AXIAL ANOMALY

Introduction

Consider a QED-like theory with an exactly massless electron,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; i\overline{\Psi}(\partial\!\!\!/ - ieA\!\!\!/)\Psi \;+\; \mathcal{L}^{\text{gauge}}_{\text{fixing}} \;+\; \mathcal{L}_{\text{ghost}} \;+\; \mathcal{L}^{\text{counter}}_{\text{terms}}. \tag{1}$$

The action of this theory is invariant under the global axial symmetry

$$\Psi(x) \;\to\; \exp(+i\theta\gamma^5)\Psi(x), \quad \Psi^\dagger(x) \;\to\; \Psi^\dagger(x)\exp(-i\theta\gamma^5), \quad \overline{\Psi}(x) \;\to\; \overline{\Psi}(x)\exp(+i\theta\gamma^5), \tag{2}$$

which leads to the *classical* conservation of the axial current $J^{\mu5} = \overline{\Psi}\gamma^\mu\gamma^5\Psi$, $\partial_\mu J^{\mu5} = 0$. However, the measure of the functional integral over the electron fields

$$\iiint\!\mathcal{D}[\overline{\Psi}(x)]\iiint\!\mathcal{D}[\Psi(x)]\;\exp\bigl(iS[\overline{\Psi},\Psi,A_\mu]\bigr) \tag{3}$$

— or rather, *the UV-regulated measure of the functional integral over the electron fields in the EM field background — is not invariant under the axial symmetry* (2). Consequently, the axial current $J^{\mu5}$ is not conserved in the quantum theory. Instead, we have the *axial anomaly* — also known as the *triangle anomaly* or the *Adler–Bell–Jackiw anomaly* —

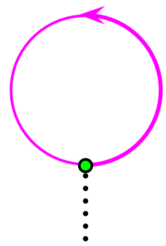
$$\left\langle \partial_\mu J^{\mu5}(x)\right\rangle \;=\; -\frac{e^2}{16\pi^2}\left\langle \epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}(x)F_{\mu\nu}(x)\right\rangle \;=\; -\frac{e^2}{2\pi^2}\left\langle \mathbf{E}\cdot\mathbf{B}(x)\right\rangle \tag{4}$$

Formally, the anomalous non-conservation of the axial current obtains as

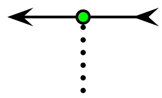
$$\left\langle \partial_\mu J^{\mu5}\right\rangle \;=\; \left\langle \partial_\mu\bigl(\overline{\Psi}\gamma^\mu\gamma^5\Psi\bigr)\right\rangle \;=\; \mathrm{Tr}\left((\partial_\mu\gamma^\mu\gamma^5)\times\frac{1}{D\!\!\!\!/}\right) \tag{5}$$

where Tr is the functional trace in the space of Dirac spinor fields and $1/D\!\!\!\!/$ is the electron propagator in the EM background. Diagrammatically, eq. (5) amounts to

$$\langle \partial_\mu J^{\mu 5}(q) \rangle \;=$$

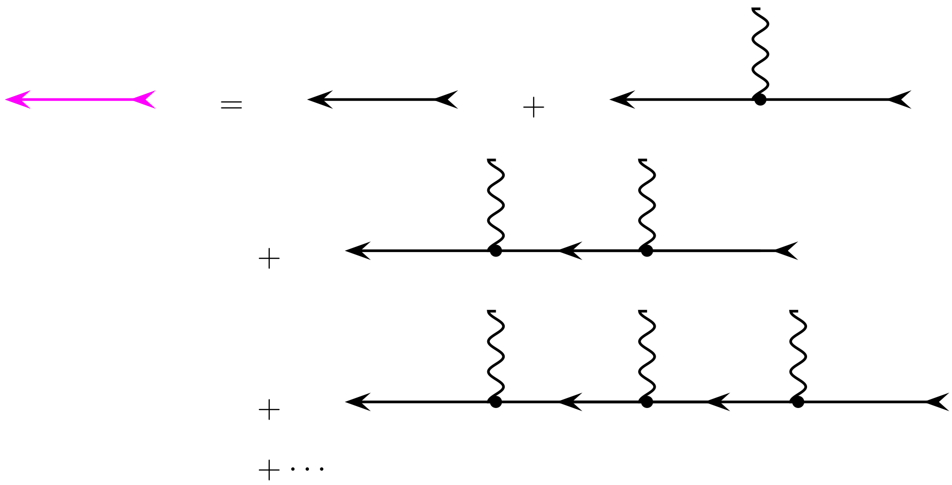


where q is the incoming momentum along the dotted line, the green vertex is



$$= \; -iq_\mu \times \gamma^\mu \gamma^5 \,, \tag{6}$$

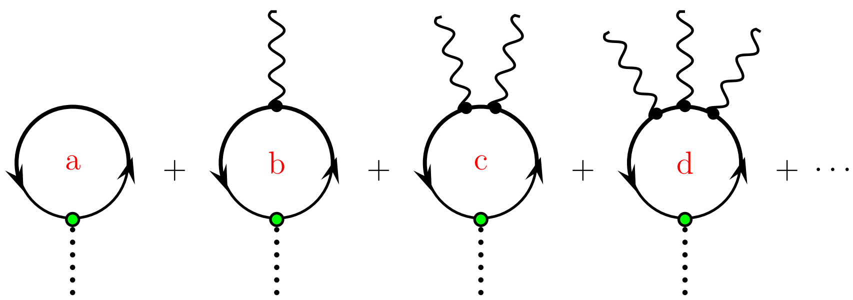
and the purple arrowed line is the electron's propagator in the EM background. In terms of the free electron propagator (denoted by a black arrowed line),



(7)

hence

$$\langle \partial_\mu J^{\mu 5}(q) \rangle \;=$$

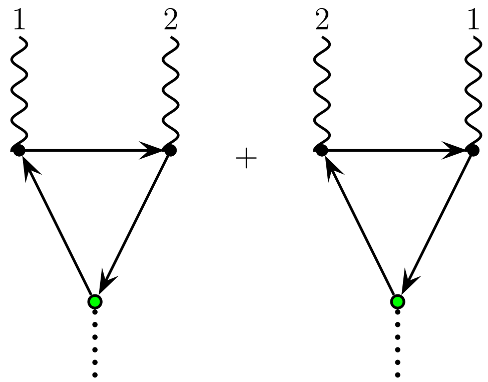


(8)

Actually, the photon-less diagram (8.a) vanishes by momentum conservation (which requires $q = 0$ and hence zero green vertex); also the Dirac trace in the numerator vanishes even before one integrates over the loop momentum. Likewise, the one-photon diagram (8.b) vanishes by Lorentz and charge-conjugation symmetries. (The photon is a C-negative vector while $\partial_\mu J^{\mu 5}$ is a C-positive pseudoscalar.) So the non-trivial contributions to the axial anomaly begin with the two-photon diagram (8.c).

Adler–Bardeen Theorem

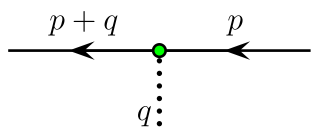
In fact, the axial anomaly comes *entirely* from the two-photon one-loop diagrams (8.c) (there are two such diagrams related by the photon permutation). By the Adler–Bardeen theorem, all the diagrams involving more than two photons cancel each other, and all the multi-loop diagrams also cancel each other. Such cancellation works similarly to the Ward–Takahashi identities for the vector current $J^\mu = \overline{\Psi}\gamma^\mu\Psi$, $\partial_\mu J^\mu = 0$, *cf.* my notes for the diagrammatic proof of the WT identities. However, for the one-loop two-photon diagrams



(9)

the proof of WT-like identities for the axial current fails due to un-regulatable UV divergence, and that's what leads to the axial anomaly.

