Higgs Mechanism

When a local rather than global symmetry is spontaneously broken, we do not get a massless Goldstone boson. Instead, the gauge field of the broken symmetry becomes massive, and the would-be Goldstone scalar becomes the longitudinal mode of the massive vector. This is the *Higgs mechanism*, and it works for both abelian and non-abelian local symmetries. In the non-abelian case, for each spontaneously broken generator T^a of the local symmetry the corresponding gauge field $A^a_\mu(x)$ becomes massive.

The Abelian Example

To understand how the Higgs mechanism works, let's start with the abelian example of a local $U(1)$ phase symmetry. The complete model comprises a complex scalar field $\Phi(x)$ of electric charge q coupled to the EM field $A^\mu(x)$; the Lagrangian is

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; D_\mu\Phi^* D^\mu\Phi \;-\; V(\Phi^*\Phi) \tag{1}$$

where

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; iqA_\mu(x)\Phi(x), \qquad D_\mu\Phi^*(x) \;=\; \partial_\mu\Phi^*(x) \;-\; iqA_\mu(x)\Phi^*(x), \tag{2}$$

and

$$V(\Phi^*\Phi) \;=\; \frac{\lambda}{2}\,(\Phi^*\Phi)^2 \;+\; m^2(\Phi^*\Phi). \tag{3}$$

Suppose $\lambda > 0$ but $m^2 < 0$, so that $\Phi = 0$ is a local maximum of the scalar potential, while the minima form a degenerate circle

$$\Phi \;=\; \frac{v}{\sqrt{2}} \times e^{i\theta}, \quad v \;=\; \sqrt{\frac{-2m^2}{\lambda}}\,, \quad \text{any real } \theta. \tag{4}$$

Consequently, the scalar field Φ develops a non-zero vacuum expectation value $\langle\Phi\rangle \neq 0$, which spontaneously breaks the $U(1)$ symmetry of the theory. Were that $U(1)$ symmetry global rather than local, is spontaneous breakdown would lead to a massless Goldstone scalar stemming from the phase of the complex field $\Phi(x)$. But for the local $U(1)$ symmetry, the phase of $\Phi(x)$ — not just the phase of the vacuum expectation value $\langle\Phi\rangle$ but the x-dependent phase of the dynamical $\Phi(x)$ field — can be eliminated by a gauge transform, so the physical consequences of the SSB are more complicated.

To see how this works, let's use polar coordinates in the scalar field space, thus

$$\Phi(x) \;=\; \frac{1}{\sqrt{2}}\,\phi_r(x)\times e^{i\Theta(x)}, \quad \text{real } \phi_r(x)>0, \quad \text{real } \Theta(x). \tag{5}$$

This field redefinition is singular when $\Phi(x)=0$, so we should never use it for theories with $\langle\Phi\rangle=0$, but it's OK for spontaneously broken theories where we expect $\Phi(x)\neq 0$ almost everywhere. In terms of the real fields $\phi_r(x)$ and $\Theta(x)$, the scalar potential depends only on the radial field ϕ_r,

$$V(\Phi) \;=\; \frac{\lambda}{8}\bigl(\phi_r^2-v^2\bigr)^2 \;+\; \text{const}, \tag{6}$$

or in terms of the radial field shifted by its VEV, $\phi_r(x)=v+\sigma(x)$,

$$\phi_r^2 \;-\; v^2 \;=\; (v+\sigma)^2 \;-\; v^2 \;=\; 2v\sigma \;+\; \sigma^2, \tag{7}$$

$$V \;=\; \frac{\lambda}{8}\bigl(2v\sigma+\sigma^2\bigr)^2 \;=\; \frac{\lambda v^2}{2}\times\sigma^2 \;+\; \frac{\lambda v}{2}\times\sigma^3 \;+\; \frac{\lambda}{8}\times\sigma^4. \tag{8}$$

At the same time, the covariant derivative $D_\mu\Phi$ becomes

$$D_\mu\Phi \;=\; \frac{1}{\sqrt{2}}\Bigl(\partial_\mu\bigl(\phi_r e^{i\Theta}\bigr) \;+\; iqA_\mu\times\phi_r e^{i\Theta}\Bigr) \;=\; \frac{e^{i\Theta}}{\sqrt{2}}\Bigl(\partial_\mu\phi_r \;+\; \phi_r\times i\partial_\mu\Theta \;+\; \phi_r\times iqA_\mu\Bigr). \tag{9}$$

Inside the big () on the RHS, the first term is real while the other two terms are imaginary, hence

$$\begin{aligned} |D_\mu\Phi|^2 \;&=\; \frac{1}{2}\Bigl|\partial_\mu\phi_r \;+\; \phi_r\times i\partial_\mu\Theta \;+\; \phi_r\times iqA_\mu\Bigr|^2 \\ &=\; \tfrac{1}{2}(\partial_\mu\phi_r)^2 \;+\; \frac{\phi_r^2}{2}\times\bigl(\partial_\mu\Theta+qA_\mu\bigr)^2 \\ &=\; \tfrac{1}{2}(\partial_\mu\sigma)^2 \;+\; \frac{(v+\sigma)^2}{2}\times\bigl(\partial_\mu\Theta+qA_\mu\bigr)^2. \end{aligned} \tag{10}$$

Altogether,

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu\sigma)^2 \;-\; V(\sigma) \;-\; \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \frac{(v+\sigma)^2}{2}\times\bigl(\partial_\mu\Theta+qA_\mu\bigr)^2. \tag{11}$$

To understand the physical content of this Lagrangian, let's expand it in powers of the fields (and their derivatives) and focus on the quadratic part describing the free particles,

$$\mathcal{L}_{\text{free}} = \tfrac{1}{2}(\partial_\mu\sigma)^2 - \frac{\lambda v^2}{2}\times\sigma^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{v^2}{2}\times\big(qA_\mu + \partial_\mu\Theta\big)^2. \tag{12}$$

The first two terms here obviously describe a real scalar particle of positive mass$^2 = \lambda v^2$. The other two terms — involving the $A_\mu(x)$ and the $\Theta(x)$ fields — seem to describe a photon and a scalar field, but in fact describe a massive vector field and no scalars!

