Form Factors

Consider elastic scattering of an electron off an atomic nucleus. For large enough electron momenta, such scattering depends not only on the net electric charge Ze of the nucleus but also on the space distribution $\rho(\mathbf{x})$ of this charge inside the nucleus. Indeed, in the first Born approximation, the (non-relativistically normalized) scattering amplitude is

$$f(\mathbf{q}) \;=\; \frac{em_e}{2\pi} \times A^0(\mathbf{q}) \tag{1}$$

where $A^0(\mathbf{q})$ is the Fourier transform of the nucleus's electrostatic potential $A^0(\mathbf{x})$. In terms of the nuclear charge distribution,

$$A^0(\mathbf{q}) \;=\; \frac{\rho(\mathbf{q})}{\mathbf{q}^2} \;=\; \frac{1}{\mathbf{q}^2} \times \int d^3\mathbf{x}\, \rho(\mathbf{x})\, e^{-i\mathbf{q}\mathbf{x}}, \tag{2}$$

which is often written as

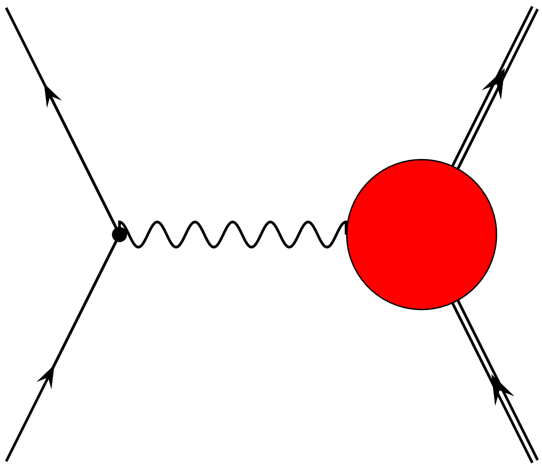
$$A^0(\mathbf{q}) \;=\; \frac{eZ}{\mathbf{q}^2} \times F(\mathbf{q}^2) \quad \text{for} \quad F(\mathbf{q}^2) \;=\; \frac{1}{eZ} \int d^3\mathbf{x}\, \rho(\mathbf{x})\, e^{-i\mathbf{q}\mathbf{x}}, \tag{3}$$

hence the scattering amplitude

$$f(\mathbf{q}^2) \;=\; \frac{e^2 Z m_e}{2\pi\mathbf{q}^2} \times F(\mathbf{q}^2). \tag{4}$$

The $F(\mathbf{q}^2)$ factor here is called the *nuclear form factor*; for small momenta $F \approx 1$, while for large momenta it probes the nuclear structure, or at least the electric charge distribution. Experimentally, we get the form factor from the scattering amplitude (4), and then use it to reconstruct the charge distribution.

Now let's probe the structure of a relativistic particle like a proton. At the tree level of QED — but non-perturbatively in strong interactions inside the proton — the scattering amplitude obtains as



thus

$$i\mathcal{M} \;=\; \frac{-i}{q^2}\times \bar{u}(e')(ie\gamma_\mu)u(e)\times \bar{u}(p')(-ie\Gamma^\mu)u(p). \tag{5}$$

where $-ie\Gamma^\mu$ is the photon-proton-proton vertex. Since the proton is is a composite particle, this vertex depends on the momenta p, p', and $q = p' - p$, which makes the $\Gamma^\mu_{\alpha\beta}(p',p)$ a kind of a form factor, or rather an array of form factors with Lorentz and Dirac indices μ, α, β. For the on-shell proton momenta p, p' and in the limit of zero photon momentum $q \to 0$, this form factor becomes 1, or rather,

$$\Gamma^\mu_{\alpha\beta} \;\to\; \gamma^\mu_{\alpha\beta}\,, \tag{6}$$

but for $q \neq 0$ the situation is much more interesting.