To see how this works, consider the green $-i\,\not{q}\gamma^5$ vertex for the $\partial_\mu J^{\mu 5}$ between two *massless* electron propagators:

$S \stackrel{\text{def}}{=}$ 

$$= \frac{i}{\not{p}+\not{q}} \times (-i\,\not{q}\gamma^5) \times \frac{i}{\not{p}}\,. \tag{10}$$

Since the massless propagators anticommute with the γ^5 — indeed,

$$\gamma^5 \times \frac{1}{\not{p}} = \gamma^5 \times \frac{\not{p}}{p^2} = -\frac{\not{p}}{p^2} \times \gamma^5 = -\frac{1}{\not{p}} \times \gamma^5, \tag{11}$$

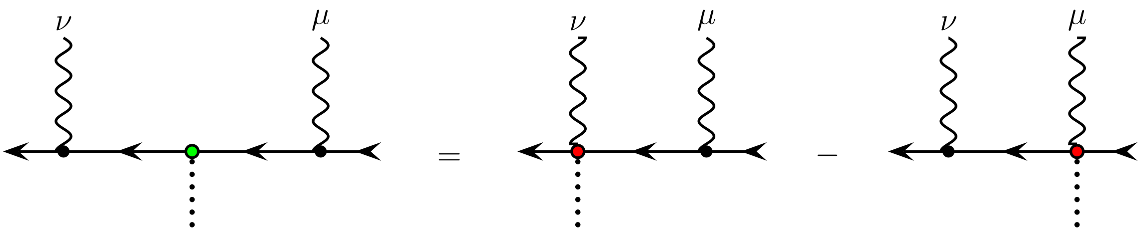
— we may rewrite the propagator-vertex-propagator combo (10) as

$$S \;=\; i\frac{1}{\not{p}+\not{q}} \times \left(\not{q} \;=\; (\not{p}+\not{q})-\not{p}\right) \times \left(\gamma^5\frac{1}{\not{p}} \;=\; -\frac{1}{\not{p}}\gamma^5\right) \;=\; -i\left(\frac{1}{\not{p}}-\frac{1}{\not{p}+\not{q}}\right)\gamma^5. \qquad (12)$$

Now let's put this combo between two photon vertices $(ie\gamma^\nu)$ and $(ie\gamma^\mu)$:

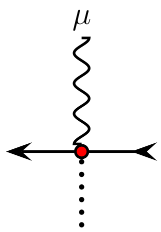
$$\begin{aligned}(ie\gamma^\nu)\times S\times(ie\gamma^\mu) \;&=\; (ie\gamma^\nu)\times(-i)\left(\frac{1}{\not{p}}-\frac{1}{\not{p}+\not{q}}\right)\gamma^5\times(ie\gamma^\mu)\\ &=\; -(ie\gamma^\nu)\frac{i}{\not{p}}\gamma^5(ie\gamma^\mu)\;+\;(ie\gamma^\nu)\frac{i}{\not{p}+\not{q}}\gamma^5(ie\gamma^\mu)\\ &=\; +(ie\gamma^\nu\gamma^5)\frac{i}{\not{p}}(ie\gamma^\mu)\;-\;(ie\gamma^\nu)\frac{i}{\not{p}+\not{q}}(ie\gamma^\mu\gamma^5).\end{aligned} \qquad (13)$$

Diagrammatically, this formula amounts to



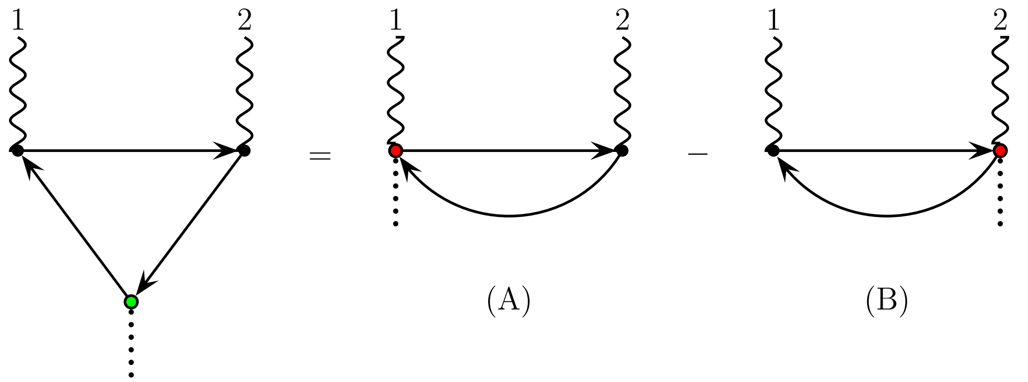
(14)

where the red vertex combines the photon and the axial current divergence into an axial-photon-like vertex



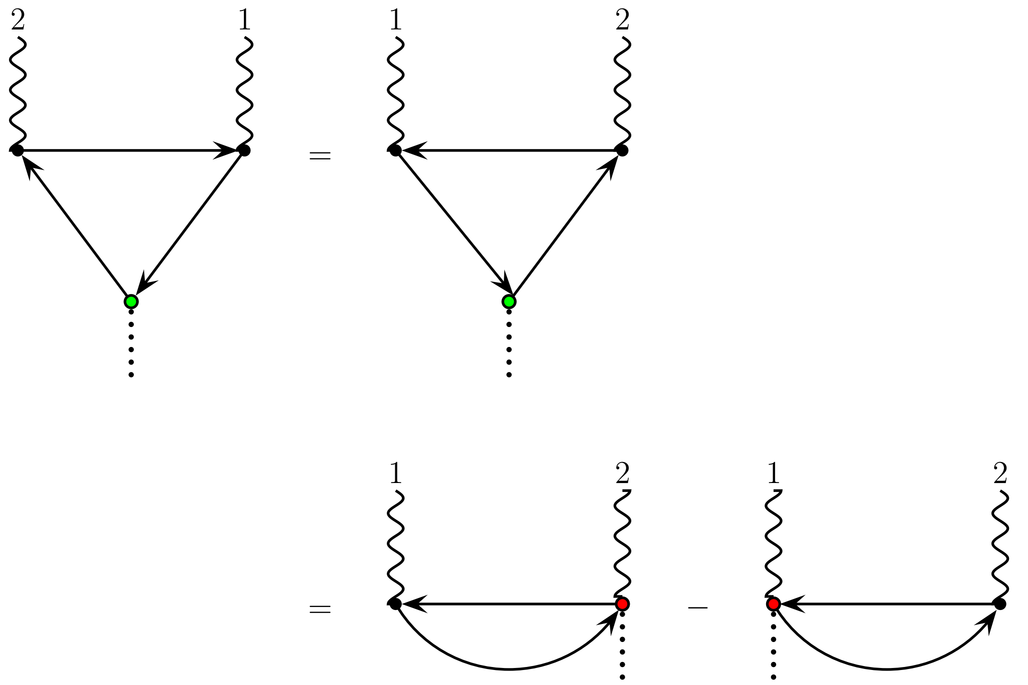
$$= \; ie\gamma^\mu\gamma^5. \qquad (15)$$

