Optical Theorem for Soft Photons

In a previous set of notes we saw how both virtual and real soft photons cause infrared divergences, but such divergences cancel out from the physically measurable cross-sections. In particular, when we scatter an electron off some heavy particle X and distinguish between the process in which a soft photon is emitted — regardless of how low its energy might be — and a process without any such photon, we get IR divergent cross-sections at order $O(\alpha^3)$,

$$\text{both}\quad \sigma_{1\,\text{loop}}(e^-X\to e^-X)\quad\text{and}\quad \sigma_{\text{tree}}(e^-X\to e^-X\gamma)\quad\text{are IR divergent}\tag{1}$$

but the combined cross-section

$$\sigma_{\text{order}\,\alpha^3}(e^-+X\to e^-+X+\text{optional}\,\gamma)\quad\text{is IR finite.}\tag{2}$$

In these notes, we shall see how making this particular combination of cross-sections makes sense from the optical theorem point of view. But to simplify the analysis, we shall focus on the s-channel pair creation process rather than a t-channel scattering, thus

$$\sigma_{\text{order}\,\alpha^3}(\mu^-+\mu^+\to e^-+e^++\text{optional}\,\gamma)\;=\;\sigma_{1\,\text{loop}}(\mu^-\mu^+\to e^-e^+)+\sigma_{\text{tree}}(\mu^-\mu^+\to e^-e^+\gamma).\tag{3}$$

In general, the optical theorem relates the imaginary part of elastic scattering amplitude in the forward direction to the total cross-section,

$$\operatorname{Im}\mathcal{M}(a+b\to a+b;\theta=0)\;=\;2E_{\text{cm}}p_{\text{cm}}\times\sigma_{\text{total}}(a+b\to\text{anything}).\tag{4}$$

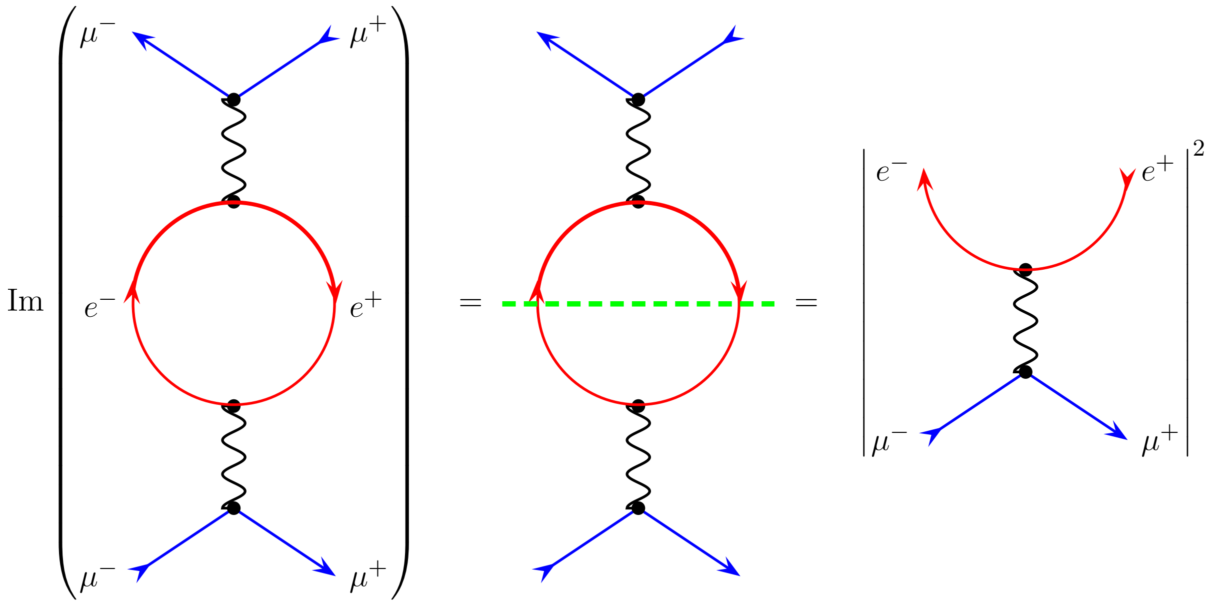
Now let's apply this to the one-loop amplitude for muon scattering; counting powers of α on both sides of the theorem, we get

