Electric Current Conservation and Ward–Takahashi Identities

Outline

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(3) Formal proof of Ward–Takahashi identities.

(4) Ward–Takahashi identities for renormalization.

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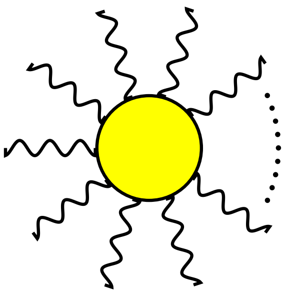
(1) Introduction

QED has a large family of Ward–Takahashi identities. Of particular importance are two series of WT identities for the off-shell amplitudes involving 0 or 2 electronic external lines and any number N of photonic external lines.

- No electrons, N photons amplitudes

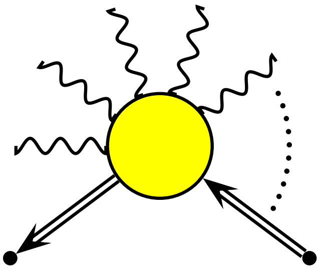


$$= iV_N^{\mu_1 \dots \mu_N}(k_1, \dots, k_N) \xrightarrow[\text{shorthand}]{} iV_N^{1,\dots,N}. \tag{1}$$

The V_N are amputated amplitudes, meaning no external leg bubbles in the diagrams, and the external legs themselves are not included in the amplitudes. Ward–Takahashi identities for the V_N are simply

$$\forall i, \quad (k_i)_{\mu_i} \times V_N^{\mu_1 \dots \mu_N}(k_1, \dots, k_N) = 0. \tag{2}$$

- 2 electrons, N photons amplitudes



$$= S_N^{\mu_1\cdots\mu_N}(p', p; k_1, \ldots k_N) \xrightarrow[\text{shorthand}]{} S_N(p', p). \tag{3}$$

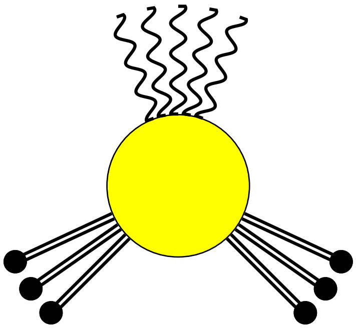
The double straight lines with valence $= 1$ vertices at their ends indicate that the S_N amplitude include the *dressed propagators* for the electrons' external legs, meaning both the free propagators and the arbitrary number of the external leg bubbles; in other words, *the electrons' external legs are not amputated.* On the other hand, all the photons' external legs are amputated: they include neither outside photon propagators nor any external leg bubbles.

Ward–Takahashi for the S_N amplitudes are recursive relations relating S_N to S_{N-1}, namely

$$\forall i, \quad (k_i)_{\mu_i} \times S_N^{\mu_1,\ldots,\mu_i,\ldots,\mu_N}(p', p) \;=\; eS_{N-1}^{\mu_1,\ldots,\not{\mu_i},\ldots,\mu_N}(p', p+k_i) - eS_{N-1}^{\mu_1,\ldots,\not{\mu_i},\ldots,\mu_N}(p'-k_i, p). \tag{4}$$

The sign convention for the external momenta here is as follows: All the photons' momenta k_i are treated as incoming, while the electron momenta follow the charge arrows: p is incoming while p' is outgoing, hence $p' - p = k_1 + \cdots + k_N$.

Besides these two series, there are Ward–Takahashi identities for amplitudes with more electronic external lines, — or for other kinds of charged particles you might want to add to basic QED. Most generally, consider any kind of QFT, with any kinds of charged and neutral fields, as long as they include the EM fields $A^\mu(x)$ coupled to the *conserved electric current,* $\partial_\mu J^\mu(x) = 0$. For any such QFT, consider an amplitude $\mathcal{S}_{N,M}$ involving N photons and M particles of other kinds, charged or neutral,



$$= \mathcal{S}^{\mu_1\ldots\mu_N}_{N,M}(p_1,\ldots,p_M;\, k_1,\ldots,k_N). \tag{5}$$

Similar to the 2-electron amplitudes (3), the N photonic external lines here are amputated while the M external lines for all other particles include the dressed propagators. However, here we all the external momenta as incoming, thus $(p_1+\cdots+p_M)+(k_1+\cdots+k_N)=0$. The Ward–Takahashi identities for the general off-shell amplitudes (5) relate them to amplitudes with one less photon, specifically

$$\begin{aligned}(k_i)_{\mu_i} \times \mathcal{S}^{\mu_1\ldots\mu_N}_{N,M}&(p_1,\ldots,p_M;\, k_1,\ldots,k_N)\\ &= -\sum_{j=1}^{M} Q_j \times \mathcal{S}^{\cdots\not\mu_i\cdots}_{N-1,M}(p_1,\ldots,p_j+k_1,\ldots,p_M;k_1,\ldots,\not k_i,\ldots,k_N)\end{aligned} \tag{6}$$

where Q_j is the electric charge of the particle $\#j$.