Now let's apply the diagrammatic equation (14) to the triangle diagrams (9). For the first triangle diagram we get



(A) (B) (16)

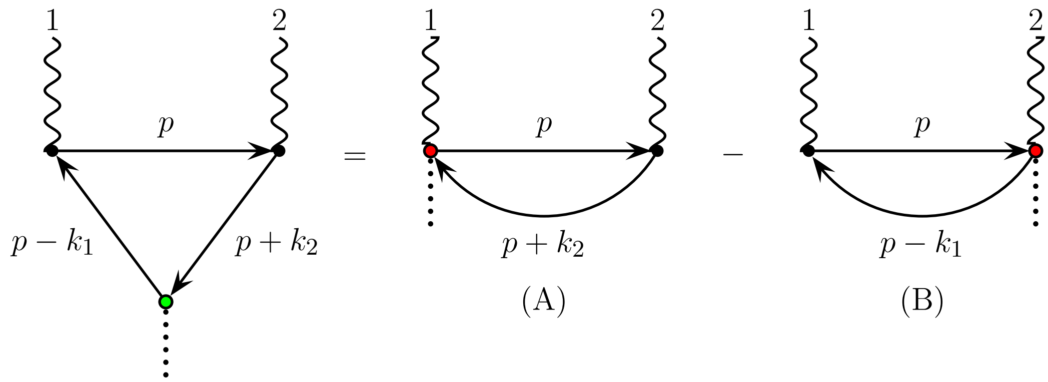
and likewise for the second diagram



(C) (D) (17)

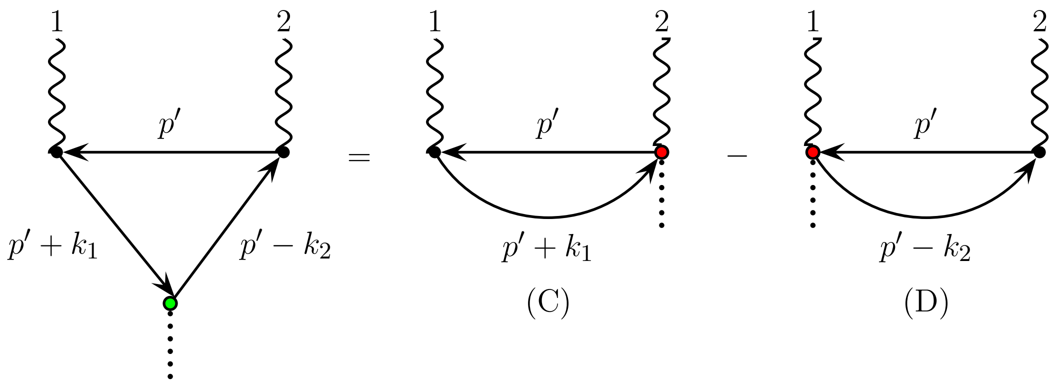
When we add the two triangle diagrams on the LHS, on the RHS we get a formal cancellation: Diagram (D) cancels diagram (A), and diagram (C) cancels diagram (B).

However, all these diagrams suffer from quadratic UV divergences, so when canceling diagrams one must be careful identifying the corresponding momenta. Let the two photons have momenta k_1 and k_2 (treated as incoming, hence $k_1 + k_2 + q = 0$), then for the first triangle diagram we have



(18)

and likewise for the second diagram



(19)

To properly cancel the diagrams for similar momenta in similar propagators, we need

$$p' = p + k_2 \quad \text{to cancel (A) and (D),} \tag{20}$$

$$p' = p - k_1 \quad \text{to cancel (B) with (C),} \tag{21}$$

and the two identifications are inconsistent since $k_1 + k_2 = -q \neq 0$! Thus, the cancellation between all 4 diagrams works only if we can UV-regulate each diagram in a manner which allows us to shift the loop momenta.

Or rather, we get a cancellation of the axial current divergence $\partial_\mu J^{\mu 5}$ at the two-photon level *only* if we can find a UV regulator with the following properties:

1. It does regulate the UV divergences of all the relevant diagrams.

2. It allows constant shifts of loop momenta, $\int d^4p \to \int d^4(p+\text{const})$.

3. It does not screw up the gauge invariance of QED (beyond the usual gauge fixing) and the electric current conservation.

4. It preserves the chiral symmetry of massless electrons and hence the relation

$$\frac{i}{\not p + \not q} \times (-i \not q \gamma^5) \times \frac{i}{\not p} \;=\; \left(\frac{i}{\not p + \not q} - \frac{i}{\not p}\right)\gamma^5. \tag{22}$$

Alas, such UV regulators do not exist!

Indeed, let's look at a few common (and not-so-common) UV regulators and see that none of them has all four of the required features:

- Wilson's hard-edge cutoff. Although it does regulate all the UV divergences, it clearly fails the requirements 2, 3, and 4.

- The dimensional regularization fulfils the requirements 1, 2, and 3, but fails the requirement 4 in a rather subtle way. Specifically, DR screws up the $\epsilon^{\alpha\beta\mu\nu}$ tensor and hence the relation of the γ^5 matrix defining the chirality to the $\epsilon^{\alpha\beta\mu\nu}\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu$. Consequently, we end up with different analytic continuations of the γ^5 matrix to $D<4$ dimensions, the $\gamma^5_{(1)}$ which anticommutes with all the γ^μ, and a different $\gamma^5_{(2)}$ in the axial symmetry current $\overline{\Psi}\gamma^\mu\gamma^5_{(2)}\Psi$. It is this difference which breaks the requirement 4.

- The Pauli–Villars cutoff also fulfils the requirements 1, 2, and 3 but breaks the chiral symmetry. Indeed, the regulated axial current becomes

$$J^{\mu 5}_{\text{reg}} \;=\; \overline{\Psi}\gamma^\mu\gamma^5\Psi \;-\; \overline{\eta}\gamma^\mu\gamma^5\eta \tag{23}$$

  where η PV compensator field — a very heavy charged Dirac field with a wrong sign of the Hilbert-space norm (and hence wrong sign of η loops). Since the η field is massive — its mass M_{PV} acts as a UV cutoff scale Λ, — its axial current is not conserved,

hence

$$\partial_\mu J^{\mu 5}_{\rm reg} \;=\; -2iM_{\rm PV}\times\overline{\eta}\gamma^5\eta \;\neq 0. \tag{24}$$

Also, while the massless electron propagator obeys the relation (22), there is no similar relation for the massive compensator field η,

$$\frac{i}{\not p+\not q-M_{\rm PV}}\times(-i\not q\gamma^5)\times\frac{i}{\not q-M_{\rm PV}} \;\neq\; \left(\frac{i}{\not p+\not q-M_{\rm PV}}-\frac{i}{\not q-M_{\rm PV}}\right)\gamma^5. \tag{25}$$

⋆ Covariant higher-derivative regulator. Unlike the other UV regulators we have considered thus far, CHD do preserve both the gauge symmetry and the axial symmetry of the massless QED. (Assuming that the covariant higher derivatives for the electron field anticommute with the γ^5, for example $\mathcal{L}\supset(-i/2\Lambda^2)\overline{\Psi}\not D^3\Psi$.) In particular, despite the modified electron propagators and the modified axial divergence vertex, the modified propagators and vertices do obey the appropriate generalization of eq. (22). The CHD regulator also allows for shifting the loop momentum variable, $\int d^4p\to\int d^4p'$, as long as the constant $p'-p\ll\Lambda$.

