Noether Theorem

Back in 1915, Emmy Noether proved the **theorem:** *For every generator of a continuous symmetry of a mechanical system there is a conserved quantity.* Eventually, the *Noether theorem* was generalized from classical mechanics to classical field theory, to quantum mechanics, and to quantum field theory. In these notes we shall focus on field theory where Noether theorem says that for every generator T^a of a continuous *global* symmetry of a field theory there is a conserved current J^μ_a, $\partial_\mu J^{\mu a} = 0$.

Let me illustrate the Noether theorem with an example: A classical theory of N real scalar fields $\Phi_a(x)$ ($a = 1, 2, \ldots, N$) with the Lagrangian density

$$\mathcal{L} \;=\; \frac{1}{2}\sum_a \partial_\mu\Phi_a\partial^\mu\Phi_a \;-\; \frac{m^2}{2}\sum_a(\Phi_a)^2 \;-\; \frac{\lambda}{4}\left(\sum_a(\Phi_a)^2\right)^2 . \tag{1}$$

This Lagrangian density — and hence the action $S = \int\mathcal{L}\,d^4x$ — is invariant under the $O(N)$ orthogonal transforms of the fields into each other,

$$\Phi'_a(x) \;=\; \sum_b R_{ab}\Phi_b(x) \quad \text{for } R \in O(N). \tag{2}$$

Indeed, orthogonality of the R_{ab} matrix, $R^\top R = 1$, implies that

$$\sum_a(\Phi'_a)^2 \;=\; \sum_a(\Phi_a)^2 \tag{3}$$

— which leaves the potential part of $\mathcal{L}$ invariant; and for a global symmetry where R_{ab} is the same for all x, the kinetic part of $\mathcal{L}$ is also invariant:

$$\partial_\mu\Phi'_a(x) \;=\; \sum_b R_{ab}\partial_\mu\Phi_b \quad\Longrightarrow\quad \sum_a\partial_\mu\Phi'_a\partial^\mu\Phi'_a \;=\; \sum_a\partial_\mu\Phi_a\partial^\mu\Phi_a\,. \tag{4}$$

The continuous subgroup of $O(N)$ is $SO(N)$ — the group of rotations in the N dimensional field space. The $SO(N)$ group is generated by the antisymmetric matrices $A_{ab} = -A_{ba}$.

Indeed, the infinitesimal $SO(N)$ rotations have form

$$\Phi'_a(x) \;=\; \Phi_a(x) \;+\; \delta\Phi_a(x), \qquad \delta\Phi_a(x) \;=\; \sum_b \epsilon_{ab}\Phi_b(x), \tag{5}$$

for infinitesimal ϵ_{ab}; in matrix form, this means $R = 1+\epsilon$. Orthogonality of such infinitesimal rotation matrix means

$$1 \;=\; R^\top R \;=\; 1 \;+\; \epsilon \;+\; \epsilon^\top \;+\; O(\epsilon^2), \tag{6}$$

hence in the infinitesimal $\epsilon \to 0$ limit

$$\epsilon^\top \;=\; -\epsilon \quad\iff\quad \epsilon_{ba} \;=\; -\epsilon_{ab}\,. \tag{7}$$

As to the finite $SO(N)$ rotations, any finite rotation R can be obtains as a sequence of n small-angle rotations $R^{1/n}$ which become infinitesimal for $n \to \infty$. Specifically,

$$\text{for } n\to\infty, \quad R^{1/n} \;=\; 1 \;+\; \left(\text{infinitesimal } \frac{A}{n}\right) \;+\; O(1/n^2) \tag{8}$$

for some antisymmetric matrix $A_{ab} = -A_{ba}$, hence

$$R \;=\; \lim_{n\to\infty}\left(1+\frac{A}{n}+O(1/n^2)\right)^n \;=\; \exp(A). \tag{9}$$

In other words, *every special orthogonal matrix R is a matrix exponential of some antisymmetric matrix A.*