Note that the basic Ward–Takahashi identities (2) and (4) are special cases of the general WT identity (6). Indeed, the purely photonic amputated amplitude $\mathcal{V}_N$ is the special case $\mathcal{S}_{N,0}$ of the amplitude (5) for $M=0$, and in this case the RHS of eq. (6) is simply zero, hence the identity (2). Likewise, the two-electron amplitudes S_N are special cases of $\mathcal{S}_{N,2}$ where the two non-photonic external legs belong to electrons. Or rather, treating both of these legs as incoming, one positron of momentum $p_2=-p'$ and one electron of momentum $p_1=+p$. Consequently, eq. (6) becomes

$$\begin{aligned}(k_i)_{\mu_i} \times \mathcal{S}^{\mu_1\ldots\mu_N}_{N,2}&(-p',p;\, k_1,\ldots,k_N)\\ &= -(+e)\times \mathcal{S}^{\cdots\not\mu_i\cdots}_{N-1,2}(-p'+k_i,p;\, k_1,\ldots,k_N)\\ &\quad -\ (-e)\times \mathcal{S}^{\cdots\not\mu_i\cdots}_{N-1,2}(-p',p+k_i;\, k_1,\ldots,k_N),\end{aligned} \tag{7}$$

and hence eq. (4).

(2) Current Conservation in Quantum Field Theories

Consider a field theory with an exact $U(1)$ phase symmetry — global or local, — and the corresponding conserved current $J^\mu(x)$, $\partial_\mu J^\mu = 0$. In the quantum theory, current quantization formally means that in the Heisenberg picture the current operator $\hat{J}^\mu(x)$ obeys the continuity equation $\partial_\mu \hat{J}^\mu = 0$. But since the current does not live in the vacuum — in any sense of the word, — in practice we measure or calculate not the current operator itself but rather its correlation functions with the other local operators of the theory, especially the quantum fields themselves. So let

$$\mathcal{G}_n^\mu(x_1,\ldots,x_n;y) = \langle\Omega|\,\mathbf{T}\hat{\varphi}_1(x_1)\cdots\hat{\varphi}_n(x_n)\times\hat{J}^\mu(y)\,|\Omega\rangle \tag{8}$$

be such a correlation function of the current $\hat{J}^\mu(y)$ with n fields $\hat{\varphi}_i(x_i)$ of any kind: charged or neutral, vectors, spinor, scalar, whatever. For example, in basic QED each $\hat{\varphi}_i$ can be any of the $\hat{A}^\nu$, $\hat{\Psi}_\alpha$, or $\hat{\overline{\Psi}}_\alpha$. (To avoid clattering my notations, I suppress the vectors, spinor, *etc.*, indices of $\mathcal{G}_n$ due to the fields $\varphi_i(x_i)$ and keep only the index μ due to the current $J^\mu(y)$.) In eq. (8), $|\Omega\rangle$ is the physical vacuum state of the theory, while all the $\hat{\phi}^a(x)$ are fully interacting fields in the Heisenberg picture of QM; likewise, the current operator $\hat{J}^\mu(y)$ is in the Heisenberg picture.

Naively, one might expect that the continuity equation $\partial_\mu \hat{J}^\mu(y) = 0$ for the current operator translates into similar continuity equations

$$\frac{\partial}{\partial y^\mu}\mathcal{G}_n^\mu(x_1,\ldots,x_n;y) \;=\; 0, \tag{9}$$

but the reality is more complicated because the time-ordering in eq. (8) does not commute with the time derivative for $\mu = 0$. Indeed, for any two local operators $\hat{A}(x)$ and $\hat{B}(y)$ we have

$$\frac{\partial}{\partial y^0}\,\mathbf{T}\Big(\hat{A}(x)\times\hat{B}(y)\Big) \;=\; \mathbf{T}\Big(\hat{A}(x)\times\frac{\partial}{\partial y^0}\hat{B}(y)\Big) \;+\; \delta(x^0-y^0)\times\big[\hat{A}(x),\hat{B}(y)\big] \tag{10}$$

where the second term stems from sudden re-ordering of the two factors when $x^0 = y^0$. In particular, for a quantum field $\hat{\varphi}(x)$ and the conserved current $\hat{J}^\mu(x)$

$$\frac{\partial}{\partial y^\mu}\,\mathbf{T}\Big(\hat{\varphi}(x)\times\hat{J}^\mu(y)\Big) \;=\; \mathbf{T}\Big(\hat{\varphi}(x)\times\partial_\mu\hat{J}^\mu(y)\Big) +\; \delta(x^0-y^0)\times\big[\hat{\varphi}(x),\hat{J}^0(y)\big], \tag{11}$$

where the first term on the RHS vanishes by the current conservation, but the second term gives rise to a singularity when $x = y$. Indeed, a field $\hat{\varphi}(x)$ of charge Q — that is, a field creating particles of charge Q and/or annihilating particles of charge $-Q$, — obeys

$$\left[\hat{\varphi}(x), \hat{Q}\right] \;=\; -Q \times \hat{\varphi}(x), \tag{12}$$

hence for a local density $\hat{J}^0(y)$ of the charge operator $\hat{Q}$

$$\left[\hat{\varphi}(x), \hat{J}^0(y)\right] \;=\; -Q \times \delta^{(3)}(\mathbf{x}-\mathbf{y}) \times \hat{\varphi}(x) \quad \text{when } x^0 \;=\; y^0. \tag{13}$$