Unfortunately, while the CHD regulates the overall UV divergences of all the multi-loop graphs, it fails to regulate the divergences of some of the one-loop graphs. In particular, it fails to regulate the triangle graphs (9) or the (A), (B), (C), (D) graphs. Indeed, if the highest covariant derivative of the electron field in the Lagrangian is D^n, then at very large loop momentum p,

$$\text{each propagator }\propto\frac{1}{p^n},\qquad\text{each one-photon vertex }\propto p^{n-1}, \tag{26}$$

and hence the superficial degree of divergence for a graphs like (A), (B), (C), (D) is

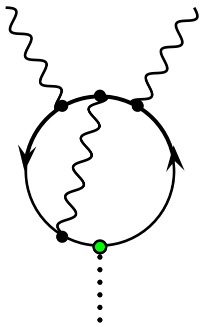
$$\mathcal{D} \;=\; 4\;+\;2\times(n-1)\;-\;2\times n\;=\;+2, \tag{27}$$

so these graphs remain quadratically divergent despite the CHD regulator.

On the other hand, by adding enough covariant derivatives of the EM field — or equivalently, ordinary derivatives of the $F_{\mu\nu}$, for example

$$\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}\big(-\partial^2/\Lambda^2\big)^n F^{\mu\nu}, \tag{28}$$

— one may regulate all the multi-loop graphs such as

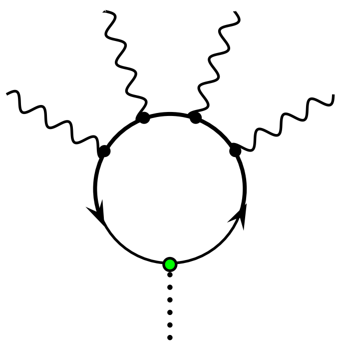


(29)

Thanks to the very existence of such a regulator, one can diagrammatically prove the Ward–Takahashi-like identities of the axial symmetry for all the multi-loop graphs. In particular, one can show that *all the multi-loop contributions to the axial current divergence $\partial_\mu J^{\mu 5}$ cancel each other.* And that's why *the axial anomaly is a purely-one-loop effect* — which is what the Adler–Bardeen theorem says.

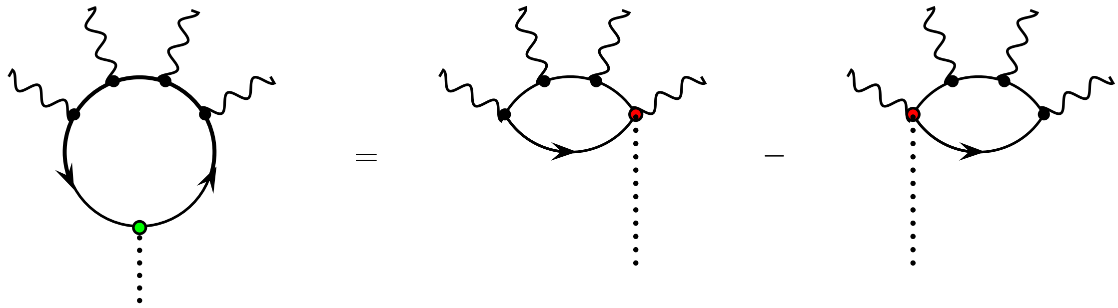
⋆ ⋆ ⋆

Let me complete this section by considering the multi-photon one-loop diagrams for the axial anomaly such as



(30)

for $n_\gamma = 4$ photons. Similarly to what we had for $n_\gamma = 2$, we may re-express any such diagram as a difference



(31)

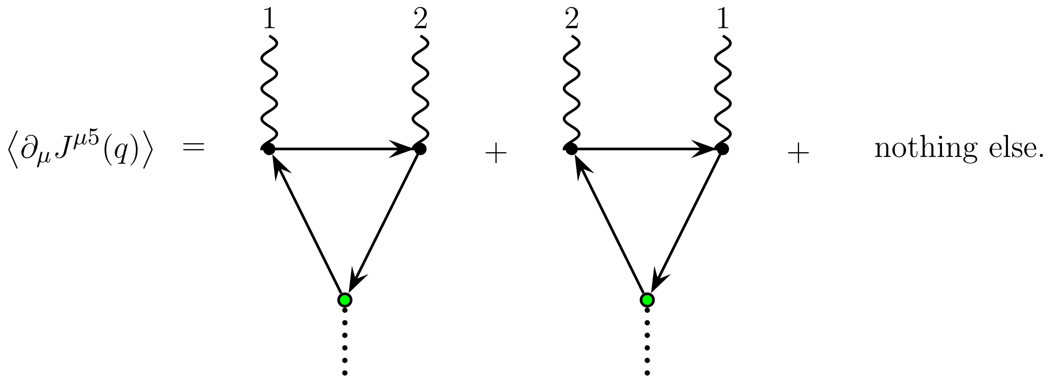
and then after we sum over $n_\gamma!$ permutations of the n_γ photons, we end up with a formal cancellation of the net amplitude. But the real cancellation happens if and only if the diagrams are UV finite, or at worst logarithmically divergent, since a $\log\Lambda$ divergence allows constant shifts of loop momenta. But for UV divergences worse than logarithmic, we would need a UV regulators that both allows momentum shifts and does not screw up the relations (31) — and alas, there are no such regulators.

So what is the superficial degree of divergence of an n_γ-photon one-loop diagram? The diagrams on the RHS of eq. (31) have n_γ vertices and n_γ fermionic propagators; in the absence of CHD regulators, the vertices are $(ie\gamma^\mu)$ or $(ie\gamma^\mu\gamma^5)$ without any powers of the loop momentum, while the propagators scale like $1/p$, hence the net superficial degree of divergence

$$\mathcal{D}\;=\;4\;-\;n_\gamma\,. \tag{32}$$

Thus, for $n_\gamma > 4$, all the diagrams are UV finite, so we may shift their respective loop momenta as we please, hence the formal cancellation becomes real cancellation. Likewise, for $n_\gamma = 4$, the diagrams are logarithmically divergent, but we may still shift their respective loop momenta as we please, so again the formal cancellation becomes real cancellation. Thus, the only possible contributions to the axial anomaly are the diagrams with $n_\gamma \leq 3$ photons.