$$\operatorname{Im}\mathcal{M}_{1\,\text{loop}}(\mu^-\mu^+\to\mu^-\mu^+)\;\propto\;\sigma_{\text{tree}}(\mu^-\mu^+\to\text{anything}),\tag{5}$$

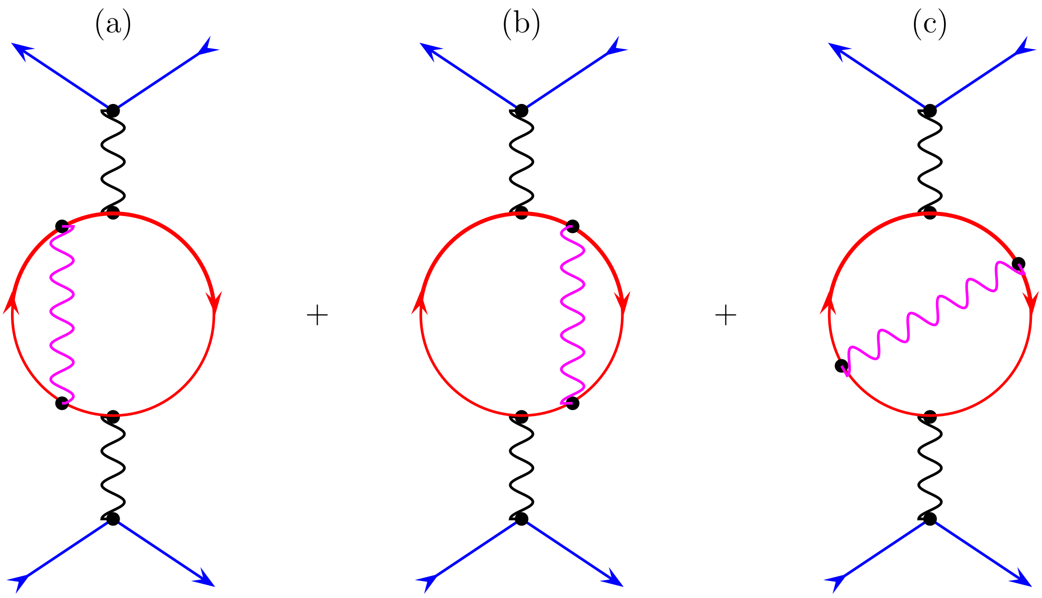
where the imaginary part obtains only from the s-channel diagrams. Moreover, the particles running around that loop are precisely the particles we pair-produce in the final state. In particular,

$$\operatorname{Im}\mathcal{M}_{1\,\text{electron loop}}(\mu^-\mu^+\to\mu^-\mu^+)\;\propto\;\sigma_{\text{tree}}(\mu^-\mu^+\to e^-e^+).\tag{6}$$

Diagrammatically, this relation follows from the Cutkosky cutting rules:



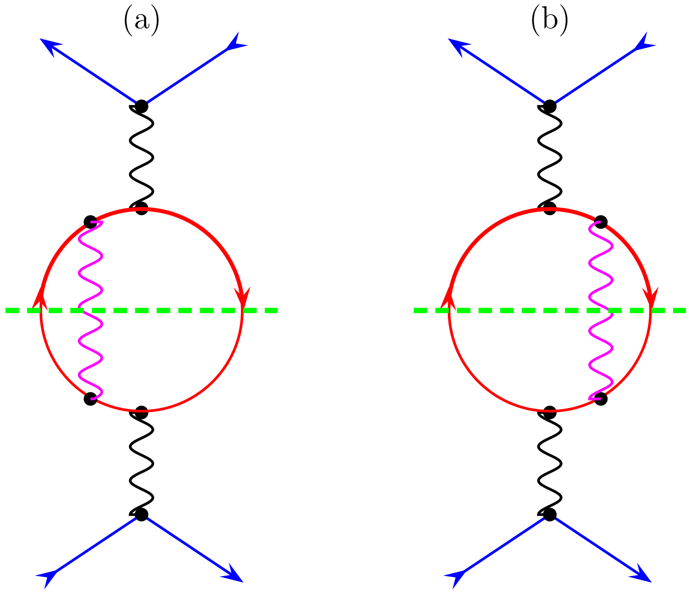
Now let's go to the next loop level and consider the two-loop amplitude for muon scattering. Specifically, let's focus on the diagrams containing a two-loop bubble — an electron loop crossed by a photon — inserted into the s-channel photon connecting initial muons to the final muons. There are three such diagrams, namely



(7)

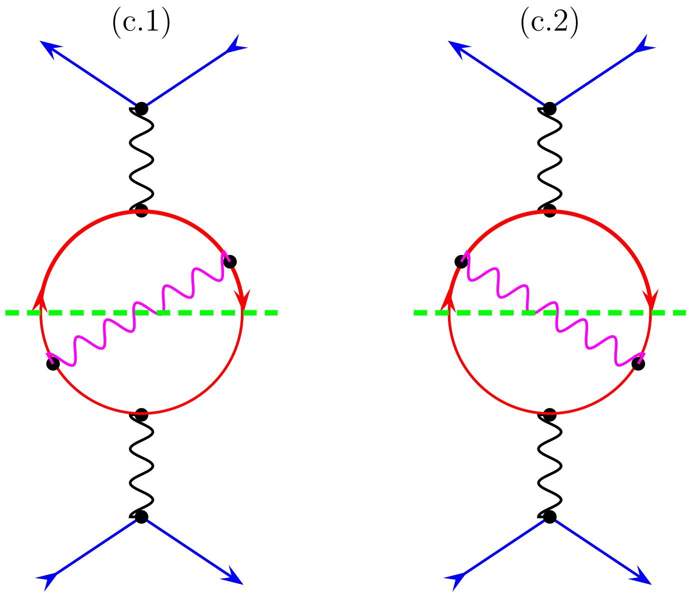
Note that all the electron propagators here (shown in red) are deeply off-shell, so there are no infrared divergences. Consequently, once we take the imaginary part of this 2-loop amplitude, the total cross-section related to it by the optical theorem should be infrared-finite.

To find which cross-sections are related to the diagrams (7) we use the Cutkosky rules: Cut a diagram in two in every way that (1) leaves each half-diagram amputated, and (2) every cut propagator can go on-shell for some values of the loop momenta. For the diagrams (a) and (b) there is only one allowed cut (for each diagram), namely



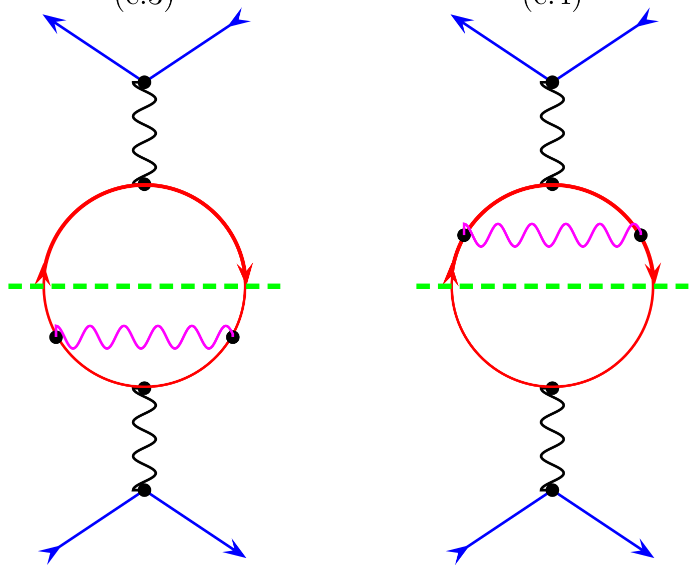
(8)