To see how this works, note the *local* $U(1)$ symmetry of the theory, which acts as

$$\begin{aligned}
A'_\mu(x) &= A_\mu(x) - \partial\Lambda(x),\\
\Phi'(x) &= \Phi(x)\times\exp(iq\Lambda(x)),\\
\sigma'(x) &= \sigma(x),\\
\Theta'(x) &= \Theta(x) + q\Lambda(x),
\end{aligned} \tag{13}$$

for an arbitrary x-dependent $\Lambda(x)$. Physically, such a local symmetry means that one of the 6 field variables at each x — the real and the imaginary parts of the $\Phi(x)$, and the 4 components of the $A^\mu(x)$ — is redundant, and we may reduce this redundancy by imposing a gauge-fixing condition such as the Coulomb gauge $\nabla\cdot\mathbf{A}(x)\equiv 0$ or the Landau gauge $\partial_\mu A^\mu(x)\equiv 0$. When we have a charged scalar field with a non-zero VEV, we may also impose a gauge-fixing condition on that scalar field (instead of the vector field $A^\mu(x)$), thus *the unitary gauge*

$$\Theta(x) = \text{phase}\big(\Phi(x)\big) = 0 \quad \forall x. \tag{14}$$

The unitary gauge is badly singular when the complex field $\Phi(x)$ fluctuates around zero, so it should never be used for the gauge symmetries which are NOT spontaneously broken. But when the symmetry IS spontaneously broken by $\langle\Phi\rangle\neq 0$ and the points where $\Phi(x)$ vanishes are few and far between (if they exist at all), the phase $\Theta(x)$ is well-defined almost everywhere, and it is easy to gauge it away by setting $\Lambda(x) = (-1/q)\Theta(x) \implies \Theta'(x) = 0$.

In the unitary gauge, the last two terms in the free Lagrangian (12) become simply

$$\mathcal{L}^{\text{massive}}_{\text{vector}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2v^2}{2}\times A_\mu A^\mu, \tag{15}$$

the Lagrangian of a massive vector field of mass $m_v = qv$. The scalar $\Theta(x)$ is gone from this Lagrangian — it was eliminated by the unitary gauge fixing. For the same reason, the

Lagrangian (15) is NOT gauge invariant — we have used up the gauge symmetry of the original theory for eliminating the $\Theta(x)$ field, and now the remaining $A^\mu(x)$ field does not have any gauge symmetry anymore.

Without the unitary gauge — or any other gauge-fixing condition — we may describe exactly the same massive vector particles using *redundant fields* $A^\mu(x)$ and $\Theta(x)$ subject to gauge symmetry

$$A'_\mu(x) \;=\; A_\mu(x) \;-\; \partial_\mu\Lambda(x), \quad \Theta'(x) \;=\; \Theta(x) \;+\; q\Lambda(x), \tag{16}$$

and a gauge-invariant free Lagrangian

$$\mathcal{L}^{\text{massive}}_{\text{vector}} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \frac{(qv)^2}{2}\times\bigl(A_\mu \,+\, q^{-1}\partial_\mu\Theta\bigr)^2. \tag{17}$$

But the $\Theta(x)$ field here is not physical, it does not give rise to any scalar particles, and its plane waves are mere gauge artefacts. The only physical particles in this system are the massive vector particles, the same as in the Θ-less unitary-gauge Lagrangian (15).

Altogether, the complete particle spectrum of the theory of $\Phi(x)$ and $A^\mu(x)$ fields with a spontaneously-broken local $U(1)$ symmetry comprises a massive real scalar $\sigma(x)$ and a massive vector. But there is NO massless Goldstone scalar!

To see what happened to the would-be Goldstone boson, let's count the degrees of freedom of the complete theory. The complex scalar field $\Phi(x)$ carries 2 degrees of freedom, while the vector field $A_\mu(x)$ subject to gauge symmetry carries another 2 DoF, for the total of 4 DoF. This means that for every momentum 3-vector $\mathbf{k}$, there should be 4 distinct 1-particle states $|\mathbf{k},??\rangle$ belonging to different particle species or different spin/polarization states. This counting should work for both spontaneously-broken or unbroken $U(1)$ symmetry, although the specific 1-particle states turn out to be quite different for the two regimes:

- The unbroken $U(1)$ regime for $m^2>0$ and $\langle\Phi\rangle=0$:
  In this regime, the $A^\mu(x)$ fields describe a massless photon, which has 2 helicity states, $\lambda=\pm1$ (but not $\lambda=0$). At the same time, the complex scalar field $\Phi(x)$ with an unbroken $U(1)$ symmetry describes 2 scalar particle species with opposite electric charges $\pm q$, the particle and the antiparticle. Altogether, for each $\mathbf{k}$ there are 4 one-particle

states: the scalar particle $\left|S^+\right\rangle$, the antiparticle $\left|S^-\right\rangle$, and two photon states $|\gamma(\lambda=+1)\rangle$ and $|\gamma(\lambda=-1)\rangle$.