Moreover, for $n_\gamma = 1$ or $n_\gamma = 3$ the net amplitude vanishes by the charge-conjugation symmetry of QED: the axial current divergence $\partial_\mu J^{\mu 5}$ is C-even while the photons are C-odd, so the number of photons contributing to the anomaly must be even. Also, for $n_\gamma = 0$ the diagram vanishes even before we integrate over the loop momentum because $q^\mu = 0$ in the green vertex, and also because the Dirac trace in the numerator vanishes due to too few γ matrices. And this leaves us with the only non-trivial contribution for $n_\gamma = 2$. Thus, *the axial anomaly of QED comes entirely from the two triangle diagrams*



(9)

Calculating the Triangle Anomaly

In this section I am going to evaluate the UV-regulated triangle diagrams (9) for the axial anomaly. People have evaluated these diagrams using a wide variety of UV regulators, — and you can find many of them in different textbooks, — but in these notes I am going to use Pauli–Villars. Thus, the regulated axial current is

$$J^{\mu 5}_{\text{reg}} = \overline{\Psi}\gamma^\mu\gamma^5\Psi - \overline{\eta}\gamma^\mu\gamma^5\eta \tag{33}$$

where η is the PV compensating field: it's a Dirac spinor, of the same electric charge $-e$ as the electron but of a very large mass M, and it's loops carry a wrong sign, hence the '−' sign of the second term in eq. (33). Classically, in the EM background

$$\not{D}\Psi = 0, \quad D_\mu\overline{\Psi}\gamma^\mu = 0, \quad \text{but} \quad \not{D}\eta = -iM\eta, \quad D_\mu\overline{\eta}\gamma^\mu = +iM\overline{\eta}, \tag{34}$$

hence

$$\partial_\mu\big(\overline{\Psi}\gamma^\mu\gamma^5\Psi\big) = \big(D_\mu\overline{\Psi}\gamma^\mu\big)\times\gamma^5\Psi - \overline{\Psi}\gamma^5\times\big(\not{D}\Psi\big) = 0 + 0 = 0 \tag{35}$$

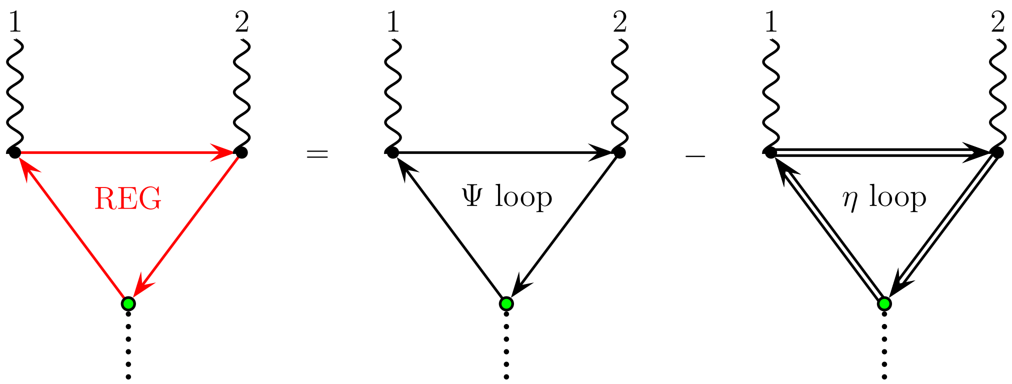
but

$$\partial_\mu\big(\overline{\eta}\gamma^\mu\gamma^5\eta\big) = \big(D_\mu\overline{\eta}\gamma^\mu\big)\times\gamma^5\eta - \overline{\eta}\gamma^5\times\big(\not{D}\eta\big) = +iM\overline{\eta}\times\gamma^5\eta - \overline{\eta}\gamma^5\times(-iM\eta) = +2iM\overline{\eta}\gamma^5\eta \tag{36}$$

instead of zero, and therefore

$$\left\langle \partial_\mu J^{\mu 5}_{\rm reg} \right\rangle \;=\; -2iM \left\langle \overline{\eta}\gamma^5\eta \right\rangle \;\neq\; 0. \tag{37}$$

Diagrammatically, we can get the same result (37) from the following argument: The PV-regulated triangle graph amounts to

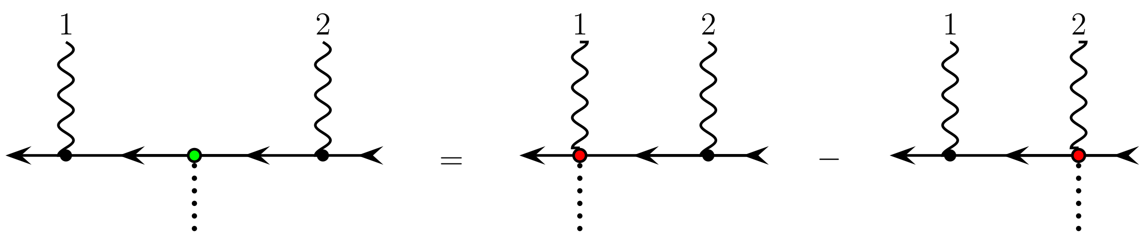


(38)

where the double line is the heavy η propagator. For a massless electron propagator we saw that

$$\frac{i}{\not p + \not q} \times (-i\,\not q\gamma^5) \times \frac{1}{\not q} \;=\; \frac{+i}{\not p + \not q} \times \gamma^5 \;+\; \gamma^5 \times \frac{i}{\not p} \tag{39}$$

and hence



(14)

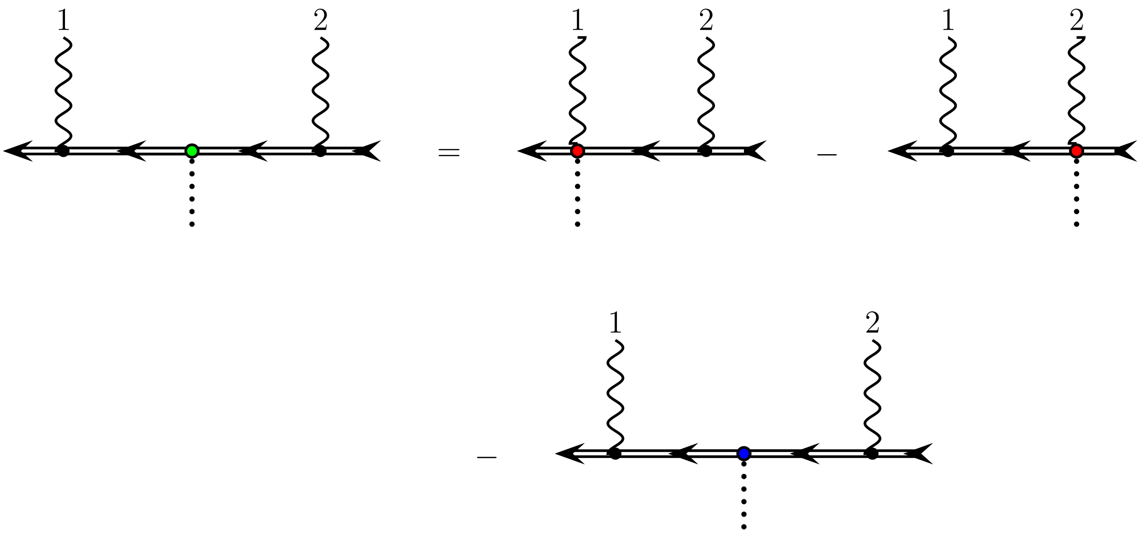
But for the massive η propagator, the algebra is more complicated:

$$\gamma^5 \times \frac{1}{\not p - M} \;=\; \gamma^5 \times \frac{\not p + M}{p^2 - M^2} \;=\; \frac{M - \not p}{p^2 - M^2} \times \gamma^5 \;=\; -\frac{1}{\not p + M} \times \gamma^5, \tag{40}$$