On the other hand, the third diagram (c) can be cut in 4 different ways:



(9)

(c.3) (c.4)

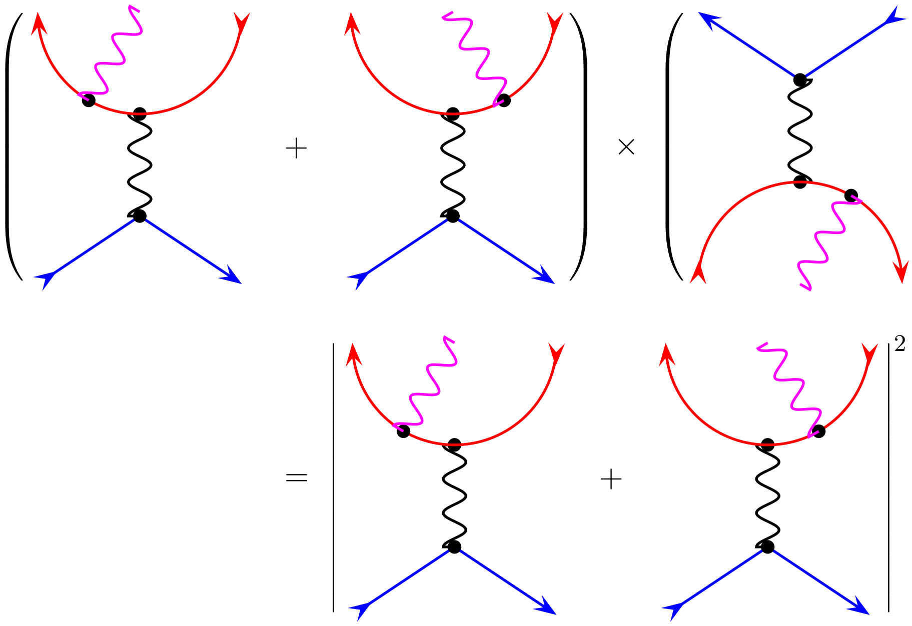


(10)

In the cut diagrams (a), (b), (c.1), and (c.2), the cut severs 3 propagators belonging respectively to an electron, a photon, and a positron. For each of these cut diagrams, we may interpret the bottom half as $\mathcal{L}(\mu^-\mu^+ \to e^-\gamma e^+)$ while the top half as

$$\mathcal{M}(e^-\gamma e^+ \to \mu^-\mu^+) \;=\; \mathcal{M}^*(\mu^-\mu^+ \to e^-\gamma e^+) \tag{11}$$

Moreover, the sum of the 4 cut diagrams (a) + (b) + (c.1) + (c.2) factorizes to



(12)

Physically, the last line here is the mod-square of the complete tree amplitude for the muon pair annihilating into electron-positron pair, with a photon emitted in the process.[⋆] Once we sum over spins of the final state particles and integrates over the phase space, we end up with the total tree-level cross-section for the $\mu^-\mu^+ \to e^-\gamma e^+$ process, thus in the context of the Optical Theorem

$$\mathrm{Im}\big[\mathcal{M}(a)+\mathcal{M}(b)+\mathcal{M}(c.1)+\mathcal{M}(c.2)\big] \;\propto\; \sigma_{\mathrm{tree}}(\mu^-\mu^+ \to e^-\gamma e^+). \tag{13}$$