- The spontaneously-broken $U(1)$ regime for $m^2<0$ and $\langle\Phi\rangle\neq 0$:
  In this regime, there is only one scalar particle species σ, but the massive vector particle has 3 spin states, $\lambda=-1,0,+1$. Again, altogether there are 4 one-particle states: the $|\sigma\rangle$, and the $|V(\lambda=+1)\rangle$, $|V(\lambda=0)\rangle$, $|V(\lambda=-1)\rangle$.
  - $\star$ But these are rather different 4 states from the unbroken $U(1)$ regime!

Now we can see what happens in the spontaneously-broken regime to the would-be Goldstone boson $\Theta(x)$: It became the longitudinal $\lambda=0$ polarization of the massive vector field! Indeed, the unbroken-symmetry regime has a massless vector without the $\lambda=0$ mode. Once the symmetry is spontaneously broken and the vector becomes massive, it has to have all 3 spin states, including the $\lambda=0$ longitudinal mode. That mode has to come from somewhere, so the Higgs mechanism 'eats up' the would-be Goldstone scalar $\Theta(x)$ and turns it into the longitudinal polarization of the massive vector!

A rigorous way to see how this works would be to start with the redundant gauge-invariant description (16) of the massive vector field, fix the Coulomb gauge $\nabla\cdot\mathbf{A}=0$ instead of the unitary gauge, expand the Lagrangian (17) into Fourier and helicity modes, eliminate the modes of the A^0 field, quantize the theory canonically, and in the process see how the $\hat{\Theta}_{\mathbf{k}}$ and the $\hat{\Pi}^{\Theta}_{\mathbf{k}}$ operators combine into the creation and annihilation operators for the longitudinally polarized vector particles. But this is a lot of work, and I am not going to do it here. Instead, I let the unitary gauge speak for the outcome of the Higgs mechanism, even if it hides the gory details of the 'eating up the Goldstone boson'.

To complete this section, let me write down the complete Lagrangian of the spontaneously-broken theory in the unitary gauge, including all the interactions of the $\sigma(x)$ fields with itself and with the massive vector field:

$$\begin{aligned}\mathcal{L} \;=\; & \tfrac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{\lambda v^2}{2}\times\sigma^2 \;-\; \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \frac{q^2v^2}{2}\times A_\mu A^\mu \\ & -\;\frac{\lambda v}{2}\times\sigma^3 \;-\; \frac{\lambda}{8}\times\sigma^4 \;+\; qv^2\times\sigma\,A_\mu A^\mu \;+\; \frac{q^2}{2}\times\sigma^2\,A_\mu A^\mu.\end{aligned} \tag{18}$$

Propagators and Gauges for Massive Vectors

The unitary gauge is very useful for identifying the physical particles of the theory. This will be particularly convenient in the Higgsed non-abelian gauge theories, as we shall see later in these notes. On the other hand, beyond the tree level of the perturbation theory, the unitary gauge makes the ultraviolet divergences of the loop graphs much worse than they would be in the theory with massless vector fields. To avoid this problem, the perturbation theory beyond the tree level is usually done using non-unitary Feynman-like R_ξ gauges I shall explain in a minute.

To see the problem with loop diagrams, consider the massive photon's propagator. In the unitary gauge, the massive photon is simply an ordinary massive vector field with free Lagrangian (15), so its propagator is just the massive vector propagator from homework#4,

$$= \frac{i}{k^2 - m^2 + i0}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right) \quad \text{for } m = qv. \tag{19}$$

Note that at large off-shell momenta $k \gg m$ this propagator behaves as $O(1/m^2)$, unlike the massless photon's propagator which behaves as $O(1/k^2)$. So when the propagator (19) is a part of some loop and we integrate over the loop momentum, we get

$$\int \frac{d^4k}{(2\pi)^4} \begin{pmatrix} \text{massive photon} \\ \text{propagator} \end{pmatrix} \times \begin{pmatrix} \text{other propagators} \\ \text{and vertices} \end{pmatrix}, \tag{20}$$

the integrand decreases for $k^\mu \to \infty$ much slower than for a similar diagram involving a massless photon. Consequently, the integral (20) generally suffers from much worse ultraviolet divergence — the divergence for $k \to \infty$ — than a similar integral in a massless QED. In the QFT (II) class we shall learn how to handle the UV divergences in QED and other *renormalizable* theories. The worse divergences due to the massive photon propagator (19) in the unitary gauge would break the renormalization techniques and make the massive theory non-renormalizable. In other words, it would be a good effective low-energy theory for the tree-level calculations but unsuitable for the loop calculations.

Fortunately, the problem is not with the massive photons *per se* but rather with the unitary gauge. Other gauges exist where the massive photon propagator behaves like $O(1/k^2)$ for large off-shell momenta, and in those gauges the UV divergences of the massive theory are similar to its massless counterpart.