hence

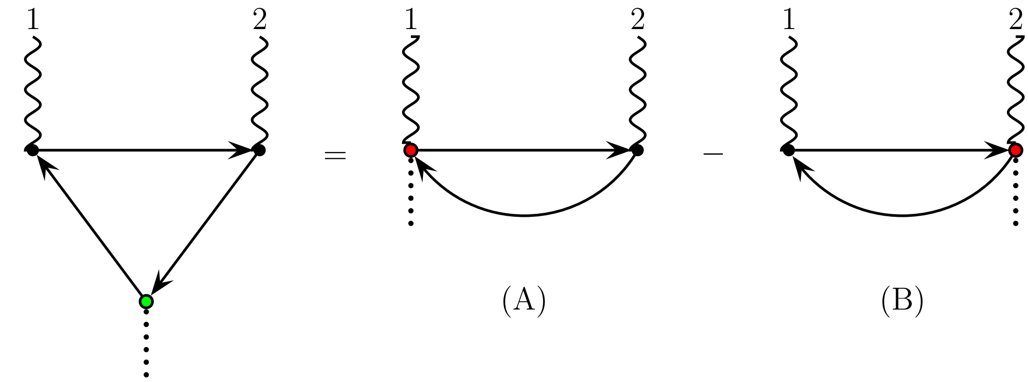
$$
\begin{aligned}
&\frac{i}{\not{p}+\not{q}-M}\times(-i\,\not{q}\gamma^5)\times\frac{i}{\not{p}-M} \;= \\
&\qquad = \; -i\frac{1}{\not{p}+\not{q}-M}\;\not{q}\,\frac{1}{\not{p}+M}\times\gamma^5 \\
&\qquad = \; -i\frac{1}{\not{p}+\not{q}-M}\Big((\not{p}+\not{q}-M)\,-\,(\not{p}+M)\,+\,2M\Big)\frac{1}{\not{p}+M}\times\gamma^5 \\
&\qquad = \; -\frac{i}{\not{p}+M}\times\gamma^5\;+\;\frac{i}{\not{p}+\not{q}-M}\times\gamma^5\;+\;\frac{i}{\not{p}+\not{q}-M}(2iM)\frac{i}{\not{p}+M}\times\gamma^5 \\
&\qquad = \; \gamma^5\times\frac{i}{\not{p}-M}\;+\;\frac{i}{\not{p}+\not{q}-M}\times\gamma^5\;-\;\frac{i}{\not{p}+\not{q}-M}\times(+2iM\gamma^5)\times\frac{i}{\not{p}+M}\,,
\end{aligned}
\tag{41}
$$

and therefore



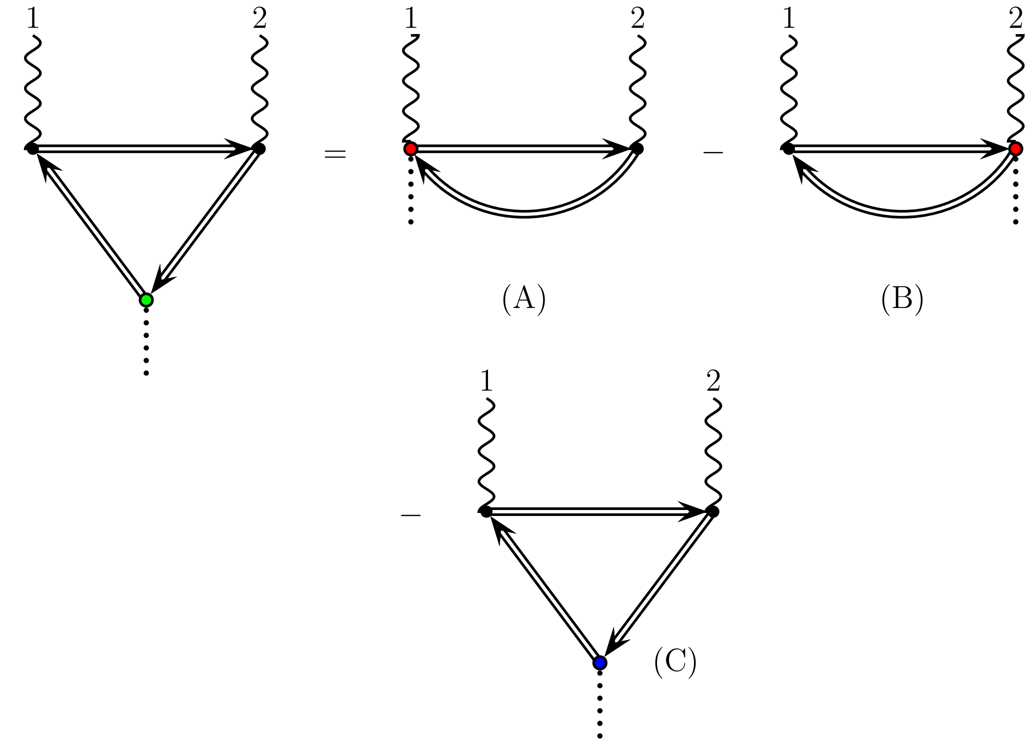
(42)

where the blue vertex's factor is $+2iM\gamma^5$. Substituting eqs. (14) and (42) into the PV-regulated triangle diagram (38), we get



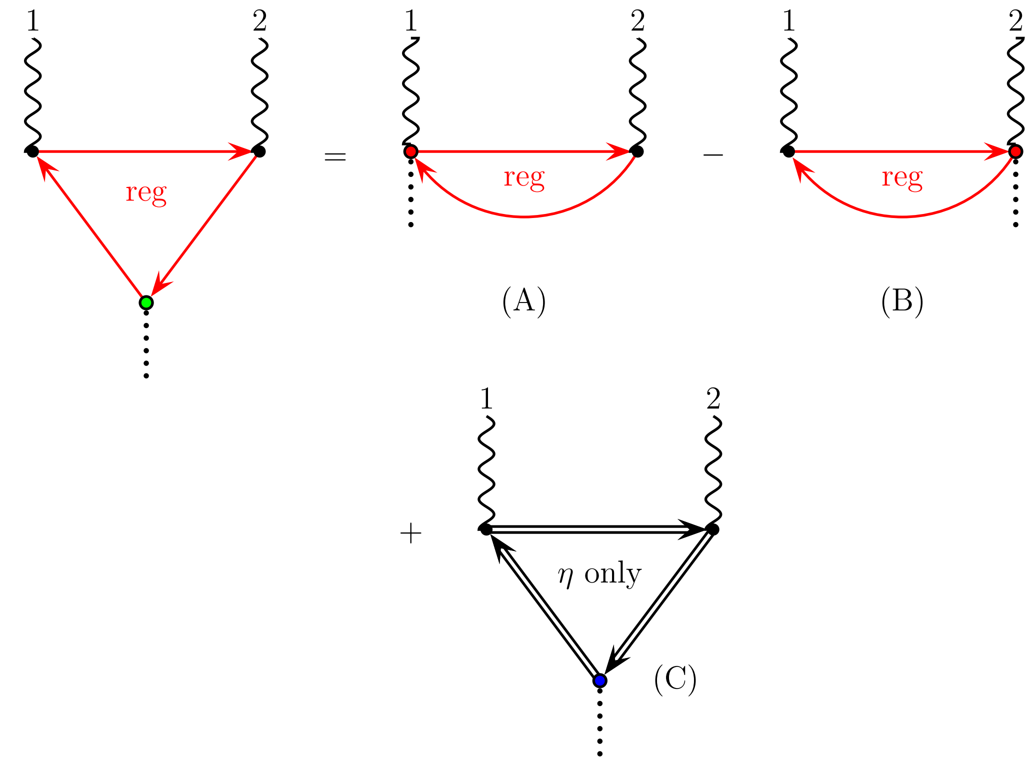
(43)

while



(44)