In the field space, the infinitesimal transforms (5) act on functions $\mathcal{F}(\Phi)$ of the fields as

$$\delta\mathcal{F}(\Phi) \;=\; \sum_{ab}\epsilon_{ab}\Phi_b\frac{\partial}{\partial\Phi_a}\mathcal{F}(\Phi) \;=\; \frac{-i}{2}\sum_{ab}\epsilon_{ab}T_{ab}\mathcal{F}(\Phi) \tag{10}$$

where

$$T_{ab} \;=\; -T_{ba} \;=\; -i\Phi_a\frac{\partial}{\partial\Phi_b} \;+\; i\Phi_b\frac{\partial}{\partial\Phi_a} \tag{11}$$

are the operators in the field space *generating* the $SO(N)$ symmetries. For $N=3$, the $SO(3)$

symmetry can be identified as the *isospin* with 3 generators

$$T^a = -i\epsilon_{abc}\Phi_b \frac{\partial}{\partial\Phi_c} \quad \langle\!\langle \text{implicit } \textstyle\sum_{bc} \rangle\!\rangle \tag{12}$$

obeying angular-momentum-like commutation relations

$$[T_a, T_b] = i\epsilon_{abc}T_c \,. \tag{13}$$

But for $N > 3$, the cross product of two isovectors yield an antisymmetric iso-tensor rather than an isovectors hence $\frac{1}{2}N(N-1)$ generators (11). And their commutator algebra has a messier form that eq. (13); instead, we have

$$[T_{ab}, T_{cd}] = -i\delta_{bc}T_{ad} + i\delta_{ac}T_{bd} + i\delta_{bd}T_{ac} - i\delta_{ad}T_{bc} \,. \tag{14}$$

Coming back to the Noether theorem, for each generator T_{ab} of the $SO(N)$ symmetry we have a conserved current

$$J^\mu_{ab} = -J^\mu_{ba} = \Phi_a\partial^\mu\Phi_b - \Phi_b\partial^\mu\Phi_a \,. \tag{15}$$

Please note: The $SO(N)$ symmetries (2) leave the action invariant regardless of the fields obeying or disobeying any equations of motions. On the other hand, the classical currents (15) are conserved $\partial_\mu J^\mu_{ab} = 0$ only when the fields do obey their equations of motion. In particular, for the $SO(N)$ invariant Lagrangian density (1), we have

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a)} = \partial^\mu\Phi_a \quad \text{while} \quad \frac{\partial\mathcal{L}}{\partial(\Phi_a)} = -\Phi^a \times \left(m^2 + \lambda\sum_c \Phi_c^2\right), \tag{16}$$

hence the classical field equations are

$$\forall a = 1, \ldots, N: \quad \partial^2\Phi_a = -\Phi^a \times \left(m^2 + \lambda\sum_c \Phi_c^2\right). \tag{17}$$

Consequently, for the fields obeying these equations

$$\partial_\mu\bigl(\Phi_a\partial^\mu\Phi_b\bigr) \;=\; (\partial_\mu\Phi_a)(\partial^\mu\Phi_b) \,+\, \Phi_a(\partial^2\Phi_b) \;=\; (\partial_\mu\Phi_a)(\partial^\mu\Phi_b) \,-\, \Phi_a\Phi_b\times\left(m^2+\lambda\sum_c\Phi_c^2\right), \tag{18}$$

where both terms are symmetric WRT $a\leftrightarrow b$, and therefore the currents (15) are conserved,

$$\partial_\mu J^\mu_{ab} \;=\; \partial_\mu\bigl(\Phi_a\partial^\mu\Phi_b\bigr) \,-\, (a\leftrightarrow b) \;=\; 0, \tag{19}$$

In the quantum field theory, the classical currents $J^\mu_{ab}(x)$ become operators

$$\begin{aligned}
\hat{\mathbf{J}}_{ab}(\mathbf{x},t) &= -\hat{\mathbf{J}}_{ba}(\mathbf{x},t) \;=\; -\hat\Phi_a(\mathbf{x},t)\nabla\hat\Phi_b(\mathbf{x},t) \,+\, \hat\Phi_b(\mathbf{x},t)\nabla\hat\Phi_a(\mathbf{x},t),\\
\hat J^0_{ab}(\mathbf{x},t) &= -\hat J^0_{ba}(\mathbf{x},t) \;=\; \hat\Phi_a(\mathbf{x},t)\hat\Pi_b(\mathbf{x},t) \,-\, \hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\mathbf{x},t).
\end{aligned} \tag{20}$$