Indeed, consider a non-unitary gauge in which the would-be Goldstone scalar $\pi(x) = \operatorname{Im}\Phi(x)$ is not restricted at all; instead, we impose a Landau gauge condition $\partial_\mu A^\mu(x) \equiv 0$ on the vector field itself. Or rather, impose a Landau-like gauge condition

$$\partial_\mu A^\mu(x) \;-\; \xi m\pi(x) \;=\; \text{given } f(x), \tag{21}$$

and then in the path-integral formulation of the theory (which we shall learn in the QFT II class), we *average* over all possible $f(x)$. The result of such gauge-averaging is equivalent to not imposing any gauge condition at all but rather adding a gauge-fixing term

$$\Delta\mathcal{L} \;=\; -\frac{1}{2\xi}\Big(\partial_\mu A^\mu \;-\; \xi m\pi\Big)^2 \tag{22}$$

to the Lagrangian of the theory. For a massless theory (no Higgs mechanism), the net quadratic Lagrangian

$$\mathcal{L}_2 \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \Delta\mathcal{L} \tag{23}$$

yields the Feynman gauge (or Feynman-like gauges for $\xi \neq 1$) for the photon propagator

$$\mu \sim\!\sim\!\sim\!\sim\!\sim \nu \;=\; \frac{i}{k^2+i0}\left(-g^{\mu\nu} \;+\; (1-\xi)\frac{k^\mu k^\nu}{k^2+i0}\right). \tag{24}$$

For the massive photon due to Higgs mechanism, we combine the vector field $A^\mu(x)$ and the would-be Goldstone scalar $\pi(x)$ with free Lagrangian

$$\begin{aligned}\mathcal{L}_2 \;&=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}(\partial_\mu\pi)^2 \;+\; \Delta\mathcal{L} \\ &=\; -\tfrac{1}{2}(\partial_\mu A_\nu)^2 \;+\; \tfrac{1}{2}(1-\xi^{-1})(\partial_\lambda A^\lambda)^2 \;+\; \tfrac{1}{2}m^2 A_\mu A^\mu \;+\; \tfrac{1}{2}(\partial_\mu\pi)^2 \;-\; \tfrac{1}{2}\xi m^2\pi^2.\end{aligned} \tag{25}$$

Consequently, we get the so-called R_ξ gauge in which the massive vector field has propagator

$$\mu \sim\!\sim\!\sim\!\sim\!\sim \nu \;=\; \frac{i}{k^2-m^2+i0}\left(-g^{\mu\nu} \;+\; (1-\xi)\frac{k^\mu k^\nu}{k^2-\xi m^2+i0}\right) \;\xrightarrow[\xi=1]{}\; \frac{-ig^{\mu\nu}}{k^2-m^2+i0}. \tag{26}$$

Unlike the unitary gauge propagator (19), this propagator behaves as $1/k^2$ for large off-shell momenta k, so the renormalization works similar to QED. (Although the technical aspects of renormalization are much more complicated in the Higgsed case, and it took much longer to prove the theory's renormalizability.)

The price of a non-unitary gauge like R_ξ is that the unphysical would-be Goldstone field $\pi(x)$ is not eliminated for the theory. Instead, it remains in the theory, couples to the physical Higgs scalar $\sigma(x)$, and perhaps to the other fields (if the theory has any). In Feynman rules, the $\pi(x)$ has a propagator

$$\cdots\cdots\cdots\cdots\cdots\cdots \;=\; \frac{i}{k^2 - \xi m^2 + i0}\,. \tag{27}$$

Note the pole at $k^2 = \xi m^2$, which is not the physical mass^2 of any particle unless $\xi = 1$; this spurious pole is needed to cancel the other unphysical effects of the massive photon propagator (26).

To summarize this section, loop calculations requires gauges like R_ξ where the Feynman rules involve both physical and unphysical fields. However, the tree-level calculation can also be done in the unitary gauge which involves only the physical fields. And the semi-classical calculations where we identify the particles and find their masses — as we shall do in a moment for the non-abelian Higgs mechanism — are best done in the unitary gauge.

Non-Abelian Higgs Mechanism

Example: SU(2) with a Higgs Doublet

To illustrate the non-abelian Higgs mechanism, consider the example of $SU(2)$ gauge theory coupled to a doublet of complex scalar fields $\Phi^i(x)$. In terms of canonically normalized fields, the Lagrangian is

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; D_\mu\Phi^*_i D^\mu\Phi^i \;-\; \frac{\lambda}{2}\left(\Phi^*_i\Phi^i \;-\; \frac{v^2}{2}\right)^2, \tag{28}$$

where

$$\begin{aligned} D_\mu\Phi^i \;&=\; \partial_\mu\Phi^i \;+\; \tfrac{i}{2}gA^a_\mu\big(\sigma^a\big)^i{}_j\Phi^j\,,\\ D_\mu\Phi^*_i \;&=\; \partial_\mu\Phi^*_i \;-\; \tfrac{i}{2}gA^a_\mu\Phi^*_j\big(\sigma^a\big)^j{}_i\,,\\ F^a_{\mu\nu} \;&=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g\epsilon^{abc}A^b_\mu A^c_\nu\,. \end{aligned} \tag{29}$$

For $v^2 > 0$ the scalar potential has a local maximum at $\Phi^i = 0$ while the minima form a spherical shell $\Phi^*_i\Phi^i = (v^2/2)$ in the $\mathbf{C}^2 = \mathbf{R}^4$ field space; all such minima are related by the

$SU(2)$ symmetries to

$$\langle\Phi\rangle \;=\; \frac{v}{\sqrt{2}}\times\begin{pmatrix}0\\1\end{pmatrix}. \tag{30}$$

Note that this vacuum expectation value spontaneously breaks the $SU(2)$ symmetry down to nothing — there is no subgroup of $SU(2)$ which leaves this VEV invariant. Consequently, we expect all 3 vector fields $A^a_\mu(x)$ to become massive.