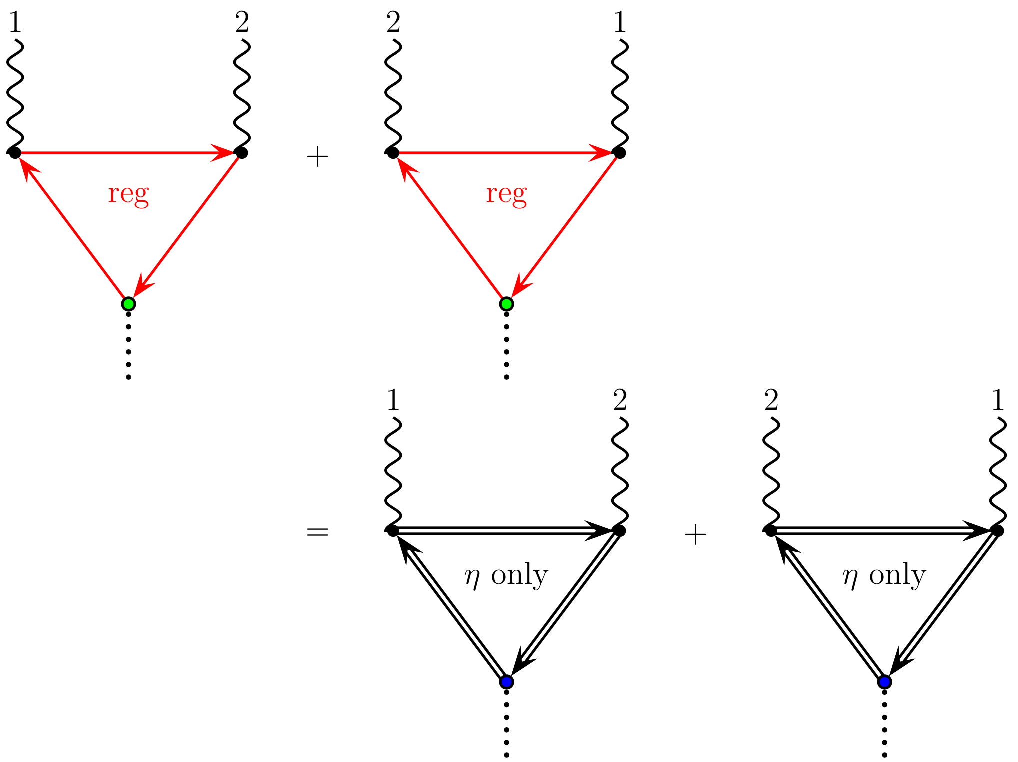
and therefore



(45)

where the diagrams (A) and (B) are PV-regulated — *i.e.*, obtain from differences between the massless and the superheavy loops, — while the diagram (C) involves only the superheavy field η.

Now let's add a similar PV-regulated triangle diagram with the two photons exchanged, $1 \leftrightarrow 2$. As we saw earlier, the diagram (A) for one photon order formally cancels the diagram (B) for the other photon order, and vice verse. For the regulated — and hence UV-finite — diagrams (A) and (B), a formal cancellation means actual cancellation, so all we are left with are the diagrams (C) for the two photon orders. Thus,



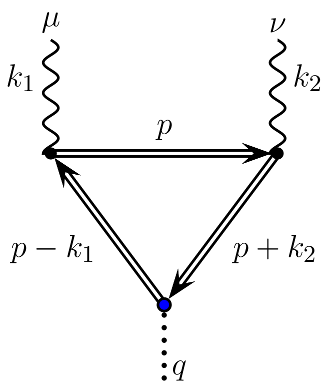
(46)

which is the diagrammatic way of expressing

$$\left\langle \partial_\mu J^{\mu 5}_{\mathrm{reg}} \right\rangle \;=\; 2iM \left\langle \overline{\eta}\gamma^5\eta \right\rangle \;\neq\; 0. \tag{37}$$

$\star \quad \star \quad \star$

After all these preliminaries, let's actually evaluate the diagrams (46):



$$\begin{aligned}
\mathcal{M} &= -\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\begin{array}{c}(2iM\gamma^5)\dfrac{i}{\not p+\not k_2-M+i0}(ie\gamma^\nu)\times\\ \times\dfrac{i}{\not p-M+i0}(ie\gamma^\mu)\dfrac{i}{\not p-\not k_1-M+i0}\end{array}\right)\\
&= +2Me^2\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu\nu}}{\mathcal{D}}
\end{aligned}\tag{47}$$

where

$$\mathcal{D} \;=\; \big((p-k_1)^2-M^2+i0\big)\times\big((p+k_2)^2-M^2+i0\big)\times\big(p^2-M^2+i0\big)\tag{48}$$

and

$$\mathcal{N}^{\mu\nu} \;=\; \mathrm{tr}\big(\gamma^5(\not p+\not k_2+M)\gamma^\nu(\not p+M)\gamma^\mu(\not p-\not k_1+M)\big).\tag{49}$$

To evaluate this Dirac trace, remember that the γ^5 must be accompanied by an even number ≥ 4 of the γ^α matrices. In our case, we have γ^ν and γ^μ, so two more should come from the three propagators' numerators, while the third propagator's numerator contributes the factor of M, thus

$$\mathcal{N}^{\mu\nu} \;=\; \mathrm{tr}\big(\gamma^5(\not p+\not k_2)\gamma^\nu\,\not p\gamma^\mu M\big)\;+\;\mathrm{tr}\big(\gamma^5(\not p+\not k_2)\gamma^\nu M\gamma^\mu(\not p-\not k_1)\big)\;+\;\mathrm{tr}\big(\gamma^5 M\gamma^\nu\,\not p\gamma^\mu(\not p-\not k_1)\big).\tag{50}$$

Next, all 4 traces here have form

$$\mathrm{tr}(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) \;=\; 4i\epsilon^{\alpha\beta\gamma\delta},\tag{51}$$

hence

$$
\begin{aligned}
\mathcal{N}^{\mu\nu} &= 4iM\epsilon^{\alpha\nu\beta\mu}(p+k_2)_\alpha p_\beta \;+\; 4iM\epsilon^{\alpha\nu\mu\beta}(p+k_2)_\alpha(p-k_1)_\beta \;+\; 4iM\epsilon^{\nu\alpha\mu\beta}p_\alpha(p-k_1)_\beta \\
&= 4iM\epsilon^{\alpha\nu\beta\mu}\Big(+(p+k_2)_\alpha p_\beta \;-\; (p+k_2)_\alpha(p-k_1)_\beta \;+\; p_\alpha(p-k_1)_\beta\Big) \\
&= 4iM\epsilon^{\alpha\nu\beta\mu}\Big((p+k_2)_\alpha k_{1\beta} \;+\; p_\alpha(p-k_1)_\beta \;=\; p_\alpha p_\beta \;+\; k_{2\alpha}k_{1\beta}\Big) \\
&\quad \langle\!\langle \text{ by antisymmetry of the } \epsilon \text{ tensor in } \alpha\leftrightarrow\beta \,\rangle\!\rangle \\
&= 4iM\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta}\,.
\end{aligned}
\tag{52}
$$

