{i\theta_s}\Psi^s,\quad \Psi^b\to e^{i\theta_b}\Psi^b. \tag{73}$$

Consequently, the matrix elements of the Cabibbo–Kobayashi–Maskawa matrix also change their phases according to

$$V_{U,D}\ \to\ \exp(i\theta_U-i\theta_D)\times V_{U,D}\,. \tag{74}$$

At the same time, the CP symmetry is also redefined to accommodate the new phases of the quark fields. In fact, this redefinition completely parallels the redefined CKM matrix so that in the new basis it is equivalent to complex conjugation of the new CKM matrix,

$$\mathbf{CP}:\ V_{U,D}\ \leftrightarrow\ V^*_{U,D}\,. \tag{75}$$

Therefore, the weak interactions are invariant under *some kind of a* **CP** *symmetry* if and only if the CKM matrix can be made real by a re-phasing (73) of the quark flavors.

For two quark doublets (u,d) and (c,s) — but no (t,b) — this is automatically true. Indeed, for two doublets the CKM matrix is a 2×2 unitary matrix, which may be parametrized by 1 real angle (the Cabibbo angle) and 3 complex phases, for example

$$V = \begin{pmatrix} e^{i(a+b+c)}\cos\theta_c & e^{i(a+b)}\sin\theta_c \\ -e^{i(a+c)}\sin\theta_c & e^{i(a)}\cos\theta_c \end{pmatrix}. \tag{76}$$

At the same time, there are 4 quark flavors whose phases we can change, but since only the differences between the quark phases affect the CKM matrix, we may adjust $4-1=3$ of its complex phases. In particular, we may set $a=b=c=0$ in eq. (76) and get a real matrix

$$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}. \tag{77}$$

Thus, for just 2 quark doublets (u,d) and (c,s), the weak interactions preserve CP!

But when Kobayashi and Maskawa did similar phase counting for theories with $n>2$ quark doublets — which back in 1973 was a sheer speculation, — they got a very different result: The unitary $n\times n$ CKM matrix is parametrized by $\frac{1}{2}n(n-1)$ real angles and $\frac{1}{2}n(n+1)$ complex phases; by changing the $2n$ quark phases we may eliminate $2n-1$ of these phase parameters, which leaves us with

$$\tfrac{1}{2}n(n+1) \;-\; (2n-1) \;=\; \tfrac{1}{2}(n-1)(n-2) \;>\; 0 \tag{78}$$

CP-violating phase parameters we cannot eliminate. In particular, for $n=3$ there is one CP-violating phase we cannot eliminate, and it is this phase which is responsible for all the CP violations in weak interactions.

In the standard convention for the CKM matrix

$$\mathrm{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \times \begin{pmatrix} \cos\theta_{13} & 0 & e^{-i\delta}\sin\theta_{13} \\ 0 & 1 & 0 \\ -e^{+i\delta}\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \times \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{27}$$

the 3 real angles are $\theta_{12}\approx\theta_{\text{Cabibbo}}$, θ_{23}, and θ_{13}, while δ is the CP-violating phase. Experimentally, this phase has a surprisingly large value $\delta\approx 65^\circ$, but the CP violation is rather weak due to smallness of the real mixing angles $\theta_{12} \approx 13^\circ$, $\theta_{23} \approx 2.4^\circ$, and especially $\theta_{13} \approx 0.21^\circ$. Indeed, in the absence of any one of these real mixing angles, the CKM matrix would be real or equivalent to real via re-phasing of the quark flavors: For $\theta_{13} = 0$ the matrix (27) would be real as it is, while for $\theta_{23} = 0$ we would have

$$\text{real } V' = \begin{pmatrix} e^{+i\delta/2} & 0 & 0 \\ 0 & e^{-i\delta/2} & 0 \\ 0 & 0 & e^{-i\delta/2} \end{pmatrix} \times V \times \begin{pmatrix} e^{-i\delta/2} & 0 & 0 \\ 0 & e^{+i\delta/2} & 0 \\ 0 & 0 & e^{+i\delta/2} \end{pmatrix}. \tag{79}$$

and likewise for $\theta_{12} =$ we would have

$$\text{real } V' = \begin{pmatrix} e^{+i\delta/2} & 0 & 0 \\ 0 & e^{+i\delta/2} & 0 \\ 0 & 0 & e^{-i\delta/2} \end{pmatrix} \times V \times \begin{pmatrix} e^{-i\delta 2/} & 0 & 0 \\ 0 & e^{-i\delta/2} & 0 \\ 0 & 0 & e^{+i\delta/2} \end{pmatrix}, \tag{80}$$

So instead of the δ phase per se, the physical CP-breaking effects of the CKM matrix depend on the flavor-phase independent Jarlskog invariant J defined as