In the process, 3 would-be Goldstone scalars should be eaten by the Higgs mechanism. Since the theory has 2 complex — or equivalently 4 real — scalars, only one real scalar should survive un-eaten. Ironically, it is this un-eaten scalar $\sigma(x)$ which is called *the physical Higgs field.*

To see how this works, let's fix the unitary gauge. Any complex doublet $\Phi^i(x)$ can be rotated by some $SU(2)$ symmetry $U(x)$ so that the upper component of the rotated $\Phi' = U\Phi$ is zero, $\Phi'^1 = 0$, while the lower component Φ'^2 is real and positive. Thus, in the unitary gauge we require

$$\begin{gathered}\operatorname{Re}\Phi^1(x) \;\equiv\; \operatorname{Im}\Phi^1(x) \;\equiv\; \operatorname{Im}\Phi^2(x) \;\equiv\; 0,\\ \text{hence}\quad \Phi(x) \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \phi_r(x)\end{pmatrix}\quad \text{for a real } \phi_r(x)>0.\end{gathered} \tag{31}$$

This gauge-fixing condition is terribly singular for $\phi_r \to 0$, so it should never be used for the unbroken-symmetry regime of the theory. But for the spontaneously broken theory where $\phi_r(x)$ fluctuates around the minimum at $\phi_r = v > 0$, the unitary gauge is OK.

In the unitary gauge, the only scalar field is the $\phi_r(x)$, or equivalently the shifted field $\sigma(x) = \phi_r(x) - v$; all the other scalar fields are frozen by the gauge-fixing conditions (31). In terms of physical Higgs field $\sigma(x)$, the scalar potential becomes

$$V \;=\; \frac{\lambda}{2}\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 \;=\; \frac{\lambda}{8}\left(2v\sigma+\sigma^2\right)^2 \;=\; \frac{\lambda v^2}{2}\times\sigma^2 + \frac{\lambda v}{2}\times\sigma^3 \;+\; \frac{\lambda}{8}\times\sigma^4 \tag{32}$$

where the first terms is the mass term, $\text{mass}^2 = \lambda v^2$, while the remaining terms are selfinteractions. More interestingly, the covariant derivative of the Higgs doublet Φ becomes

$$D_\mu \Phi \;=\; \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\ \partial_\mu\sigma\end{pmatrix} + \frac{ig}{2}A^3_\mu \times \begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ v+\sigma\end{pmatrix} \right.$$
$$\left. +\; \frac{ig}{2}A^1_\mu \times \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ v+\sigma\end{pmatrix} + \frac{ig}{2}A^2_\mu \times \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\ v+\sigma\end{pmatrix}\right]$$
$$=\; \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{i}{2}g\bigl(A^1_\mu - iA^2_\mu\bigr)\times(v+\sigma)\\ \partial_\mu\sigma \,-\, \frac{i}{2}g\,A^3_\mu\times(v+\sigma)\end{pmatrix}, \tag{33}$$

hence

$$D_\mu\Phi^\dagger\, D^\mu\Phi \;=\; \tfrac{1}{2}\left|\tfrac{i}{2}g\bigl(A^1_\mu - iA^2_\mu\bigr)\times(v+\sigma)\right|^2 \;+\; \tfrac{1}{2}\left|\partial_\mu\sigma \,-\, \tfrac{i}{2}g\,A^3_\mu\times(v+\sigma)\right|^2$$
$$=\; \frac{g^2(v+\sigma)^2}{8}\times\Bigl(\bigl(A^1_\mu\bigr)^2 \,+\, \bigl(A^2_\mu\bigr)^2\Bigr) \;+\; \frac{g^2(v+\sigma)^2}{8}\times\bigl(A^3_\mu\bigr)^2 \;+\; \tfrac{1}{2}(\partial_\mu\sigma)^2. \tag{34}$$

The last term here is the kinetic term for the Higgs scalar $\sigma(x)$, while the rest of the bottom line are mass terms for the vector fields and the interaction terms between the vectors and the σ. Curiously, we get the same mass and similar interactions for all 3 vector fields A^a_μ:

$$\mathcal{L} \;\supset\; \frac{g^2(v+\sigma)^2}{8}\,A^a_\mu A^{a\mu} \;=\; \frac{M^2}{2}\times A^a_\mu A^{a\mu} \;+\; \frac{g^2 v}{4}\times\sigma A^a_\mu A^{a\mu} \;+\; \frac{g^2}{8}\times\sigma^2 A^a_\mu A^{a\mu} \tag{35}$$

where

$$M^2 \;=\; \frac{g^2v^2}{4}\,. \tag{36}$$

Example: SU(2) with a Higgs Triplet

Now consider an example of a partially broken gauge symmetry, an $SU(2)$ Higgsed down to a $U(1)$ subgroup, or equivalently $SO(3)\to SO(2)$. This time, the scalar fields $\Phi^a(x)$ are real and form a triplet of the $SU(2)$ rather than a doublet. Thus,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; \tfrac{1}{2}D_\mu\Phi^a D^\mu\Phi^a \;-\; \frac{\lambda}{8}\bigl(\Phi^a\Phi^a \,-\, v^2\bigr)^2, \tag{37}$$

where

$$D_\mu\Phi^a \;=\; \partial_\mu\Phi^a \;-\; g\epsilon^{abc}A^b_\mu\Phi^c\,, \qquad F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; g\epsilon^{abc}A^b_\mu A^c_\nu\,. \tag{38}$$

Again, for $v^2>0$ the scalar potential $V(\Phi)$ has a degenerate family of minima which form a spherical shell $\Phi^a\Phi^a = v^2$ in the scalar field space $\mathbf{R}^3$, and all such minima are equivalent by $SU(2) \cong SO(3)$ symmetries to

$$\langle\Phi\rangle \;=\; \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}. \tag{39}$$

This time, this vacuum expectation value is invariant under an $SO(2)$ subgroup of the $SO(3)$, — or equivalently under an $U(1)$ subgroup of the $SU(2)$. Specifically, it's the $SO(2) \cong U(1)$ generated by the T^3, the third component of the isospin $\mathbf{T}$. Consequently, out of the 3 vector fields A^a_μ, we expect the A^3_μ to remain massless while the other 2 fields $A^{1,2}_\mu$ should become massive.