In particular, the net charges Q_{ab} become operators

$$\hat Q_{ab}(t) \;=\; -\hat Q_{ba}(t) \;=\int\! d^3\mathbf{x}\,\hat J^0_{ab}(\mathbf{x},t) \;=\int\! d^3\mathbf{x}\,\Bigl(\hat\Phi_a(\mathbf{x},t)\hat\Pi_b(\mathbf{x},t) \,-\, \hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\mathbf{x},t)\Bigr) \tag{21}$$

in the Hilbert space of the quantum theory. (Which is the Fock space of N species of identical spinless bosons.) In your next homework (set#4, problem 1), you will learn that these charge operators are conserved in the quantum way — they commute with the Hamiltonian operator $\hat H$. Moreover, the charges (21) obey the commutation relations (14) of the $SO(N)$ generators,

$$\Bigl[\hat Q_{ab},\hat Q_{cd}\Bigr] \;=\; -i\delta_{[c[b}\hat Q_{a]d]} \;\equiv\; -i\delta_{bc}\hat Q_{ad} \,+\, i\delta_{ac}\hat Q_{bd} \,+\, i\delta_{bd}\hat Q_{ac} \,-\, i\delta_{ad}\hat Q_{bc}\,, \tag{22}$$

and they act on the quantum fields $\hat\Phi_a(x)$ similarly to the classical generators T_{ab} acting on the classical fields,

$$\Bigl[\hat Q_{ab},\hat\Phi_c(x)\Bigr] \;=\; -i\delta_{bc}\hat\Phi_a(x) \,+\, i\delta_{ac}\hat\Phi_b(x), \tag{23}$$

$$\text{hence for } \hat V \;=\; \frac12\sum_{ab}A_{ab}\hat Q_{ab} \quad \langle\!\langle \text{ where } A_{ba}=-A_{ab}\,\rangle\!\rangle \tag{24}$$

$$\left[\hat{V},\hat{\Phi}_c\right] \;=\; -iA_{cd}\hat{\Phi}_d \quad \langle\!\langle\text{implicit }\textstyle\sum_d\rangle\!\rangle. \tag{25}$$

In the quantum theory, the operators $\hat{Q}_{ab}$ *represent* the symmetry generators T_{ab}, and that's why they must obey similar commutation relations. As to the finite $SO(N)$ 'rotations' of the field space, they are *represented by unitary operators*

$$\hat{U}(R) \;=\; \exp\left(\frac{-i}{2}\sum_{ab}A_{ab}\hat{Q}_{ab}\right) \quad \text{for } R \;=\; \exp(A), \tag{26}$$

and the similarity of the commutations relations of the generators T_{ab} and the charges $\hat{Q}_{ab}$ assures that

$$\hat{U}(R_2R_1) \;=\; \hat{U}(R_2)\hat{U}(R_1) \tag{27}$$

In the Schrödinger picture of the quantum theory, the symmetry operators (26)act on the quantum states

$$|\psi\rangle \;\to |\psi'\rangle \;=\; \hat{U}\,|\psi\rangle\,. \tag{28}$$

In your next homework#4, you will see how this symmetry acts on the multi-particle states: It rotates by R the species index of each particle but makes no other changes:

$$\hat{U}\,|n:(\mathbf{p}_1,a_1),\ldots,(\mathbf{p}_n,a_n)\rangle \;=\; \sum_{b_1,\ldots,b_n} R_{a_1,b_1}\cdots R_{a_n,b_n}\,|n:(\mathbf{p}_1,b_1),\ldots,(\mathbf{p}_n,b_n)\rangle\,. \tag{29}$$

In the Heisenberg picture, the symmetry operators leaves the quantum states as they are but instead they act on the operators as

$$\hat{\mathcal{O}}' \;=\; \hat{U}\hat{\mathcal{O}}\hat{U}^\dagger, \tag{30}$$

and in the homework#4 you will see that the (26) operator $\hat{U}(R)$ acts on the quantum fields precisely as the $SO(N)$ symmetry R, namely

$$\hat{U}\hat{\Phi}_a(x)\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{\Phi}_b\,. \tag{31}$$