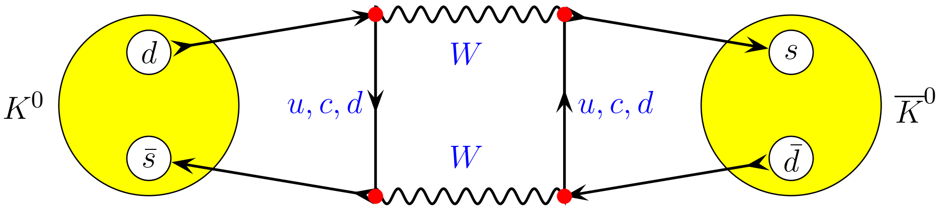
$$\begin{gathered} \text{for } U, U' = u, c, t \text{ but } U \neq U',\ \text{and } D, D' = d, s, b \text{ but } D \neq D' : \\ \operatorname{Im}\bigl(V_{U,D} V_{U',D'} V^*_{U,D'} V^*_{U',D}\bigr) \;=\; \pm J. \end{gathered} \tag{81}$$

In the standard convention

$$J \;=\; \cos\theta_{12}\sin\theta_{12} \times \cos\theta_{23}\sin\theta_{23} \times \cos^2\theta_{13}\sin\theta_{13} \times \sin(\delta_{13}), \tag{82}$$

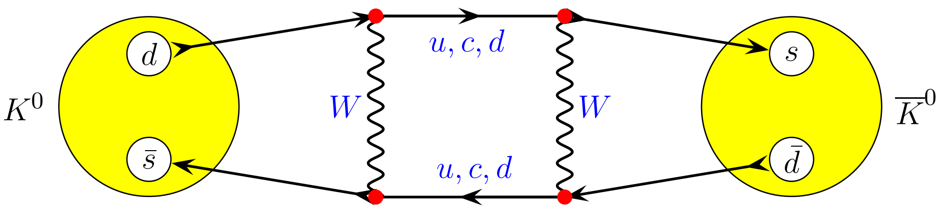
and its experimental value is $J = (3.08 \pm 0.15) \times 10^{-5}$. It's the smallness of this invariant which makes the CP violating effects much weaker than the rest of the weak interaction!

For example, consider the imaginary part of the $\left\langle \overline{K}^0 \middle| \widehat{\mathcal{M}} \middle| K^0 \right\rangle$ amplitude, which is responsible for most of CP violation in the kaon system. The leading contribution to this amplitude comes from the GIM (Glashow–Iliopoulos–Maiani) box diagrams



(83)

and



(84)

In both diagrams, each of the 4 vertices includes a CKM matrix element for the appropriate flavors. Specifically, let U and U' denote the flavors of two charge $= +\frac{2}{3}$ quark propagators in each diagram, then

$$\mathcal{M} \;=\; C \sum_{U,U'=u,c,t} V_{U,d}V^*_{U,s} \times V_{U',d}V^*_{U',s} \times \mathcal{F}\big(m(U), m(U')\big) \tag{85}$$

where

$$C \;=\; O\left(\frac{\alpha_2^2 f_K^2 M_K^2}{M_W^2}\right) \;=\; O(\alpha_2 G_F f_K^2 M_K^2) \tag{86}$$

is the common factor for all quark flavors in the loop, and

$$\mathcal{F}\big(m(U), m(U')\big) \;=\; \int\limits_0^\infty \frac{M_W^2\, x^2\, dx}{(x+M_W^2)^2(x+m^2(U))(x+m^2(U'))} \tag{87}$$

is the quark-mass-dependent factor for each combination of the U and U' flavors.

By unitarity of the CKM matrix U,

$$\sum_{U=u,c,t} V_{U,d}V^*_{U,s} \;=\; 0, \tag{88}$$

so without the mass-dependent factors $\mathcal{F}$ the net $K^0 \leftrightarrow \overline{K}^0$ amplitude (85) would vanish. To emphasise the importance of the quark mass differences, we may rewrite this amplitude as

$$\mathcal{M} \;=\; C\sum_{U,U'=u,c,t} V_{U,d}V^*_{U,s}\times V_{U',d}V^*_{U',s}\times S\bigl(m(U),m(U')\bigr) \tag{89}$$

where

$$\begin{aligned} S\bigl(m(U),m(U')\bigr) \;&=\; \mathcal{F}\bigl(m(U),m(U')\bigr) \;-\; \mathcal{F}\bigl(m(U),m(u)\bigr) \\ &\qquad -\; \mathcal{F}\bigl(m(u),m(U')\bigr) +\; \mathcal{F}\bigl(m(u),m(u)\bigr) \\ &\approx\; \mathcal{F}\bigl(m(U),m(U')\bigr) \;-\; \mathcal{F}\bigl(m(U),0\bigr) \;-\; \mathcal{F}\bigl(0,m(U')\bigr) +\; \mathcal{F}\bigl(0,0\bigr), \end{aligned} \tag{90}$$

the bottom line here stemming from the negligibly small up-quark's mass. In light of eqs. (90),

$$S\bigl(m(U),m(U')\bigr) \;=\; 0 \quad \text{for} \quad U=u \quad \text{or} \quad U'=u \quad \text{or both}, \tag{91}$$

and therefore

$$\mathcal{M} \;=\; C\Bigl(\bigl(V_{cd}V^*_{cs}\bigr)^2\times S(m_c,m_c) \,+\, 2\bigl(V_{cd}V^*_{cs}\bigr)\bigl(V_{td}V^*_{ts}\bigr)\times S(m_c,m_t) \,+\, \bigl(V_{td}V^*_{ts}\bigr)^2\times S(m_t,m_t)\Bigr). \tag{92}$$

