DIMENSIONAL REGULARIZATION

The dimensional regularization of ultraviolet divergences involves analytic continuation of the Euclidean momentum integrals to momentum spaces of non-integer dimensions $D < 4$ — which makes the integrals finite — and then taking the limit $D \to 4$ (from below). Thus,

$$\int\limits_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4}\, f(k_E) \;=\; \int \frac{\mu^{4-D}\, d^D k_E}{(2\pi)^D}\, f(k_E), \tag{1}$$

where μ is the reference energy scale at which the spherical momentum-space shell dk_e^{rad} has the same volume in D dimensions as in 4 dimensions. At much larger loop momenta, the dk_e^{rad} shell's volume becomes smaller in $D < 4$ dimensions than in 4 dimensions:

$$\begin{aligned} d^4 k_E \;\sim\; (k_e^{\text{rad}})^3\, dk_e^{\text{rad}} \;\longrightarrow\; \mu^{4-D} \times (k_e^{\text{rad}})^{D-1}\, dk_e^{\text{rad}} \;=\; \left(\frac{\mu}{k_e^{\text{rad}}}\right)^{4-D} \times (k_e^{\text{rad}})^3\, dk_e^{\text{rad}} \\ \ll\; (k_e^{\text{rad}})^3\, dk_e^{\text{rad}}, \end{aligned} \tag{2}$$

and that's what regularized the UV divergence of the integral (1).

Let's take a closer look at the UV-regulating factor (marked in red in eq. (2)). For $D = 4 - 2\epsilon$,

$$\left(\frac{\mu}{k_e^{\text{rad}}}\right)^{4-D=2\epsilon} \;=\; \left(\frac{k_e^2}{\mu^2}\right)^{-\epsilon} \;=\; \exp\left(-\epsilon \times \log\frac{k_e^2}{\mu^2}\right), \tag{3}$$

which becomes small when

$$\log\frac{k_e^2}{\mu^2} \;\sim\; \frac{1}{\epsilon} \quad\Longrightarrow\quad k_e^2 \;\sim\; \mu^2 \times \exp(1/\epsilon). \tag{4}$$

Thus, the effective UV cutoff scale^2 in dimensional regularization is

$$\Lambda_{\text{DR}}^2 \;=\; \mu^2 \times \exp(1/\epsilon) \;\gg\; \mu^2. \tag{5}$$

In practice, one usually sets the reference energy scale μ in the ball park of the energy scale of the amplitude in question, for example $\mu \sim |q_{\text{net}}|$; consequently, for $\epsilon \to +0$ we have $\Lambda_{\text{DR}} \gg \mu$ and hence $\Lambda_{\text{DR}} \gg$ energy scale of the amplitude.

Now consider a generic logarithmically divergent momentum integral; for most regularization schemes, this means

$$\text{regulated integral} \;=\; (\text{constant } C) \times \log\frac{\Lambda^2}{m^2} \;+\; \text{finite}. \tag{6}$$

For the dimensional regularization, the effective UV cutoff scale is as in eq. (5), so we expect

$$\text{regulated integral} \;=\; (\text{same constant } C) \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{m^2}\right) \;+\; \text{finite}. \tag{7}$$

Thus, we may identify the coefficient C of the $(1/\epsilon)$ pole obtaining from dimensional regularization with the coefficient of $\log\Lambda^2$ in the other regularization schemes.

Integrals over Momentum Spaces of Non-Integer Dimensions

Before we can use dimensional regularization, we need to learn how to perform integrals over (Euclidean) momentum spaces of non-integer dimensions D. Let's start with the Gaussian integrals

$$\int\frac{d^Dk_E}{(2\pi)^D}\,\exp(-tk_E^2). \tag{8}$$

For any integer dimension D, $k_E^2 = k_1^2 + k_2^2 + \cdots + k_D^2$, hence

$$\exp(-tk_E^2) \;=\; \prod_{i=1}^{D}\exp(-tk_i^2) \tag{9}$$

and therefore

$$\begin{aligned}\int\frac{d^Dk_E}{(2\pi)^D}\,\exp(-tk_E^2) \;&=\; \prod_{i=1}^{D}\int\limits_{-\infty}^{+\infty}\frac{dk_i}{2\pi}\,\exp(-tk_i^2)\\ &=\; \left[\int\limits_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\exp(-tk^2)\right]^D\\ &=\; \left[\frac{1}{2\pi}\times\sqrt{\frac{\pi}{t}} \;=\; \frac{1}{\sqrt{4\pi t}}\right]^D\\ &=\; (4\pi t)^{-D/2}.\end{aligned} \tag{10}$$

Let's analytically continue this formula to the non-integer D. In other words, we let

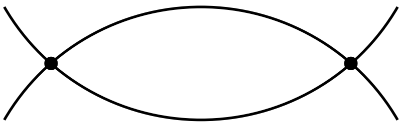
$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2) \; = \; (4\pi t)^{-D/2} \tag{11}$$

for any D, integer or non-integer, real or complex. For non-integer D this formula maybe thought as a *definition* of the Gaussian integral over a non-integer-dimensional space.