Going back to the classical fields, for $N=2$ the two real fields $\Phi_1(x)$ and Φ_2 can be reorganized into a complex field $\Phi(x)$ and its complex conjugate $\Phi^*(x)$. In terms of these complex fields, the $SO(2)$ symmetry becomes the *phase symmetry*

$$\Phi'(x) \;=\; e^{-i\theta}\Phi(x), \quad \Phi^{*\prime}(x) \;=\; e^{+i\theta}\Phi^{*}(x). \tag{32}$$

In the quantum theory, this phase symmetry is generated by the charge $\hat{Q} = \hat{Q}_{21} = -\hat{Q}_{12}$, specifically

$$\begin{aligned} \exp(+i\theta\hat{Q})\hat{\Phi}(x)\exp(-i\theta\hat{Q}) \;&=\; e^{-i\theta}\hat{\Phi}(x), \\ \exp(+i\theta\hat{Q})\hat{\Phi}^{\dagger}(x)\exp(-i\theta\hat{Q}) \;&=\; e^{+i\theta}\hat{\Phi}^{\dagger}(x). \end{aligned} \tag{33}$$

In the particle language, the charge $\hat{Q}$ counts the net number of particles minus the number of antiparticles,

$$\hat{Q} \;=\; \hat{N}_{\text{particles}} \;-\; \hat{N}_{\text{antiparticles}} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\left(\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^{\dagger}_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right). \tag{34}$$

The proof is a part of homework#4.

Proof of the Noether Theorem

Let's prove the Noether theorem for the classical field theory. To simplify out notations, let ϕ_a run over all the fields of the theory, including the scalar fields, the components of the vector fields, *etc.*, *etc.* Any continuous symmetry of the field systems is generated by an infinitesimal symmetry of the form

$$\phi'_a(x) \;=\; \phi_a(x) \;+\; i\epsilon T\phi_a(x) \tag{35}$$

where ϵ is an infinitesimal parameter and T is the generator of the symmetry. T acts as some kind of an operator in the field space, usually a linear operator like $T\phi_a = \sum_b T_{ab}\phi_b(x)$ for some matrix T_{ab}, but it can also be a non-linear operator, and/or or involve the derivatives ∂_μ for symmetries acting on the spacetime coordinates x^μ.

Under the infinitesimal transforms (35), the action

$$S[\phi(x)] \;=\; \int d^4x\,\mathcal{L}(\phi,\partial\phi) \tag{36}$$

should remain invariant, $\delta S = 0$, which leaves us with two options for the Lagrangian density $\mathcal{L}(\phi,\partial\phi)$: Either it remains invariant, $\delta\mathcal{L} = 0$, or else $\mathcal{L}$ changes by a total spacetime derivative,

$$\delta\mathcal{L}(\phi,\partial\phi) \;=\; \epsilon\times\partial_\mu I^\mu(\phi,\partial\phi) \tag{37}$$

for some vector-valued function of the fields ϕ_a and their derivatives $\partial_\mu\phi_a$.

On the other hand, given the action (35) on the classical fields, its action on the Lagrangian density $\mathcal{L}(\phi,\partial\phi)$ follows from its dependence on the fields and their derivatives, thus

$$\begin{aligned}
\delta\mathcal{L} &= \sum_a\left(\frac{\partial\mathcal{L}}{\partial\phi_a}\times\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\times\partial_\mu\delta\phi_a\right)\\
&= \sum_a\left(\frac{\partial\mathcal{L}}{\partial\phi_a}\times\epsilon iT\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\times\epsilon i\partial_\mu T\phi_a\right)\\
&\quad\langle\!\langle\,\text{integrating the second term by parts}\,\rangle\!\rangle\\
&= \epsilon\sum_a\left(\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right)\times iT\phi_a\\
&\qquad + \epsilon\,\partial_\mu\left(\sum_a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\times iT\phi_a\right).
\end{aligned}\tag{38}$$

When the fields happen to obey their equations of motion

$$\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) \;=\; 0, \tag{39}$$

every term on the second-from-the-bottom line of eq. (38) vanishes and we are left with the bottom line only, thus

$$\delta\mathcal{L} \;=\; \epsilon\times\partial_\mu\left(\sum_a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\times iT\phi_a\right). \tag{40}$$

Comparing this formula with eq. (37), we immediately see that

$$\epsilon\times\partial_\mu\left(\sum_a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\times iT\phi_a\right) \;=\; \epsilon\times\partial_\mu I(\phi,\partial\phi). \tag{41}$$

Therefore, if we *define* the current J^μ according to

$$J^\mu \;\stackrel{\text{def}}{=}\; \sum_a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\times iT\phi_a \;-\; I^\mu(\phi,\partial\phi), \tag{42}$$

then this current is conserved, $\partial_\mu J^\mu(x)=0$ when the fields obey their equations of motion.