Plugging this formula into eq. (11), we immediately arrive at

$$\frac{\partial}{\partial y^\mu}\mathbf{T}\Big(\hat{\varphi}(x) \times \hat{J}^\mu(y)\Big) \;=\; 0 \;+\; \delta^{(4)}(x-y) \times -Q \times \hat{\varphi}(x). \tag{14}$$

In the same way, for multiple fields inside the times ordering $\mathbf{T}$ we get

$$\frac{\partial}{\partial y^\mu}\mathbf{T}\Big(\hat{\varphi}_1(x_1)\cdots\hat{\varphi}_n(x_n) \times \hat{J}^\mu(y)\Big) \;=\; \mathbf{T}\Big(\hat{\varphi}_1(x_1)\cdots\hat{\varphi}_n(x_n)\Big) \times \sum_{j=1}^{n}(-Q[\varphi_j]) \times \delta^{(4)}(x_j - y). \tag{15}$$

Consequently, the correlation functions

$$\mathcal{G}_n^\mu(x_1,\ldots,x_n;y) = \;\langle\Omega|\,\mathbf{T}\hat{\varphi}_1(x_1)\cdots\hat{\varphi}_n(x_n) \times \hat{J}^\mu(y)\,|\Omega\rangle \tag{8}$$

of the conserved current obey

$$\frac{\partial}{\partial y^\mu}\mathcal{G}_n^\mu(x_1,\ldots,x_n;y) \;=\; -\mathcal{F}_n(x_1,\ldots,x_n) \times \sum_{j=1}^{n} Q[\varphi_j] \times \delta^{(4)}(x_j - y) \tag{16}$$

where

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \langle\Omega|\,\mathbf{T}\hat{\varphi}_1(x_1)\cdots\hat{\varphi}_n(x_n)\,|\Omega\rangle \tag{17}$$

is the correlation function of the same n fields but without the current operator $J^\mu(y)$. The delta-function terms on the RHS of eq. (16) are called *the contact terms* because they show up only when the current operator comes into direct contact with a field operator, *i.e.* acts at exactly the same spacetime point $y = x_j$. As we shall see in the next section, it is the contact terms which are responsible for the non-zero RHS of the Ward–Takahashi identities (4) or (6).

To complete this section, let's Fourier transform the contact terms — and hence eq. (16) — to the momentum space. For $n = 1$, a contact term of the form $f(x) \times \delta^{(4)}(x - y)$ transforms to

$$\begin{aligned} F(p,k) &= \int d^4x\, e^{ipx} \int d^4y\, e^{iky} \times f(x)\delta^{(4)}(x-y) \\ &= \int d^4x\, e^{ipx} \times e^{ikx} \times f(x) \\ &= f(p+k), \end{aligned} \tag{18}$$

which depends only on the sum $p + k$ instead of separate dependence on the two momenta. Likewise, for $n > 1$ a contact term of the form $f(x_1, \ldots, x_n) \times \delta^{(4)}(x_j - y)$ Fourier transforms to $f(p_1, \ldots, p_j + k, \ldots, p_n)$, hence the RHS of eq. (16) transforms to

$$-\sum_{j=1}^{n} Q[\varphi_j] \times \mathcal{F}_n(p_1, \ldots, p_j + k, \ldots, p_n). \tag{19}$$

At the same time, the LHS of eq. (16) Fourier transforms to

$$-ik_\mu \times \mathcal{G}_n^\mu(p_1, \ldots, p_n; k), \tag{20}$$

so the entire eq. (16) becomes

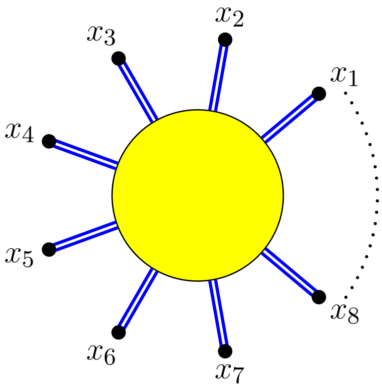
$$k_\mu \times \mathcal{G}_n^\mu(p_1, \ldots, p_n; k) \;=\; -i\sum_{j=1}^{n} Q[\varphi_j] \times \mathcal{F}_n(p_1, \ldots, p_j + k, \ldots, p_n). \tag{21}$$

(3) Formal Proof of Ward–Takahashi Identities

Eqs. (16) and (21) from the previous section apply to any conserved current $\hat{J}^\mu(y)$ in any QFT. Now let's apply them to the electric current in QED, or perhaps a larger theory including QED, like the Standard Model.