In the process, the Higgs mechanism should eat 2 real scalar fields. Since we only have 3 real scalars to begin with, only one scalar should survive un-eaten — the Physical Higgs field $\sigma(x)$.

To see how this works, we fix the unitary gauge

$$\Phi^1(x) \;\equiv\; \Phi^2(x) \;\equiv\; 0, \qquad \Phi^3(x) \;>\; 0. \tag{40}$$

As usual, this gauge is badly singular at $\Phi = 0$, but it's OK for the $\Phi(x) \approx \langle\Phi\rangle \neq 0$. Shifting the $\Phi^3(x)$ by the VEV, we get $\Phi^3(x) = v + \sigma(x)$, where $\sigma(x)$ is the physical Higgs scalar — and also the only scalar remaining in the theory in the unitary gauge.

In terms of the $\sigma(x)$, the scalar potential becomes

$$V(\sigma) \;=\; \frac{\lambda}{8}\left(2v\sigma + \sigma^2\right)^2 \;=\; \frac{\lambda v^2}{2} \times \sigma^2 + \frac{\lambda v}{2} \times \sigma^3 \;+\; \frac{\lambda}{8} \times \sigma^4, \tag{41}$$

where the first term on the RHS gives the Higgs scalar mass$^2 = \lambda v^2$. More interestingly, the covariant derivative of the scalar triple $\Phi^a(x)$ becomes

$$D_\mu \Phi^a = \begin{pmatrix} 0 \\ 0 \\ \partial_\mu \sigma \end{pmatrix} - g \begin{pmatrix} A^1_\mu \\ A^2_\mu \\ A^3_\mu \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ v+\sigma \end{pmatrix}$$

$$\langle\!\langle \text{ where } \times \text{ is the cross product of two isovectors } \rangle\!\rangle \tag{42}$$

$$= \begin{pmatrix} -gA^2_\mu(v+\sigma) \\ +gA^1_\mu(v+\sigma) \\ \partial_\mu\sigma \end{pmatrix},$$

hence the covariant kinetic terms for the scalars become

$$\tfrac{1}{2} D_\mu \Phi^a D^\mu \Phi^a = \tfrac{1}{2}(\partial_\mu \sigma)^2 + \frac{g^2(v+\sigma)^2}{2} \times \Big(\big(A^1_\mu\big)^2 + \big(A^2_\mu\big)^2 \Big). \tag{43}$$

As usual, the first term here is the kinetic term for the physical Higgs scalar σ, while the second term contains the mass terms for the vector fields,

$$\mathcal{L} \supset \frac{M^2}{2} \times \Big(\big(A^1_\mu\big)^2 + \big(A^2_\mu\big)^2 \Big), \quad M^2 = g^2 v^2, \tag{44}$$

but only for the A^1_μ and the A^2_μ — the third vector $A^3_\mu(x)$ remains massless.

The massless vector $A^3_\mu(x)$ is the gauge field of the un-Higgsed $SO(2) \cong U(1)$ subgroup of the $SO(3) \cong SU(2)$. Interpreting this gauge field as the EM field and hence the rescaled generator $Q = gT^3$ as the electric charge operator, we find that the physical Higgs field is electrically neutral while the massive vector fields have electric charges $q = \pm g$. To be precise, the massive vector fields of definite charges are not the A^1_μ and the A^2_μ themselves but rather their linear combination

$$W^+_\mu = \frac{1}{\sqrt{2}}\big(A^1_\mu - iA^2_\mu\big) \quad \text{and} \quad W^-_\mu = \frac{1}{\sqrt{2}}\big(A^1_\mu + iA^2_\mu\big) \quad \text{of charges } q = \pm g. \tag{45}$$

For completeness sake, let's re-express the theory at hand (usually called the *Georgi–Glashow* *model*) in terms of the physical fields of definite charges. Using $U(1)$–covariant derivatives

$$\widetilde{D}_\mu W^\pm_\nu \;=\; \partial_\mu W^\pm_\nu \;\pm\; igA^3_\mu W^\pm_\nu\,, \tag{46}$$

we have

$$W^\pm_{\mu\nu} \;\stackrel{\rm def}{=}\; \frac{1}{\sqrt{2}}\bigl(F^1_{\mu\nu} \mp iF^2_{\mu\nu}\bigr) \;=\; \widetilde{D}_\mu W^\pm_\nu \;-\; \widetilde{D}_\nu W^\pm_\mu\,, \tag{47}$$

but

$$F^3_{\mu\nu} \;=\; \widetilde{F}_{\mu\nu} \;+\; 2g\,{\rm Im}\bigl(W^+_\mu W^-_\nu\bigr) \quad \text{where} \quad \widetilde{F}_{\mu\nu} \;=\; \partial_\mu A^3_\nu \;-\; \partial_\nu A^3_\mu\,. \tag{48}$$