As to the non-Gaussian momentum integrals, we should re-express them in terms of Gaussian integrals and then use eq. (11) for non-integer D. For example, consider the dimensionally regulated momentum integral

$$I \; = \int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} \; = \int \frac{\mu^{4-D}\, d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} \tag{12}$$

which appears in the context of the one-loop Feynman diagram



$$\mathcal{F}(t) \; = \; \frac{\lambda^2}{2} \int_0^1 dx \int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x)]^2} \tag{13}$$

for $\Delta(x) = m^2 - tx(1-x)$. Using the Γ-function integral

$$\int_0^\infty dt\, t^{n-1} \times \exp(-t(k_E^2 + \Delta)) \; = \; \frac{\Gamma(n)}{[k_E^2 + \Delta]^n} \tag{14}$$

for $n = 2$, we let

$$\frac{1}{[k_E^2 + \Delta]^2} \; = \; \frac{1}{\Gamma(2) = 1! = 1} \times \int_0^\infty dt\, t \times \exp(-t(k_E^2 + \Delta)) \tag{15}$$

and consequently

$$\int\frac{d^Dk_E}{(2\pi)^D}\,\frac{1}{[k_E^2+\Delta]^2} \;=\; \int\frac{d^Dk_E}{(2\pi)^D}\int\limits_0^\infty dt\, t\times\exp(-t(k_E^2+\Delta))$$
$$\langle\!\langle \text{ changing the order of integration } \rangle\!\rangle$$
$$= \int\limits_0^\infty dt\, te^{-t\Delta}\times\int\frac{d^Dk_E}{(2\pi)^D}\,e^{-tk_E^2}$$
$$\langle\!\langle \text{ using eq. (11) } \rangle\!\rangle \tag{16}$$
$$= \int\limits_0^\infty dt\, te^{-t\Delta}\times(4\pi t)^{-D/2} \;=\; (4\pi)^{-D/2}\int\limits_0^\infty dt\, t^{1-(D/2)}\times e^{-t\Delta}$$
$$= (4\pi)^{-D/2}\times\Gamma(2-(D/2))\Delta^{(D/2)-2}.$$

Note that on the penultimate line here, the integrand behaves as $t^{1-(D/2)}$ for $t\to 0$. Consequently, the integral converges whenever (this power of t) >-1, which means $D<4$. Or for complex D, whenever $\mathrm{Re}(D)<4$. Physically, the $t\to 0$ limit corresponds to $k_E^2\to\infty$, so the convergence/divergence of the $\int dt$ integral at $t\to 0$ corresponds to the UV convergence/divergence of the original momentum integral.

Anyhow, for $D=4-2\epsilon$ eq. (16) becomes

$$\mu^{4-D}\times\int\frac{d^Dk_E}{(2\pi)^D}\,\frac{1}{[k_E^2+\Delta]^2} \;=\; \frac{(4\pi\mu^2)^\epsilon}{16\pi^2}\times\Gamma(\epsilon)\times\Delta^{-\epsilon} \tag{17}$$

and hence

$$\mathcal{F}(t) \;=\; \frac{\lambda^2}{32\pi^2}\int\limits_0^1 dx\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^\epsilon. \tag{18}$$

Note that this is a finite formula for $\epsilon>0$ (*i.e.*, for $D<4$), but it becomes singular in the $\epsilon\to 0$ limit because the $\Gamma(\epsilon)$ function has a pole at $\epsilon=0$.