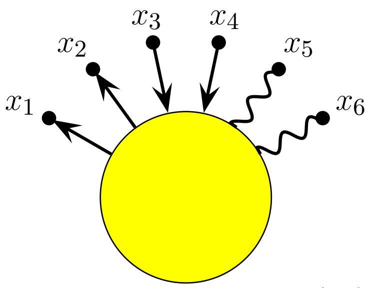
Diagrammatically, the correlation function $\mathcal{F}_n(x_1, \ldots, x_n)$ is the net coordinate-space amplitude of all the Feynman diagrams with n *un-amputated* external legs of appropriate kinds,

$$i\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \tag{22}$$



For example, in basic QED (the EM and electron fields, and nothing else)

$$i\mathcal{F}_6^{\cdots}(\cdots) \;=\; i\,\langle\Omega|\,\mathbf{T}\hat{\Psi}(x_1)\hat{\Psi}(x_2)\hat{\overline{\Psi}}(x_3)\hat{\overline{\Psi}}(x_4)\hat{A}^{\kappa}(x_5)\hat{A}^{\lambda}(x_6)\,|\Omega\rangle \;=\; \tag{23}$$

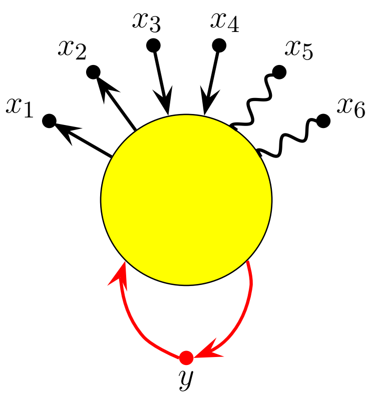


Now consider the correlation functions

$$\mathcal{G}_n^{\mu}(x_1,\ldots,x_n;y) = \;\langle\Omega|\,\mathbf{T}\hat{\varphi}^{a_1}(x_1)\cdots\hat{\varphi}^{a_n}(x_n)\times\hat{J}^{\mu}(y)\,|\Omega\rangle \tag{8}$$

of the electric current $\hat{J}^{\mu}(y)$ with the quantum fields. In basic QED $\hat{J}^{\mu}(y) = -e\hat{\overline{\Psi}}(y)\gamma^{\mu}\hat{\Psi}(y)$, so in the Feynman rules for the correlation functions, $\hat{J}^{\mu}(y)$ becomes an external vertex of valence $= 2$ connected to 2 electron lines, one for the $\hat{\Psi}(y)$ and the other for the $\hat{\overline{\Psi}}(y)$. For example,

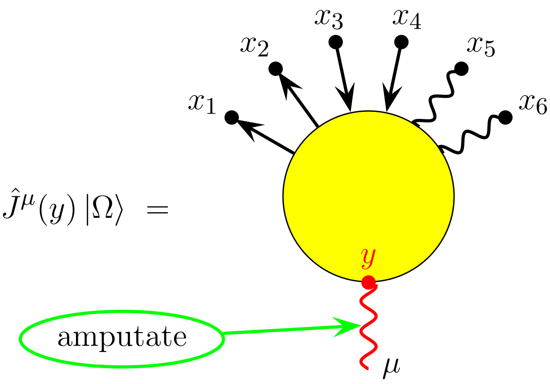
$$\begin{aligned} &i\mathcal{G}_6^{\mu}(x_1,\ldots,x_6;y) \;= \\ &=\; i\,\langle\Omega|\,\mathbf{T}\,\hat{\Psi}(x_1)\hat{\Psi}(x_2)\hat{\overline{\Psi}}(x_3)\hat{\overline{\Psi}}(x_4)\hat{A}^{\lambda}(x_5)\hat{A}^{\kappa}(x_6)\,\hat{J}^{\mu}(y)\,|\Omega\rangle \;=\end{aligned} \tag{24}$$



Note that the Dirac indexology of the bottom vertex at y is $(-e\gamma^\mu)_{\alpha\beta}$ — which is exactly similar to the photon's vertex $(ie\gamma^\mu)_{\alpha\beta}$, apart from the overall factor of i. Consequently, each diagram contributing to a $\mathcal{G}_n$ correlation function can be interpreted as $i\times$ the diagram with an extra external EM field $\hat{A}^\mu(y)$, except that we do not have the dressed propagator for that extra EM field. In other words, the external leg for that extra photon is amputated, unlike the un-amputated external legs for the other n fields.

Altogether, the G_n correlation function of n fields plus electric current is $i\times$ amplitude with n un-amputated external legs plus one amputated leg for an extra photon, for example

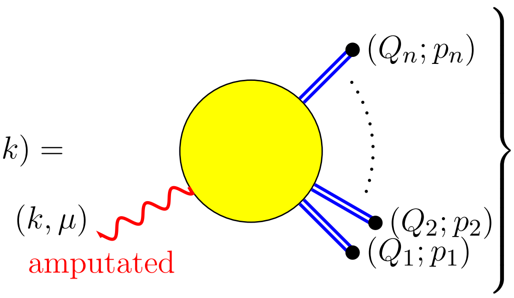
$$\mathcal{G}_6^\mu(x_1,\ldots,x_6;y) \;=\; = \;\langle\Omega|\,\mathbf{T}\,\hat{\Psi}(x_1)\hat{\Psi}(x_2)\hat{\overline{\Psi}}(x_3)\hat{\overline{\Psi}}(x_4)\hat{A}^\lambda(x_5)\hat{A}^\kappa(x_6)\,\hat{J}^\mu(y)\,|\Omega\rangle \;=$$



(25)