Let's start our analysis of the $\Gamma^\mu_{\alpha\beta}(p',p)$ array of functions of momenta with its Lorenz symmetry. Keeping the Dirac indices implicit, the $\Gamma^\mu(p',p)$ is a Lorentz vector valued function of three independent true vectors, p^μ, p'^μ, and γ^μ, hence it must have form

$$\Gamma^\mu(p',p) \;=\; A\times p^\mu \;+\; B\times p'^\mu \;+\; C\times\gamma^\mu \tag{7}$$

where A, B, and C are scalar functions of the Lorentz invariants p^2, p'^2, (pp'), $\not{p}$, and $\not{p}'$. Note that $\not{p}$ and $\not{p}'$ do not commute, so functions of both of these variables are subject to ordering ambiguities. Likewise, in the $C\gamma^\mu$ term there are ordering ambiguities between the γ^μ and the $\not p$ and $\not p'$. However, thanks to the anticommutation relations

$$\not p\,\not p \;=\; p^2, \quad \not p'\,\not p' \;=\; p'^2, \quad \{\not p,\not p'\} \;=\; 2(pp'), \quad \{\not p,\gamma^\mu\} \;=\; 2p^\mu, \quad \{\not p',\gamma^\mu\} \;=\; 2p'^\mu, \tag{8}$$

all or these ordering issues can be resolved by bringing all products of $\not p$, $\not p'$, and γ^μ to some standard order, for example

$$\begin{aligned}\Gamma^\mu \;=\; &\big(A_1 + A_2\not p + A_3\not p' + A_4\not p'\not p\big)\times p^\mu + \big(B_1 + B_2\not p + B_3\not p' + B_4\not p'\not p\big)\times p'^\mu \\ &+\; \big(C_1\gamma^\mu + C_2\gamma^\mu\not p + C_3\not p'\gamma^\mu + C_4\not p'\gamma^\mu\not p\big),\end{aligned} \tag{9}$$

where $A_1,\ldots,C_4$ are scalar functions of the ordinary (non-matrix) variables p^2, p'^2, and (pp') only.

Next, let's put the incoming and the outgoing protons on mass shell, but keep the virtual photon off-shell. This fixes $p^2 = p'^2 = M^2$ and makes the $A_1,\ldots,C_4$ functions of a single independent variable (pp'), or equivalently $q^2 = 2M^2 - 2(pp')$. Moreover, going on-shell involves sandwiching the Γ^μ matrix between the external leg factors for the incoming and the outgoing proton,

$$\bar u(p')\times\Gamma^\mu\times u(p). \tag{10}$$

In this context, we may drastically simplify eq. (9) by using

$$\not p\times u(p) \;=\; Mu(p) \quad\text{and}\quad \bar u(p')\times\not p' \;=\; M\bar u(p'), \tag{11}$$

so that a rightmost factor $\not p$ or a leftmost factor of $\not p'$ in some term for Γ^μ can be replaced with the proton's mass M. Thus, *in the on-shell context of* (10),

$$\Gamma^\mu(p',p) \;\cong\; A(q^2)\times p^\mu \;+\; B(q^2)\times p'^\mu \;+\; C(q^2)\times\gamma^\mu \tag{12}$$

$$\text{for}\quad A \;=\; A_1 \;+\; MA_2 \;+\; MA_3 \;+\; M^2A_4\,, \tag{13}$$

$$B \;=\; B_1 \;+\; MB_2 \;+\; MB_3 \;+\; M^2B_4\,, \tag{14}$$

$$C \;=\; C_1 \;+\; MC_2 \;+\; MC_3 \;+\; M^2C_4\,. \tag{15}$$

We may further simplify eq. (12) for the on-shell protons by using the on-shell Ward identity

$$q_\mu \times \bar{u}(p')\Gamma^\mu u(p) \;=\; 0. \tag{16}$$

To satisfy this identity, we need $A(q^2) = B(q^2)$ while $C(q^2)$ remains unconstrained. Indeed, for the Γ^μ as in eq. (12),

$$q_\mu \times \bar{u}(p')\Gamma^\mu u(p) \;=\; q_\mu\big(Ap^\mu \;+\; Bp'^\mu\big) \times \bar{u}(p')u(p) \;+\; C \times \bar{u}(p')\,\not{q}u(p), \tag{17}$$