Let's take a closer look at this pole using $\Gamma(x+1)=x\times\Gamma(x)$. In particular, for $x=\epsilon\to 0$,

$$\begin{aligned}\Gamma(\epsilon) \;&=\; \frac{\Gamma(\epsilon+1)}{\epsilon} \;=\; \frac{1}{\epsilon}\left(\Gamma(1)\,+\,\epsilon\times\Gamma'(1)\,+\,\frac{\epsilon^2}{2}\Gamma''(1)\,+\,\cdots\right)\\ &=\; \frac{1}{\epsilon}\,-\,\gamma_E\,+\,\frac{\pi^2+6\gamma_E^2}{12}\times\epsilon\,+\,O(\epsilon^2)\end{aligned} \tag{19}$$

where $\gamma_E \approx 0.5772$ is the Euler–Mascheroni constant. At the same time,

$$\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^{\epsilon} = \exp\left(\epsilon\times\log\frac{4\pi\mu^2}{\Delta(x)}\right) = 1 + \epsilon\times\log\frac{4\pi\mu^2}{\Delta(x)} + \frac{\epsilon^2}{2}\times\log^2\frac{4\pi\mu^2}{\Delta(x)} + O(\epsilon)^3, \quad (20)$$

hence

$$\Gamma(\epsilon)\times\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^{\epsilon} = \frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{\Delta(x)} + O(\epsilon). \quad (21)$$

In dimensional regularization, positive powers of $\epsilon \to 0$ correspond to negative powers of $\log\Lambda^2_{\rm UV} \to \infty$. And although such negative powers of $\log\Lambda^2_{\rm UV}$ go to zero much slower than the negative powers of the $\Lambda^2_{\rm UV}$ itself, they do eventually go to zero in the very-large-UV-cutoff-scale limit. Consequently, in dimensional regularization we neglect all *positive* powers of ϵ in various amplitudes (but only in the net product of all the factors). Thus, in eq. (18) we approximate

$$\Gamma(\epsilon)\times\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^{\epsilon} \underset{\epsilon\to 0}{\longrightarrow} \frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{\Delta(x)} \quad (22)$$

and hence

$$\mathcal{F}_{\rm DR}(t) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{\Delta(x)}\right). \quad (23)$$

Finally, using

$$\log\frac{4\pi\mu^2}{\Delta(x)} = \log\frac{4\pi\mu^2}{m^2} - \log\frac{\Delta(x) = m^2 - tx(1-x)}{m^2} \quad (24)$$

we arrive at

$$\begin{aligned}\mathcal{F}_{\rm DR}(t) &= \frac{\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m^2} - \int_0^1 dx\log\frac{m^2 - tx(1-x)}{m^2}\right)\\ &= \frac{\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m^2} - J(t/m^2)\right).\end{aligned} \quad (25)$$

In class we have evaluated the same one-loop diagram (13) using Wilson's hard-edge cutoff and got

$$\mathcal{F}(t) \;=\; \frac{\lambda^2}{32\pi^2}\left(\log\frac{\Lambda^2_{\rm HE}}{m^2}\;-\;1\;-\;J(t/m^2)\right). \tag{26}$$

Likewise, in your homework#13 you should have obtained

$$\begin{aligned}\mathcal{F}(t) \;&=\; \frac{\lambda^2}{32\pi^2}\left(\log\frac{\Lambda^2_{\rm PV}}{m^2}\;-\;J(t/m^2)\right)\\ &=\; \frac{\lambda^2}{32\pi^2}\left(\log\frac{\Lambda^2_{\rm HD}}{m^2}\;-\;2\;-\;J(t/m^2)\right)\end{aligned} \tag{27}$$

for the respectively Pauli–Villars and higher-derivative UV regulators. Consequently, all these cutoffs yield exactly the same result provided we identify

$$\log\Lambda^2_{\rm HE}\;-\;1\;=\;\log\Lambda^2_{\rm PV}\;=\;\log\Lambda^2_{\rm HD}\;-\;2, \tag{28}$$

or equivalently

$$\Lambda^2_{\rm HE}\;=\;\exp(1)\times\Lambda^2_{\rm PV}\,,\quad \Lambda^2_{\rm HD}\;=\;\exp(2)\times\Lambda^2_{\rm PV}\,. \tag{29}$$

Likewise, the dimensional regularization's result (25) becomes similar to that of all the other cutoffs when we identify

$$\frac{1}{\epsilon}\;-\;\gamma_E\;+\;\log(4\pi\mu^2)\;=\;\log\Lambda^2_{\rm HE}\;-\;1\;=\;\log\Lambda^2_{\rm PV}\;=\;\log\Lambda^2_{\rm HD}\;-\;2, \tag{30}$$

or equivalently

$$\mu^2\times\exp(1/\epsilon)\;=\;\frac{\exp(\gamma_E)}{4\pi}\times\Lambda^2_{\rm PV}\;=\;\frac{\exp(\gamma_E-1)}{4\pi}\times\Lambda^2_{\rm HE}\;=\;\frac{\exp(\gamma_E-2)}{4\pi}\times\Lambda^2_{\rm HD}\,. \tag{31}$$