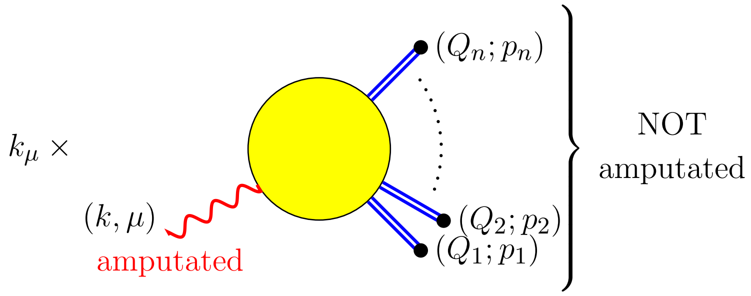
Now consider a general QFT including the QED — anything ranging from QED with extra charged leptons to the Standard Model, — pretty musc any QFT which includes the massless EM fields coupled to a conserved electric current. In any such theory, the correlation functions of the electric current with n quantum fields have similar relations to the amplitudes with n un-amputated legs and one amputated leg for an extra photon,

$$\mathcal{G}_n^\mu(p_1,\ldots,p_n;k) =$$

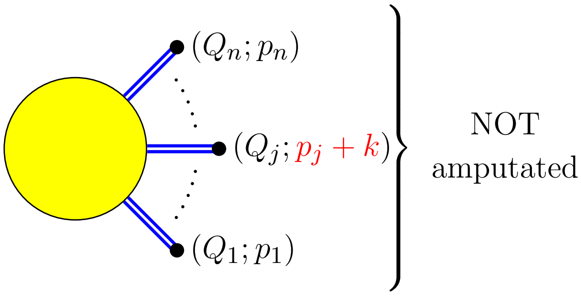


NOT amputated (26)

Eq. (21) relates such amplitudes (multiplied by the k_μ) to the amplitudes $\mathcal{G}_n$ without the extra photon. Diagrammatically, this equation reads



$$= \; -\sum_{j=1}^{n} Q_j \times$$

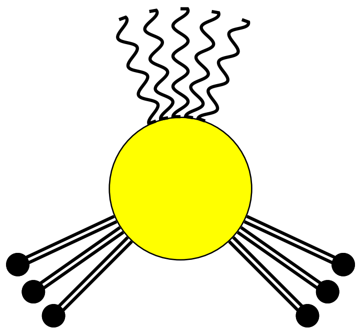


(27)

This diagrammatic equation is very similar to the general Ward–Takahashi identity

$$(k_i)_{\mu_i} \times \mathcal{S}^{\mu_1\ldots\mu_{N+1}}_{N+1,M}(p_1,\ldots,p_M;\,k_1,\ldots,k_{N+1}) \\ = \; -\sum_{j=1}^{M} Q_j \times \mathcal{S}^{\cdots\not{\mu_i}\cdots}_{N,M}(p_1,\ldots,p_j+k_i,\ldots,p_M;k_1,\ldots,\not{k}_i,\ldots,k_{N+1}), \qquad (6)$$

except that the amplitudes



$$= \; i\mathcal{S}^{\mu_1\ldots\mu_N}_{N,M}(p_1,\ldots,p_M;\,k_1,\ldots,k_N). \qquad (28)$$

has all of their photonic external legs amputated, while the external legs for all other particle species — charged or neutral — remain un-amputated. However, it is easy to translate eq. (21) or eq. (27) to the language of the $S_{N,M}$ amplitudes by simply factoring out the dressed propagators for all the photons on both sides of the equation. Indeed, let N out of n particles of the un-amputated n-particle amplitude $\mathcal{F}_n$ be photons while the remaining $M = n - N$ particles belong to other species, charged or neutral, then

$$\mathcal{F}_n^{\mu_1,\ldots,\mu_N}(p_1,\ldots,p_M;\,k_1,\ldots,k_N) \;=\; S_{N,M}^{\nu_1,\ldots,\nu_N}(p_1,\ldots,p_M;\,k_1,\ldots,k_N)\times\prod_{i=1}^{N}\binom{\text{dressed}}{\text{prop}}_{\nu_i}^{\mu_i}(k_i). \tag{29}$$

Likewise, for the $\mathcal{G}_n$ amplitude involving an extra photon

$$\begin{aligned} i\mathcal{G}_n^{\mu_1,\ldots,\mu_N;\mu}(p_1,\ldots,p_M;\,k_1,\ldots,k_N;\,k) \;&=\; S_{N+1,M}^{\nu_1,\ldots,\nu_N;\mu}(p_1,\ldots,p_M;\,k_1,\ldots,k_N;\,k)\times \\ &\qquad\times\prod_{i=1}^{N}\binom{\text{dressed}}{\text{prop}}_{\nu_i}^{\mu_i}(k_i). \end{aligned} \tag{30}$$

Note that there are only N dressed photon propagators in the is formula because the extra photon's propagator is already amputated.