Consequently, the Lagrangian of the whole model — the kinetic terms, the mass terms, and the interactions — can be expressed as

$$\begin{aligned}\mathcal{L} \;=&\; \tfrac{1}{2}(\partial_\mu\sigma)^2 \;-\; \tfrac{1}{2}M^2_\sigma\times\sigma^2 \;-\; \tfrac{1}{4}\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu} \;-\; \tfrac{1}{2}W^+_{\mu\nu}W^{-\mu\nu} \;+\; M^2_W W^+_\mu W^{-\nu}\\ &-\; \frac{\lambda v}{2}\times\sigma^3 \;-\; \frac{\lambda}{8}\times\sigma^4 \;+\; 2gv\times\sigma\times W^+_\mu W^{-\mu} \;+\; g^2\times\sigma^2\times W^+_\mu W^{-\mu}\\ &-g\times\widetilde{F}_{\mu\nu}\times{\rm Im}\bigl(W^{+\mu}W^{-\nu}\bigr) \;-\; g^2\times\Bigl({\rm Im}\bigl(W^{+\mu}W^{-\nu}\bigr)\Bigr)^2.\end{aligned} \tag{49}$$

General Case

Let's take a closer look at eqs. (34) and (43), and focus on the mass terms for the vector fields. In both cases, we start with the kinetic terms for the original scalar fields $\Phi_i(x)$ or $\Phi^a(x)$, fix the unitary gauge, work through the algebra, and eventually obtain the kinetic term for the physical Higgs field σ, the mass terms for the vector fields — or some of the vector fields — and the interactions between the massive vectors and the Higgs σ. But if all we want are the mass terms for the vectors, we may simply freeze $\sigma(x) \equiv 0$: This would eliminate the interactions with the σ as well as the $\frac{1}{2}(\partial_\mu\sigma)^2$ term, and all we would have left are the mass terms for the massive vectors.

Note that freezing $\sigma(x) \equiv 0$ is equivalent to freezing all the scalars at their vacuum expectation values, $\Phi(x) \equiv \langle\Phi\rangle$. Consequently, to get the vector's masses we do not need to go through the details of the unitary gauge fixing, all we need are the scalar VEVs, then the kinetic terms for the frozen scalars

$$(D_\mu\langle\Phi\rangle)^\dagger(D_\mu\langle\Phi\rangle) \quad \text{or} \quad \tfrac{1}{2}(D_\mu\langle\Phi\rangle)^2$$

become the mass terms for the vectors. For example, for the $SO(3)$ triplet of real scalar fields from the second example

$$D_\mu\langle\Phi\rangle^a \;=\; -g\epsilon^{abc}A^b_\mu \times v\delta^{c3} \;=\; -gv\epsilon^{ab3}\times A^b_\mu, \tag{50}$$

$$\begin{aligned}\mathcal{L}^{\text{vector}}_{\text{mass}} &= \tfrac{1}{2}\big(D_\mu\langle\Phi\rangle^a\big)^2 \;=\; \tfrac{1}{2}(gv)^2\times\epsilon^{ab3}\epsilon^{ac3}A^b_\mu A^{c\mu}\\ &= \tfrac{1}{2}\big(M \;=\; gv\big)^2\times\big(A^1_\mu A^{1\mu} \,+\, A^2_\mu A^{2\mu}\big).\end{aligned} \tag{51}$$

Likewise, for the $SU(2)$ doublet of complex scalar fields from the first example,

$$D_\mu\langle\Phi\rangle^i \;=\; \frac{ig}{2}\left(A^a_\mu\sigma^a\right)^i{}_j\times\frac{v}{\sqrt{2}}\,\delta^j_2 \;=\; \frac{igv}{2\sqrt{(2)}}\times\left(A^a_\mu\sigma^a\right)^i{}_2\,, \tag{52}$$

$$D_\mu\langle\Phi\rangle^*_i \;=\; -\frac{igv}{2\sqrt{2}}\times\left(A^a_\mu\sigma^a\right)^2{}_i\,, \tag{53}$$

$$\begin{aligned}\mathcal{L}^{\text{vector}}_{\text{mass}} &= D_\mu\langle\Phi\rangle^*_i\; D^\mu\langle\Phi\rangle^i \;=\; \frac{g^2v^2}{8}\times\left(A^a_\mu\sigma^a\right)^2{}_i\left(A^{b\mu}\sigma^b\right)^i{}_2\\ &= \frac{g^2v^2}{8}\times A^a_\mu A^{b\mu}\times\Big[\left(\sigma^a\sigma^b\right)^2{}_2 \;=\; \delta^{ab} \,-\, i\epsilon^{ab3}\Big]\\ &= \frac{g^2v^2}{8}\times A^a_\mu A^{b\mu}\times\delta^{ab} \qquad \langle\!\langle\,\text{since } A^a_\mu A^{b\mu} \text{ is symmetric in } a\leftrightarrow b.\,\rangle\!\rangle\\ &= \frac{M^2}{2}\times A^a_\mu A^{a\mu} \qquad \text{for } M \;=\; \frac{gv}{2}\,.\end{aligned} \tag{54}$$

This recipe — freezing $\Phi(x)\equiv\langle\Phi\rangle$ to find the vector masses — applies to any kind of gauge theory with scalars in any kinds of multiplets. Indeed, consider a general gauge symmetry G with generators $\hat{T}^a$ and gauge fields $A^a_\mu(x)$ ($a = 1,\ldots,\dim(G)$). Let scalars $\Phi^\alpha(x)$ belonging to some multiplet (m) of G develop non-zero vacuum expectation values $\langle\Phi^\alpha\rangle \neq 0$. Then the covariant derivatives of these scalars

$$D_\mu\Phi^\alpha(x) \;=\; \partial_\mu\Phi^\alpha(x) \;+\; igA^a_\mu(x)\times\left(T^a_{(m)}\right)^\alpha_{\;\beta}\Phi^\beta(x) \tag{55}$$

become in the unitary gauge

$$D_\mu\Phi^\alpha(x) \;=\; D_\mu\langle\Phi\rangle^\alpha \;+\; \text{terms involving the physical scalars} \tag{56}$$

where

$$D_\mu\langle\Phi\rangle^\alpha \;=\; igA^a_\mu(x)\times\left(T^a_{(m)}\right)^\alpha_{\;\beta}\langle\Phi\rangle^\beta\,. \tag{57}$$