This completes the proof of the Noether theorem for the classical field theory. And along with the proof, we have also learned how to construct the conserved current for a given infinitesimal symmetry. As an example, let's go back to the $SO(N)$– symmetric theory of N scalar fields with Lagrangian density

$$\mathcal{L} \;=\; \frac{1}{2}\sum_a (\partial_\mu \Phi_a)(\partial^\mu \Phi_a) \;-\; V\left(\sum_a \Phi_a^2\right). \tag{43}$$

The infinitesimal $SO(N)$ symmetries act on the fields according to

$$\delta\Phi_a(x) \;=\; \sum_b \epsilon_{ab}\Phi_b(x) \tag{44}$$

for some infinitesimal antisymmetric matrix ϵ_{ab}. Interpreting this transform as $\delta\Phi_c(x) = \epsilon \times iT\Phi_c(x)$, we have

$$\epsilon \times iT \;=\; \frac{1}{2}\sum_{ab} \epsilon_{ab} iT_{ab} \tag{45}$$

where

$$iT_{ab}\Phi_c \;=\; \delta_{bc}\Phi_a \;-\; \delta_{ac}\Phi_b\,, \tag{46}$$

exactly as in eq. (11). Now let's find the Noether current for each generator $T_{ab} = -T_{ba}$. Since the symmetries (44) leave invariant not just the action but the Lagrangian density $\mathcal{L}$, we do not need the I^μ term in eq. (42). In other words, we let $I^\mu(\Phi, \partial\Phi) = 0$, which leaves us with

$$\begin{aligned} J^\mu_{ab} \;&=\; \sum_c \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_c)} \times iT_{ab}\Phi_c \\ &=\; \sum_c (\partial^\mu\Phi_c) \times (\delta_{bc}\Phi_a \;-\; \delta_{ac}\Phi_b) \\ &=\; (\partial^\mu\Phi_b) \times \Phi_a \;-\; (\partial^\mu\Phi_a) \times \Phi_b \\ &=\; \Phi_a\partial^\mu\Phi_b \;-\; \Phi_b\partial_\mu\Phi_a\,, \end{aligned} \tag{47}$$

exactly as in eq. (15). And as we saw earlier, these currents are indeed conserved when the fields obey their equations of motion.

In the special case of $N = 2$, we may recast the $SO(2)$ symmetry as a phase symmetry of a single complex field $\Phi(x)$ with Lagrangian density

$$\mathcal{L} \;=\; (\partial_\mu\Phi^*)(\partial^\mu\Phi) \;-\; V\bigl(\Phi^*\Phi\bigr), \tag{48}$$

$$\Phi'(x) \;=\; e^{-i\theta}\Phi(x), \quad \Phi^{*\prime}(x) \;=\; e^{+i\theta}\Phi^*(x), \quad \mathcal{L}' \;=\; \mathcal{L}.$$

The infinitesimal phase symmetry corresponds to $\theta = \epsilon$, thus

$$\delta\Phi(x) \;=\; -i\epsilon\Phi(x), \quad \delta\Phi^*(x) \;=\; +i\epsilon\Phi^*(x), \tag{49}$$

which in terms of the T generator means

$$T\Phi(x) \;=\; -\Phi(x), \quad T\Phi^*(x) \;=\; +\Phi^*(x). \tag{50}$$

In the eq. (42) for the Noether current, the invariance of $\mathcal{L}$ under the symmetry means $I^\mu = 0$, which leaves us with

$$J^\mu \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\times iT\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\times iT\Phi^* \;=\; (\partial^\mu\Phi^*)\times -i\Phi + (\partial^\mu\Phi)\times +i\Phi^* \;=\; -2\,\mathrm{Im}\bigl(\Phi^*\partial^\mu\Phi\bigr). \tag{51}$$

And back in the homework set#1 we saw that this current is indeed conserved when the fields obey their equations of motion.