Finally, we may rewrite eq. (21)as

$$\begin{aligned} & k_\mu\times i\mathcal{G}_n^{\mu_1,\ldots,\mu_N;\mu}(p_1,\ldots,p_M;\,k_1,\ldots,k_N;\,k) \\ &\quad = \;-\sum_{j=1}^{M} Q_j\times\mathcal{F}_n^{\mu_1,\ldots,\mu_N}(p_1,\ldots,p_j+k,\ldots,p_M;\,k_1,\ldots,k_N) \end{aligned} \tag{31}$$

where the sum on the RHS is limited to the non-photonic external lines since the photons have $Q_j = 0$. Consequently, plugging eqs. (30) and (29) into this formula, we obtain

$$\begin{aligned} & k_\mu\times S_{N+1,M}^{\nu_1,\ldots,\nu_N;\mu}(p_1,\ldots,p_M;\,k_1,\ldots,k_N;\,k)\times\prod_{i=1}^{N}\binom{\text{dressed}}{\text{prop}}_{\nu_i}^{\mu_i}(k_i) \\ &\quad = \;-\sum_{j=1}^{M} Q_j\times S_{N,M}^{\nu_1,\ldots,\nu_N}(p_1,\ldots,p_j+k,\ldots,p_M;\,k_1,\ldots,k_N)\times\prod_{i=1}^{N}\binom{\text{dressed}}{\text{prop}}_{\nu_i}^{\mu_i}(k_i), \end{aligned} \tag{32}$$

where the product of N dressed photon propagators is exactly the same on both sides of the equation. Consequently, dropping this product on both sides, we finally arrive at the general

Ward–Takahashi identity for the $S_{M,N}$ amplitudes,

$$\begin{aligned} k_\mu \times S^{\nu_1,\ldots,\nu_N;\mu}_{N+1,M}(p_1,\ldots,p_M;\, k_1,\ldots,k_N;\, k) \\ = \, -\sum_{j=1}^{M} Q_j \times S^{\nu_1,\ldots,\nu_N}_{N,M}(p_1,\ldots,p_j+k,\ldots,p_M;\, k_1,\ldots,k_N). \end{aligned} \tag{33}$$

In this formula, we have contracted the $S_{N+1,M}$ amplitude with the last photon's momentum $k_\mu = (k_{N+1})_\mu$. By Bose symmetry, we can use any other photon to get a similar result, thus

$$\begin{aligned} (k_i)_{\mu_i} \times S^{\mu_1,\ldots,\mu_{N+1}}_{N+1,M}(p_1,\ldots,p_M;\, k_1,\ldots,k_{N+1}) \\ = \, -\sum_{j=1}^{M} Q_j \times S^{\mu_1,\ldots,\not\mu_i,\ldots,\mu_{N+1}}_{N,M}(p_1,\ldots,p_j+k_i,\ldots,p_M;\, k_1,\ldots,\not k_i,\ldots,k_{N+1}). \end{aligned} \tag{34}$$

Quod erat demonstrandum.

(4) Ward–Takahashi Identities and QED Renormalization

Earlier in class we saw (*cf.* my notes on QED Feynman rules and renormalization) how the Ward–Takahashi identities for the no-electron, 2-photon amplitude $\mathcal{V}_2^{\mu\nu} = \Sigma_\gamma^{\mu\nu}$ and no-electron, 4-photon amplitude $\mathcal{V}_4^{\kappa\lambda\mu\nu}$ allow for complete cancellation of all the UV divergences of QED by just 4 counterterms $\delta_1, \delta_2, \delta_3, \delta_m$ which actually exist in QED. And that is what makes QED a truly renormalizable theory.

In this section, we are going to see how the Ward–Takahashi identity for the 2-electron, 1-photon amplitude $S_1^\mu(p', p; k)$ relates the electron vertex renormalization factor Z_1 to the electron field strength renormalization factor Z_2. Specifically, we shall see that

$$Z_1 \;=\; Z_2 \quad \text{(both infinite and finite parts),} \tag{35}$$

or in terms of the counterterms $\delta_1 = \delta_2$, exactly. The identity (35) is often called *the Ward identity.*

For simplicity, let us work in the bare perturbation theory where the electric charges of various particles are the bare charges rather than the physical charges. In particular, for the

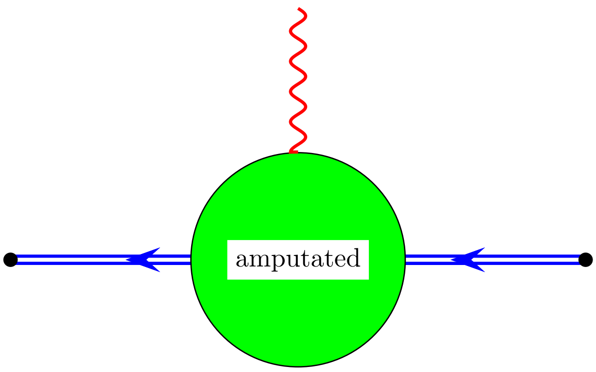
2-electron, 1-photon amplitude we have

$$k_\mu \times S_1^\mu(p', p; k) \;=\; e_b S_0(p', p + k = p') \;-\; e_b S_0(p' - k = p, p). \tag{36}$$

On the RHS of this formula, the S_0 is the un-amputated 2-electron, no-photon amplitude, which is nothing but the electron's two-point correlation function also known as the dressed electron propagator,

$$S_0(p' = p) \;=\; \mathcal{F}_2(p) \;=$$  $$= \frac{i}{\not{p} - m_b - \Sigma(\not{p}) + i\epsilon}\,. \tag{37}$$