In eq. (56), the terms involving the physical scalars — and the physical scalar fields themselves — depend on the details of the unitary gauge fixing. On the other hand, the covariant derivatives of the VEVs (57) depend only on the VEVs themselves. Moreover, such derivatives are linear functions of the vector fields with constant coefficients, so their squares become quadratic mass terms for the vectors,

$$\begin{aligned} D^\mu\langle\Phi\rangle^*_\alpha\,D_\mu\langle\Phi\rangle^\alpha \;&=\; -igA^a_\mu\times\langle\Phi\rangle^*_\beta\left(T^a_{(m)}\right)^\beta_{\;\alpha}\times igA^{b\mu}\times\left(T^a_{(m)}\right)^\alpha_{\;\gamma}\langle\Phi\rangle^\gamma \\ &=\; A^a_\mu A^{b\mu}\times g^2\,\langle\Phi\rangle^*_\beta\left(T^a_{(m)}T^b_{(m)}\right)^\beta_{\;\gamma}\langle\Phi\rangle^\gamma \\ &\qquad\langle\!\langle\text{ by } a\leftrightarrow b \text{ symmetry of the } A^a_\mu A^{b\mu}\,\rangle\!\rangle \\ &=\; \tfrac{1}{2}A^a_\mu A^{b\mu}\times g^2\,\langle\Phi\rangle^*_\beta\left\{T^a_{(m)},T^b_{(m)}\right\}^\beta_{\;\gamma}\langle\Phi\rangle^\gamma\,. \end{aligned} \tag{58}$$

In other words,

$$\mathcal{L}^{\text{vector}}_{\text{masses}} \;=\; \tfrac{1}{2}\left(M^2_V\right)^{ab}\times A^a_\mu A^{b\mu}\,, \tag{59}$$

where the mass^2 matrix for the gauge fields obtains as

$$\left(M^2_V\right)^{ab} \;=\; g^2\,\langle\Phi\rangle^\dagger_\beta\left\{T^a_{(m)},T^b_{(m)}\right\}^\beta_{\;\gamma}\langle\Phi\rangle^\gamma \;\equiv\; g^2\,\langle\Phi\rangle^\dagger\left\{T^a_{(m)},T^b_{(m)}\right\}\langle\Phi\rangle\,. \tag{60}$$

To be precise, eq. (60) applies to Higgs VEVs belonging to a single multiplet of complex scalars. For a multiplet of real scalars, there is an extra factor $\frac{1}{2}$ due to different normalization of the VEVS, and for several Higgs multiplets with non-zero VEVs, the general formula is

$$\left(M_V^2\right)^{ab} \;=\; g^2 \sum_{\Phi\in(m)}^{\substack{\text{complex}\\ \text{Higgs}\\ \text{multiplets}}} \langle\Phi\rangle^\dagger \bigl\{T^a_{(m)},T^b_{(m)}\bigr\}\langle\Phi\rangle \;+\; g^2 \sum_{\Phi\in(m)}^{\substack{\text{real}\\ \text{Higgs}\\ \text{multiplets}}} \tfrac{1}{2}\,\langle\Phi\rangle^\top \bigl\{T^a_{(m)},T^b_{(m)}\bigr\}\langle\Phi\rangle\,. \tag{61}$$

In general, such mass^2 matrix is not diagonal, and we need to diagonalize in order to find the physical vector masses. For example, in the Glashow–Weinberg–Salam theory of the weak and EM interactions — it's explained in the next set of notes — the mass matrix mixes the $SU(2)$ gauge field W^3_μ and the $U(1)$ gauge field B_μ, and the mass eigenstates are the massless EM field A_μ and the massive neural field Z_μ involved in the weak interactions.

An additional complication of the GWS theory — or any other theory with non-simple gauge group $G = G_1\times G_2\times\cdots$ — are different gauge couplings g for different factors G. In this case, the g^2 factor in eq. (61) for the mass^2 matrix element $(M^2)^{ab}$ should be replaced with $g(a)\times g(b)$ where $g(a)$ is the coupling of the gauge group factor containing the generator T^a, and likewise for the $g(b)$. Thus, the most general formula for the vector mass matrix stemming from the Higgs mechanism is

$$\left(M_V^2\right)^{ab} \;=\; g(a)\,g(b)\times\left[\sum_{\Phi\in(m)}^{\substack{\text{complex}\\ \text{Higgs}\\ \text{multiplets}}} \langle\Phi\rangle^\dagger \bigl\{T^a_{(m)},T^b_{(m)}\bigr\}\langle\Phi\rangle \;+\; \frac{1}{2} \sum_{\Phi\in(m)}^{\substack{\text{real}\\ \text{Higgs}\\ \text{multiplets}}} \langle\Phi\rangle^\top \bigl\{T^a_{(m)},T^b_{(m)}\bigr\}\langle\Phi\rangle\right]. \tag{62}$$

In my notes on the GWS theory we shall see how this works in detail, and how the gauge couplings affect the eigenstates of the mass matrix.