where

$$\bar{u}(p')\,\not{q}\gamma^\mu u(p) \;=\; \bar{u}(p')\,\not{p}'u(p) \;-\; \bar{u}(p')\,\not{p}u(p) \;=\; M\bar{u}(p')u(p) \;-\; \bar{u}(p')u(p)M \;=\; 0, \tag{18}$$

while

$$q_\mu(p'+p)^\mu \;=\; (p'-p)_\mu(p'+p)^\mu \;=\; p'^2 \;-\; p^2 \;=\; M^2 \;-\; M^2 \;=\; 0, \tag{19}$$

$$q_\mu(p'-p)^\mu \;=\; q_\mu q^\mu \;=\; q^2 \;\neq 0, \tag{20}$$

and hence

$$\begin{aligned} q_\mu\big(Ap^\mu \;+\; Bp'^\mu\big) \;&=\; \frac{A+B}{2} \times (p'+p)^\mu q_\mu \;+\; \frac{B-A}{2} \times (p'-p)^\mu q_\mu \\ &=\; 0 \;+\; \frac{B-A}{2} \times q^2. \end{aligned} \tag{21}$$

Altogether,

$$q_\mu \times \bar{u}(p')\Gamma^\mu u(p) \;=\; \frac{B-A}{2} \times q^2 \times \bar{u}(p')u(p), \tag{22}$$

and that's why we need $A = B$ for any q^2.

The bottom line is, for the on-shell protons (but an off-shell photon), the vertex function becomes

$$\Gamma^\mu(p',p) \;\cong\; A(q^2) \times (p'+p)^\mu \;+\; C(q^2) \times \gamma^\mu, \tag{23}$$

which involves two independent form factors $A(q^2)$ and $C(q^2)$. Of course, one may always trade A and C for any two independent linear combinations of A and C, and people usually do that using the *Gordon identity*

$$2M \times \bar{u}'\gamma^\mu u \;=\; (p'+p)_\mu \times \bar{u}'u \;+\; iq_\nu \times \bar{u}'\sigma^{\mu\nu}u. \qquad (24)^\star$$

Thanks to this identity, we may rewrite eq. (23) as

$$\Gamma^\mu(p',p) \;\cong\; F_{\rm electric}(q^2) \times \frac{(p'+p)^\mu}{2M} \;+\; F_{\rm magnetic}(q^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2M} \qquad (28)$$

for

$$F_{\rm el}(q^2) \;=\; 2MA(q^2) \;+\; C(q^2) \quad \text{and} \quad F_{\rm mag}(q^2) \;=\; C(q^2). \qquad (29)$$

The names 'electric' and 'magnetic' for the form factors $F_{\rm el}(q^2)$ and $F_{\rm mag}(q^2)$ reflect what these form factors probe in the non-relativistic limit. Specifically, the $F_{\rm el}(q^2)$ probes the electric charge distribution inside the proton, and also the distribution of the electric current due to the proton's overall motion. In particular, for $q^2 \to 0$, it probes the net electric charge of the proton, which requires $F_{\rm el}(0) = 1$. More generally, for a particle of electric charge Q,

$$e \times F_{\rm el}(q^2) \;\underset{q^2\to 0}{\longrightarrow}\; Q. \qquad (30)$$

For example, for a neutron $F_{\rm el}(0) = 0$. However, despite zero net charge the neutron has a non-trivial charge distribution inside it, hence $F_{\rm el}(q^2) \neq 0$ for $q^2 \neq 0$.