The S_1 amplitude on the LHS of eq. (36) is more complicated: it has two un-amputated electron legs and one amputated photon leg. In other words, the S_1^μ comprises a completely amputated core plus two dressed propagators for the incoming and outgoing electrons,

$$S_1^\mu(p', p) \;=$$ 

(38)

Moreover, any amputated core with just 3 external legs is automatically one particle irreducible (1PI), so the green disk in the schematic diagram (38) is the 1PI dressed electron-electron-photon vertex $ie_b\Gamma^\mu(p', p)$; note that in the bare perturbation theory the electric charge e_b here is the bare charge rather than the physical charge e we would use in the counterterm perturbation theory. Thus, spelling the diagram (38) as a formula for the S_1^μ we get

$$S_1^\mu(p', p) \;=\; S_0(p') \times ie_b\Gamma^\mu(p', p) \times S_0(p). \tag{39}$$

Now let's plug this formula into the WT identity (36), which gives us

$$S_0(p') \times ie_b k_\mu \Gamma^\mu(p', p) \times S_0(p) \;=\; e_b S_0(p') \;-\; e_b S_0(p). \tag{40}$$

Note bare electric charge factors e_b on both sides of this formula, so dropping these factors (and also dividing by i) we get

$$S_0(p') \times k_\mu \Gamma^\mu(p', p) \times S_0(p) \;=\; -iS_0(p') \;+\; iS_0(p). \tag{41}$$

Next, let's divide both sides of this equation by the $S_0(p')$ on the left and by the $S_0(p)$ on the right; this gives us

$$\begin{aligned} k_\mu \Gamma^\mu(p', p) \;&=\; \frac{-i}{S_0(p)} \;+\; \frac{i}{S_0(p')} \\ &=\; -\big(\not{p} - m_b - \Sigma(\not{p})\big) \;+\; \big(\not{p}' - m_b - \Sigma(\not{p}')\big) \end{aligned} \tag{42}$$

where the second equality follows from eq. (37) for the dressed electron propagator S_0.

Now let's take the limit of a small photon momentum $k = p' - p \to 0$ while both the incoming and the outgoing electron momenta go on-shell, $\not{p} \to M_{\text{phys}}$ and $\not{p}' \to M_{\text{phys}}$. In the on-shell limit,

$$\text{for } \not{p} \to M_{\text{ph}}: \quad \frac{1}{\not{p} - m_b - \Sigma(\not{p})} \;\to\; \frac{Z_2}{\not{p} - M_{\text{ph}}} \;+\; \text{finite}, \tag{43}$$

$$\text{hence} \quad \big(\not{p} - m_b - \Sigma(\not{p})\big) \;\to\; \frac{(\not{p} - M_{\text{ph}})}{Z_2} \;+\; O\big((\not{p} - M_{\text{ph}})^2\big), \tag{44}$$

and likewise for the outgoing electron, hence

$$-\big(\not{p} - m_b - \Sigma(\not{p})\big) \;+\; \big(\not{p}' - m_b - \Sigma(\not{p}')\big) \;\to\; \frac{-\not{p} + \not{p}' = \not{k}}{Z_2} \;+\; O((\not{p} - M) \times k) \;+\; O(k^2)\,. \tag{45}$$

Plugging this limit into eq. (42), and comparing the leading orders in k on both sides of the equation, we arrive at

$$k_\mu \times \Gamma^\mu(\text{on shell } p' = p) \;=\; \frac{\not{k} = k_\mu \gamma^\mu}{Z_2} \tag{46}$$

and therefore

$$\Gamma^\mu(\text{on shell } p' = p) \;=\; \frac{\gamma^\mu}{Z_2}\,. \tag{47}$$

To relate this formula to the electric charge renormalization, consider how we measure the physical electron's charge $-e_{\text{ph}}$ in terms of QED. Basically, we let an on-shell electron emit or absorb a zero-momentum photon and measure the amplitude

$$\mathcal{M} \;=\; e_{\text{phys}} \times \epsilon_\mu \bar{u}' \gamma^\mu u. \tag{48}$$

In the bare perturbation theory, the scattering amplitudes beyond the tree level obtain as

$$i\mathcal{M} \;=\; \prod_{\substack{\text{eternal}\\ \text{legs}}} \sqrt{Z} \times \sum \binom{\text{amputated}}{\text{diagrams}}, \tag{49}$$

see my notes on the LSZ reduction formula for the explanation. In particular, for the electron-electron-photon 'scattering' amplitude we have

$$i\mathcal{M} \;=\; Z_2\sqrt{Z_3} \times \left(\sum \binom{\text{amputated}}{\text{diagrams}}\right) \times \binom{\text{spin/polarization}}{\text{factors } \epsilon_\mu, \bar{u}', u}. \tag{50}$$

Since any amputated diagram with just 3 external legs is 1PI, the sum of amputated diagrams here amounts to the dressed vertex $ie_b\Gamma^\mu$ for the appropriate momenta: $k = 0$ and on-shell $p' = p$. Consequently,

$$i\mathcal{M} \;=\; Z_2\sqrt{Z_3} \times ie_{\text{bare}} \times \epsilon_\mu \bar{u}' \Gamma^\mu(\text{on shell } p' = p) u, \tag{51}$$

and comparing this formula to eq. (48) we get

$$e_{\text{phys}} \times \gamma^\mu \;=\; Z_2\sqrt{Z_3} \times e_{\text{bare}} \times \Gamma^\mu(\text{on shell } p' = p). \tag{52}$$