Now consider the two remaining cut diagrams (10), namely (c.3) and (c.4). On these diagrams, only two electron propagators are cut but not the photon propagator, so the bottom halves of these diagrams correspond to the $\mu^-\mu^+ \to e^-e^+$ annihilation without the photons, while the top halves correspond to the reverse process. Also, in the (c.3) diagram the bottom half is one-loop while the top half is tree-level, while in the (c.4) diagram it's the other way around. Altogether,

$$\mathrm{Im}\big[\mathcal{M}(c.3)+\mathcal{M}(c.4)\big] \;\propto\; \mathcal{M}_{1\,\mathrm{loop}}(\mu^-\mu^+ \to e^-e^+)\times\mathcal{M}^*_{\mathrm{tree}}(\mu^-\mu^+ \to e^-e^+) \;+\; \text{complex conjugate}, \tag{14}$$

which is the cross term in $|\mathcal{M}_{\mathrm{tree}} + \mathcal{M}_{1\,\mathrm{loop}}|^2$. Integrating this cross-term over the phase space yields the one-loop corrections to the pair production cross-section, thus in the context of the Optical Theorem

$$\mathrm{Im}\big[\mathcal{M}(c.3)+\mathcal{M}(c.4)\big] \propto\; \sigma_{1\,\mathrm{loop}}(\mu^-\mu^+ \to e^-e^+). \tag{15}$$

Altogether, the 6 cut diagrams (a), (b), and (c.1–4) comprise the imaginary part of two-loop elastic $\mu^-\mu^+ \to \mu^-\mu^+$ amplitude, or rather of the part of that amplitude which involves the electron-photon double loop,

$$\begin{aligned}\mathrm{Im}\big[\mathcal{M}(a)+\mathcal{M}(b)+\mathcal{M}(c.1)+\mathcal{M}(c.2)+\mathcal{M}(c.3)+\mathcal{M}(c.4)\big] &= \mathrm{Im}\Big(\mathcal{M}(a+b+c)\Big)\\ &= \mathrm{Im}\left(\mathcal{M}^{ee\gamma}_{2\,\mathrm{loops}}(\mu^-\mu^+ \to \mu^-\mu^+)\right).\end{aligned} \tag{16}$$

By the optical theorem, the imaginary part of that amplitude is related to the combined $O(\alpha^3)$

⋆ For simplicity, we neglect the diagrams where the photon is emitted by a muon rather than an electron, since the corresponding amplitudes are suppressed by the relative factors $O(m_e/m_\mu) \ll 1$.

cross-section (13) plus (15),

$$\operatorname{Im}\left(\mathcal{M}^{ee\gamma}_{2\,\mathrm{loops}}(\mu^-\mu^+\to\mu^-\mu^+)\right)\ \propto\ \sigma_{\mathrm{tree}}(\mu^-\mu^+\to e^-\gamma e^+)\ +\ \sigma_{1\,\mathrm{loop}}(\mu^-\mu^+\to e^-e^+). \tag{17}$$

The combined amplitude on the LHS does not suffer from IR divergences, so the combined cross-section on the RHS is guaranteed to be IR finite. And indeed, in the previous set of notes we saw that the IR divergences cancel out from this combined cross-section. However, Cutkosky cutting a diagram in different ways can introduce spurious IR singularities since the cut propagators are on-shell. When we sum over all the allowed ways to cut the amplitude, such spurious singularities must cancel each other, but the individual terms can be divergent. Consequently, when the optical theorem relates different cuttings (or different groups of cutting) to different cross-sections, those cross-sections turn out to be IR-divergent, and only the combined cross-section remains IR-finite.

In QED, the IR divergences come from the soft photons, real or virtual. To cancel such divergences, one must total up cross-sections involving the same net number of soft photons, thus

$$\sum_{L=0}^{N}\sigma\left[{L\ \text{virtual} \atop \text{soft photons}}\right](X\to Y+(N-L)\ \text{real soft photons}). \tag{18}$$

In class, I have shown how this works for $N=1$. To see how this works for larger numbers of soft photons, please read §6.5 of the Peskin and Schroeder textbook.