Note: It is this amplitude — or rather its magnitude — which is responsible for the neutral Kaon oscillations,

$$2M_K\times\delta M(K_L,K_S) \;=\; |\mathcal{M}|, \tag{93}$$

while its imaginary part

$$\operatorname{Im}\mathcal{M} \;=\; C\left(\begin{aligned} S(m_c,m_c)\times\operatorname{Im}\bigl[\bigl(V_{cd}V^*_{cs}\bigr)^2\bigr] \,+\, 2S(M_c,m_t)\times\operatorname{Im}\bigl[\bigl(V_{cd}V^*_{cs}\bigr)\bigl(V_{td}V^*_{ts}\bigr)\bigr] \\ +\, S(m_t,m_t)\times\operatorname{Im}\bigl[\bigl(V_{td}V^*_{ts}\bigr)^2\bigr] \end{aligned}\right) \tag{94}$$

is responsible for the CP violation in the neutral kaon decays.

By unitarity of the CKM matrix,

$$V_{ud}V^*_{us} \;+\; V_{cd}V^*_{cs} \;+\; V_{td}V^*_{ts} \;=\; 0, \tag{95}$$

and since the first term on the LHS happens to be real (in the standard convention for the

CKM matrix), we may parametrize the other two terms as

$$V_{cd}V_{cs}^* = a + ic, \quad V_{td}V_{ts}^* = b - ic \tag{96}$$

for some real a, b, c. Consequently,

$$\begin{aligned} \operatorname{Im}\left[\left(V_{cd}V_{cs}^*\right)^2\right] &= 2ac, \\ \operatorname{Im}\left[\left(V_{cd}V_{cs}^*\right)\left(V_{td}V_{ts}^*\right)\right] &= (a-b)c, \\ \operatorname{Im}\left[\left(V_{td}V_{ts}^*\right)^2\right] &= -2bc, \\ J = \operatorname{Im}\left[\left(V_{cd}V_{cs}^*\right)\left(V_{td}V_{ts}^*\right)^*\right] &= (a+b)c, \end{aligned} \tag{97}$$

and therefore

$$\operatorname{Im}\mathcal{M} = J \times 2C\left(\frac{a}{a+b}S(m_c,m_c) + \frac{a-b}{a+b}\times S(m_c,m_t) - \frac{b}{a+b}\times S(m_t,t_t)\right). \tag{98}$$

Likewise, the CP violation in the b-flavored or c-flavored neutral mesons is also proportional to the Jarlskog invariant J. Since this invariant happens to be rather small, $J \approx 3 \cdot 10^{-5}$, the CP violating effects are rather weak relative to the other one-loop-level weak interactions. And that why CP is a good approximate symmetry of the Standard Model.

Strong CP Violation

In principle, the $SU(3)_C \times SU(2)_W \times U(1)_Y$ Standard Model can have two separate sources of CP violation: (1) The imaginary part of the CKM matrix, which breaks the CP symmetry of the weak interactions; (2) The instanton angle Θ of QCD, which can break the CP symmetry of the strong interactions. In the terms of the QCD Lagrangian, the instanton angle appears as a CP-odd term

$$\mathcal{L}_{\text{QCD}} \supset \frac{i\Theta}{32\pi^2}\operatorname{tr}\left(\epsilon^{\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu}\right) = \frac{i\alpha_{\text{QCD}}\Theta}{16\pi}\epsilon^{\kappa\lambda\mu\nu}F^a_{\kappa\lambda}F^a_{\mu\nu}. \tag{99}$$

The instanton angle is beyond the scope of this class, but I am going to explain it in an extra lecture sometimes in April. Meanwhile, you can read the 2018 TASI Lectures on the Strong CP Problem and Axions by Prof. Anson Hook at the University of Maryland.

For the moment, let me say that a non-zero instanton angle has no perturbative effects at any loop order, but it has strong non-perturbative effects on the QCD bound states such as nucleons. In particular, the neutron gets an electric dipole moment

$$d_n \;=\; e\Theta\times\begin{cases}O(\alpha_{\rm QCD}\times\text{neutron's radius}),\\ \text{best estimate } 4.5\cdot 10^{-15}\text{ cm}.\end{cases} \tag{100}$$

But despite diligent experimental attempts to detect and measure this dipole moment, it turns out to be way too small; the current upper limit is $|d_n| < e\times 1.8\cdot 10^{-26}$ cm. In terms of the instanton angle, this limit gives $|\Theta| < 4\cdot 10^{-12}$, so for all practical purposes $\Theta = 0$ and QCD happens to have perfect CP symmetry.

There are many theories as to why Θ happens to vanish, most likely being some kind of a Peccei–Quinn symmetry, see Roberto Peccei's lecture notes for the explanation. But this subject gets way beyond the scope of this class, so let me stop here.