Moreover, by definition of the electric charge renormalization factor Z_1,

$$Z_2\sqrt{Z_3} \times e_{\text{bare}} \;=\; Z_1 \times e_{\text{phys}} \tag{53}$$

(*cf.* my notes on QED Feynman rules), so eq. (52) becomes

$$e_{\text{phys}} \times \gamma^\mu \;=\; Z_1 \times e_{\text{phys}} \times \Gamma^\mu(\text{on shell } p' = p) \tag{54}$$

and therefore

$$\Gamma^\mu(\text{on shell } p' = p) = \frac{\gamma^\mu}{Z_1}. \tag{55}$$

Comparing this formula to eq. (47), we immediately see that we must have

$$Z_1 \;=\; Z_2\,, \tag{35}$$

quod erat demonstrandum.

⋆ ⋆ ⋆

In terms of the electric charge renormalization, the Ward identity (35) reduces eq. (52) to simply

$$e_{\rm phys} \;=\; \sqrt{Z_3}\times e_{\rm bare}\,. \tag{56}$$

Thus, the electric charge renormalization in QED stems solely from the EM field renormalization, regardless to what happens to the electron field. Moreover, the bare and the physical EM fields are related to each other by the same $\sqrt{Z_3}$ factor,

$$A^\mu_{\rm bare} \;=\; \sqrt{Z_3}\times A^\mu_{\rm phys}\,, \tag{57}$$

hence eq. (57) leads to

$$e_{\rm phys}A^\mu_{\rm phys} \;=\; e_{\rm bare}A^\mu_{\rm bare}\,. \tag{58}$$

Consequently, the gauge-covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ works in exactly the same way in terms of bare or physical fields and couplings. Thus, the gauge-covariant kinetic term for the electron field in the physical Lagrangian

$$\mathcal{L}_{\rm phys} \;\supset\; \overline{\Psi}(i\gamma^\mu D_\mu)\Psi \tag{59}$$

in the bare Lagrangian simply gets multiplied by the overall factor Z_2,

$$\mathcal{L}_{\rm bare} \;\supset\; Z_2\times\overline{\Psi}(i\gamma^\mu D_\mu)\Psi, \tag{60}$$

but the covariant derivative D_μ remains unchanged.

In QED with multiple charged fermions, each fermion species gets its own bare mass $m_{b,i}$, and its own field and coupling renormalization factors Z^1_i and Z^2_i, but for each species $Z^1_i = Z^2_i$. Consequently, the renormalized electric charges of all species remain exactly the same multiples of the physical charge unit:

$$\text{given } q_i^{\text{bare}} = n_i \times e^{\text{bare}}, \quad \text{we get } q_i^{\text{phys}} = n_i \times e^{\text{phys}} \quad \text{for same } e_{\text{phys}} = \sqrt{Z_3} \times e_{\text{bare}}. \tag{61}$$

If we add a charged scalar field to QED, its renormalization is governed by similar Ward identities. The gauge-covariant kinetic term in the physical Lagrangian for such scalar is

$$\mathcal{L}_{\text{phys}} \supset (D_\mu \Phi^*)(D_\mu \Phi) = (\partial_\mu \Phi^*)(\partial^\mu \Phi) + eA_\mu \times (-i\Phi^* \partial^\mu \Phi + i\Phi \partial^\mu \Phi^*) + e^2 A_\mu A^\mu \times \Phi^* \Phi, \tag{62}$$

which in the bare Lagrangian becomes

$$\mathcal{L}_{\text{bare}} \supset Z_2 \times (\partial_\mu \Phi^*)(\partial^\mu \Phi) + Z_1^{1\gamma} \times eA_\mu \times (-i\Phi^* \partial^\mu \Phi + i\Phi \partial^\mu \Phi^*) + Z_1^{2\gamma} \times e^2 A_\mu A^\mu \times \Phi^* \Phi. \tag{63}$$

A priori, we should have 3 field and coupling renormalization factors here, Z_2, $Z_1^{1\gamma}$, and $Z_1^{2\gamma}$, but the Ward identity for the scalar field makes them identically equal,

$$Z_2 = Z_1^{1\gamma} = Z_1^{2\gamma}. \tag{64}$$

Consequently, the 3 terms in the bare Lagrangian (63) can be reassembled into a gauge-invariant combination

$$\mathcal{L}_{\text{bare}} \supset Z_2 \times (D_\mu \Phi^*)(D^\mu \Phi). \tag{65}$$

Finally, in terms of the counterterm perturbation theory, all these Ward identities become

$$\forall \text{ charged fermion field } \Psi_i(x), \quad \delta_2^{(i)} = \delta_1^{(i)}, \tag{66}$$

$$\forall \text{ charged scalar field } \Phi_i(x), \quad \delta_2^{(i)} = \delta_{1(1\gamma)}^{(i)} = \delta_{1(2\gamma)}^{(i)}. \tag{67}$$