Stress Energy Tensor

Now let's change our focus from the internal symmetries of a field theory to the spacetime symmetries, namely the translations of space and time:

⋆ Active translations $x'^\nu = x^\nu + d^\nu$, $\phi'_a(x') = \phi_a(x)$.

∗ Passive translations $\phi'_a(x) = \phi_a(x + d)$.

The active and the passive translations differ by the sign of the displacement vector d^ν, so they have the same 4 generators $\tilde{P}^\nu$. Specifically, consider a passive translation by infinitesimal displacement $d^\nu = \epsilon^\nu$, then

$$\phi'_a(x) \;=\; \phi_a(x+\epsilon) \;=\; \phi_a(x) \;+\; \epsilon^\nu \times \partial_\nu \phi_a(x) \;+\; O(\epsilon^2), \tag{52}$$

which we interpret as

$$\delta\phi_a(x) \;=\; \epsilon^\nu \times -i\tilde{P}_\nu \phi_a(x') \;+\; O(\epsilon^2) \quad \text{for} \quad \tilde{P}_\nu \phi_a(x') \;=\; i\partial_\nu \phi_a(x'). \tag{53}$$

The 4 generators $\tilde{P}_\nu$ give rise to 4 conserved currents $J^\mu_{(\nu)}$ where μ is the current index (density v. flow density) while (ν) is the generator index (energy v. momentum); conservation means zero divergence WRT the current index, thus $\partial_\mu J^\mu_{(\nu)}(x) = 0$. Physically, the currents $J^\mu_{(\nu)}$ of the translation symmetries are components of the stress-energy tensor, $J^\mu_{(\nu)} = T^\mu_{\ \nu}$, and the net charges

$$P_\nu \;=\; \int d^3\mathbf{x}\, T^0_{\ \nu} \tag{54}$$

comprise the net energy-momentum of the whole field configuration.

Let's derive the Noether stress-energy tensor from the Lagrangian of a generic field theory. While the net action $S = \int \mathcal{L} d^4x$ is invariant under the translations of space and time, the Lagrangian density is not invariant. Instead, it becomes translated with the fields,

$$\mathcal{L}(\phi', \partial\phi')@x \;=\; \mathcal{L}(\phi, \partial_\phi)@(x+\epsilon) \;=\; \mathcal{L}(\phi, \partial_\phi)@x \;+\; \epsilon^\nu \times \partial_\nu\big(\mathcal{L}(\phi, \partial\phi)\big)@x \;+\; O(\epsilon^2), \tag{55}$$

which we interpret as $\delta\mathcal{L} = \epsilon^\nu \times \partial_\nu \mathcal{L}$. In terms of the $I^\mu_{(\nu)}$ term in the Noether current $J^\mu_{(\nu)}$, we have

$$\delta\mathcal{L} \;=\; \epsilon^\nu \times \partial_\mu\big(\delta^\mu_\nu \mathcal{L}\big) \quad\Longrightarrow\quad I^\mu_{(\nu)} \;=\; \delta^\mu_\nu \mathcal{L}\,. \tag{56}$$

Consequently, the Noether formula (42) gives us

$$T^\mu_{\ \nu} \;=\; J^\mu_{(\nu)} \;=\; \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \times \Big(-i\tilde{P}_\nu\phi_a \;=\; \partial_\nu\phi_a\Big) \;-\; \delta^\mu_\nu \mathcal{L}. \tag{57}$$

Or after raising the ν index,

$$T^{\mu\nu}_{\text{Noether}} \;=\; \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \times \partial^\nu\phi_a \;-\; g^{\mu\nu}\times\mathcal{L}\,. \tag{58}$$

For example, consider the theory of N scalar fields with the Lagrangian density of the form

$$\mathcal{L} \;=\; \frac{1}{2}\sum_a(\partial_\mu\Phi_a)(\partial^\mu\Phi_a) \;-\; V(\Phi_1,\ldots,\Phi_N) \tag{59}$$

for any potential $V(\Phi_1,\ldots,\Phi_N)$. For this Lagrangian

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a)} \;=\; \partial^\mu\Phi_a, \tag{60}$$

hence Noether stress-energy tensor (58) becomes

$$T^{\mu\nu}_{\text{Noether}} \;=\; \sum_a(\partial^\mu\Phi_a)(\partial^\nu\Phi_a) \;-\; g^{\mu\nu}\mathcal{L}. \tag{61}$$

Note the symmetry of this stress-tensor, $T^{\mu\nu} = T^{\nu\mu}$.