As to the magnetic form factor $F_{\rm mag}(q^2)$, it probes the distribution of the particle's magnetic moment due to spin. In the $q^2 \to 0$ limit, it governs the interaction of the spin

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⋆ The Gordon identity follows from

$$\gamma^\mu\gamma^\nu \;=\; g^{\mu\nu} \;-\; i\sigma^{\mu\nu}, \quad \gamma^\nu\gamma^\mu \;=\; g^{\mu\nu} \;+\; i\sigma^{\mu\nu}, \qquad (25)$$

hence

$$\gamma^\mu \not{p} \;=\; p^\mu \;-\; ip_\nu\sigma^{\mu\nu}, \quad \not{p}'\gamma^\mu \;=\; p'^\mu \;+\; ip'_\nu\sigma^{\mu\nu}, \qquad (26)$$

and therefore

$$2M \times \bar{u}'\gamma^\mu u \;=\; \bar{u}'\not{p}'\gamma^\mu u \;+\; \bar{u}'\gamma^\mu \not{p}u \;=\; (p'+p)^\mu \times \bar{u}'u \;+\; i(p'-p)_\nu \times \bar{u}'\sigma^{\mu\nu}u. \qquad (27)$$

with virtual photons contained in a uniform magnetic field, thus

$$\hat{H} \supset -\frac{F_{\text{mag}}(0)}{Mc}\,\mathbf{S}\cdot\mathbf{B}. \tag{31}$$

In other words, the spinorial magnetic moment of the particle is

$$\mu = \frac{\hbar}{2Mc}\,F_{\text{mag}}(0), \tag{32}$$

which means that gyromagnetic moment $g = 2F_{\text{mag}}(0)$. In particular, for the proton $F_{\text{mag}}(0) \approx +2.792$ while for the neutron $F_{\text{mag}}(0) \approx -1.832$.

Another useful basis for the two form factors of a relativistic fermion is

$$\Gamma^{\mu}(p',p) = F_1(q^2)\times\gamma^{\mu} + F_2(q^2)\times\frac{i\sigma^{\mu\nu}q_{\nu}}{2M} \tag{33}$$

for

$$F_1 = F_{\text{el}} \quad\text{and}\quad F_2 = F_{\text{mag}} - F_{\text{el}}\,. \tag{34}$$

Let's apply this basis to the electron's form factors. Semiclassically, the electron is a point-like particle obeying the Dirac equation; in terms of Feynman diagrams, this corresponds to the tree-level electron-electron-photon vertex being simply $ie\gamma^{\mu}$ regardless of the momenta. In other words, $\Gamma^{\mu}_{\text{tree}} = \gamma^{\mu}$ ⋆ and hence

$$\forall q^2: \quad F_1^{\text{tree}} = 1, \quad F_2^{\text{tree}} = 0. \tag{35}$$

Or in terms of the electric and the magnetic form factors, $F_{\text{el}}^{\text{tree}} = 1$, $F_{\text{mag}}^{\text{tree}} = 1$, hence gyromagnetic ratio $g^{\text{tree}} = 2$.

But at higher orders of the perturbation theory — *i.e.*, beyond the tree level, — the electron is no longer pointlike, or rather not exactly pointlike. Instead, the point-like single-electron quantum state $|e^-\rangle$ of the free theory mixes us with states containing additional photons and/or electron-positron pairs, $|e^-\gamma\rangle$, $|e^-\gamma\gamma\rangle$, $|e^-e^-e^+\rangle$, *etc.*, *etc.* The

⋆ For the electron — and also for the muon and the tau lepton — one usually defines the photon vertex factor as $+ie\Gamma^{\mu}$ rather than $-ie\Gamma^{\mu}$.

extra electron-positron pairs give the physical electron a non-trivial space distribution of its electric charge and magnetic moment, hence non-trivial

$$F_1(q^2) \not\equiv 1, \quad F_2(q^2) \not\equiv 0. \tag{36}$$

Of course, the net electric charge cannot be modified by quantum corrections, hence for $q^2 = 0$, $F_1(0) = 1$, exactly; but for $q^2 \neq 0$ we generally have $F_1(q^2) = 1 + O(\alpha)$. As to the F_2 form factor, it does not need to vanish for $q^2 = 0$; instead, it gives us the anomalous magnetic moment of the electron

$$a = \frac{g-2}{2} = F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2). \tag{37}$$

I shall explain how this works at the one-loop level in my notes on the dressed QED vertex.