The trouble with the Noether formula (58) for the stress-energy tensor is that for the non-scalar fields — vector fields, tensor fields, spinor fields, *etc.*, — it gives us an asymmetric stress-energy tensor, $T^{\mu\nu}_{\text{Noether}} \neq T^{\nu\mu}_{\text{Noether}}$. Indeed, in the next homework set#4 you shall see that for the electromagnetic fields, the $T^{\mu\nu}_{\text{Noether}}$ tensor is asymmetric; it also is not gauge invariant, which is a separate kind of bad. But an asymmetric stress-energy tensor is bad enough by itself: You cannot use $T^{\mu\nu} \neq T^{\nu\mu}$ in the Einstein equations of General Relativity. Moreover, an asymmetric stress-energy tensor is bad for the angular momentum conservation. As explained in some detail in the Weinberg's textbook, the currents of the Lorentz symmetries $\delta L^{\mu\nu}$ have form

$$\mathcal{M}^{\lambda,\mu\nu}(x) \;=\; x^\mu T^{\lambda\nu}(x) \;-\; x^\nu T^{\lambda\mu}(x), \qquad \mathcal{M}^{\lambda,\mu\nu}(x) \;=\; -\mathcal{M}^{\lambda,\nu\mu}(x), \tag{62}$$

and consequently these currents are not conserved for asymmetric stress-energy tensors,

$$\partial_\lambda\mathcal{M}^{\lambda,\mu\nu} \;=\; T^{\mu\nu} \;-\; T^{\nu\mu} \;\neq\; 0 \quad \text{for asymmetric} \quad T^{\mu\nu} \;\neq\; T^{\nu\mu}. \tag{63}$$

To make the $T^{\mu\nu}$ tensor symmetric, we add a total divergence to the Noether's stress-energy tensor,

$$T^{\mu\nu}_{\text{phys}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda K^{\lambda\mu,\nu}(\phi, \partial\phi), \tag{64}$$

where $K^{\lambda\mu,\nu}$ is some kind of a three-index tensor made from the fields and their derivatives; it also must be antisymmetric in its first two indices, $K^{\lambda\mu,\nu} = -K^{\mu\lambda,\nu}$. For any such $K^{\lambda\mu,\nu}$, the 'corrected' stress-energy tensor is just as conserved as the Noether's stress-energy tensor,

$$\partial_\mu\, T^{\mu\nu}_{\text{phys}} = \partial_\mu T^{\mu\nu}_{\text{Noether}} = 0(\text{hopefully}),$$

because

$$\partial_\mu \partial_\lambda K^{\lambda\mu,\nu} = 0 \quad \text{due to } K^{\lambda\mu,\nu} = -K^{\mu\lambda,\nu}. \tag{65}$$

Also, for the fields which vanish at spatial infinity fast enough, the corrected stress-energy tensor yields the same net energy-momentum as the Noether tensor,

$$P^\nu_{\text{net}} = \int d^3\mathbf{x}\, T^{0\nu}_{\text{phys}} = \int d^3\mathbf{x}\, T^{0\nu}_{\text{Noether}} \tag{66}$$

because

$$\Delta P^\nu = \int d^3\mathbf{x}\, \partial_\lambda K^{\lambda 0,\nu} = \int d^3\mathbf{x}\, \nabla^i K^{i0,\nu} = \oint_{\text{space}\,\infty} d^2\text{Area}\, n^i K^{i0,\nu} \longrightarrow 0 \tag{67}$$

when $K^{i0,\nu}(\phi, \partial\phi)$ decreases at $r \to \infty$ faster than $1/r^2$.

The specific form of $K^{\lambda\mu,\nu}$ as a function of the fields and their derivatives is chosen such as to make the 'corrected' stress-energy tensor (64) symmetric. For example, in the homework#4 you will see that the correction

$$K^{\lambda\mu,\nu} = -K^{\mu\lambda,\nu} = F^{\mu\lambda} A^\nu \tag{68}$$

symmetrizes the electromagnetic stress-energy tensor.