Note that this numerator is independent of the loop momentum p, hence

$$
\mathcal{M} \;=\; 2e^2M\times\mathcal{N}^{\mu\nu}\times\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{\mathcal{D}}\,. \tag{53}
$$

At very large momenta, the integrand here behaves like $(1/\mathcal{D})\sim 1/p^6$, so the remaining momentum integral is UV finite. To evaluate it, we introduce the Feynman parameters $x+y+z=1$, thus

$$
\frac{1}{\mathcal{D}} \;=\; \int\limits_0^1 d^3(x,y,z)\,\delta(x+y+z-1)\,\frac{2}{[\ell^2-\Delta+i0]^3} \tag{54}
$$

where

$$
\ell^2 \;-\; \Delta(x,y,z) \;=\; z(p^2-M^2) \;+\; x((p+k_2)^2-M^2) \;+\; y((p-k_1)^2-M^2) \tag{55}
$$

and hence

$$
\ell \;=\; p \;+\; xk_2 \;-\; yk_1 \tag{56}
$$

while

$$
\Delta(x,y,z) \;=\; M^2 \;-\; xzk_2^2 \;-\; yzk_1^2 \;-\; xyq^2. \tag{57}
$$

Consequently,

$$\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{\mathcal{D}}\;=\int\limits_0^1 d^3(x,y,z)\,2\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta+i0]^3} \tag{58}$$

where

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta+i0]^3}\;=\int\frac{d^4\ell_E}{(2\pi)^4}\frac{-i}{[\ell_E^2+\Delta]^3}\;=\;\frac{-i}{16\pi^2}\int\limits_0^\infty\frac{\ell_e^2\,d(\ell_e^2)}{[\ell_e^2+\Delta]^3}\;=\;\frac{-i}{16\pi^2}\times\frac{1}{2\Delta}\,. \tag{59}$$

Moreover,

$$\Delta(x,y,z)\;=\;M^2\;-\;O(k^2\,\mathrm{or}\,q^2). \tag{60}$$

where M is the mass of the PV compensator, which acts as the UV cutoff. We presume $M\gg$ any component of the external momenta $k_{1,2}^\mu$, which gives us

$$\frac{1}{\Delta(x,y,z)}\;=\;\frac{1}{M^2}\;+\;\frac{O(k^2\,\mathrm{or}\,q^2)}{M^4}\;\approx\;\frac{1}{M^2} \tag{61}$$

and therefore

$$\int\limits_0^1 d^3(x,y,z)\,2\delta(x+y+z-1)\,\frac{1}{\Delta(x,y,z)}\;=\;\frac{1}{M^2}\;+\;\frac{O(k^2\,\mathrm{or}\,q^2)}{M^4}\,. \tag{62}$$

Consequently, in eq. (53)

$$\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{\mathcal{D}}\;=\;\frac{-i}{32\pi^2M^2}\left(1\,+\,\frac{O(k^2\,\mathrm{or}\,q^2)}{M^2}\right) \tag{63}$$

and hence

$$\begin{aligned}\mathcal{M}\;&=\;2e^2M\times\Big(\mathcal{N}^{\mu\nu}\;=\;4iM\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta}\Big)\times\frac{-i}{32\pi^2M^2}\left(1\,+\,\frac{O(k^2\,\mathrm{or}\,q^2)}{M^2}\right)\\&=\;+\frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta}\times\left(1\,+\,\frac{O(k^2\,\mathrm{or}\,q^2)}{M^2}\right)\\&\xrightarrow[M\to\infty]{}\;+\frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta}\,.\end{aligned} \tag{64}$$

Or rather,

$$\mathcal{M}(1^{\rm st}\ \text{diagram}) \;=\; +\frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta}\,. \tag{65}$$

The second diagram (46) is related to the first diagram by exchanging the two photons, thus $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. It is easy to see that the first diagram's amplitude (65) is actually symmetric between the two photons:

$$\begin{aligned} \mathcal{M}(2^{\rm nd}\ \text{diagram}) &= +\frac{e^2}{4\pi^2}\epsilon^{\alpha\mu\beta\nu}k_{1\alpha}k_{2\beta} \\ \langle\!\langle\, \text{renaming } \alpha \leftrightarrow \beta \,\rangle\!\rangle \quad &= +\frac{e^2}{4\pi^2}\epsilon^{\beta\mu\alpha\nu}k_{1\beta}k_{2\alpha} \;=\; +\frac{e^2}{4\pi^2}\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta} \\ &= \mathcal{M}(1^{\rm st}\ \text{diagram}), \end{aligned} \tag{66}$$

and thus the net anomaly amplitude is simply

$$\mathcal{M}^{\mu\nu}_{\rm net} \;=\; 2\times\frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta}. \tag{67}$$

Finally, let us re-interpret the amplitude (67) in terms of photon fields, or rather Fourier-transformed photon fields $A^\mu(k)$. Since the final state has two identical photons, we should multiply the axial anomaly amplitude by $\frac{1}{2}A^\mu(k_1)A^\nu(k_2)$, thus

$$\begin{aligned} \big\langle(\partial_\mu J^{\mu5})(q)\big\rangle &= \mathcal{M}^{\mu\nu}_{\rm net}\times\tfrac{1}{2}A_\mu(k_1)A_\nu(k_2) \\ &= \frac{e^2}{2\pi^2}\,\epsilon^{\alpha\nu\beta\mu}k_{2\alpha}k_{1\beta}\times\tfrac{1}{2}A_\mu(k_1)A_\nu(k_2) \\ &= \frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}\,\big(k_{2\alpha}A_\nu(k_2)\big)\big(k_{1\beta}A_\mu(k_1)\big) \\ &= -\frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}\times(\partial_\alpha A_\nu)(k_1)\times(\partial_\beta A_\mu)(k_2), \end{aligned} \tag{68}$$

which after Fourier transforming to the coordinate space becomes

$$\big\langle\partial_\mu J^{\mu5}(x)\big\rangle \;=\; -\frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}\times\partial_\alpha A_\nu(x)\times\partial_\beta A_\mu(x). \tag{69}$$

Since the ϵ tensor is separately antisymmetric in $\alpha \leftrightarrow \nu$ and $\beta \leftrightarrow \mu$, we may rewrite this formula as

$$\begin{aligned}
\left\langle\partial_\mu J^{\mu 5}\right\rangle &= -\frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\mu}\times\tfrac{1}{2}(\partial_\alpha A_\nu-\partial_\nu A_\alpha)\times\tfrac{1}{2}(\partial_\beta A_\mu-\partial_\mu A_\beta)\\
&= -\frac{e^2}{16\pi^2}\,\epsilon^{\alpha\nu\beta\mu}F_{\alpha\nu}F_{\beta\mu}\,.
\end{aligned}\tag{70}$$

And this completes my calculation of the axial anomaly.

Let me complete this section with a few words about the ϵFF combination on the RHS of eq. (70). In the non-relativistic terms,

$$\epsilon^{\alpha\nu\beta\mu}F_{\alpha\nu}F_{\beta\mu} \;=\; 8\mathbf{E}\cdot\mathbf{B}.\tag{71}$$

Similar to $F^{\alpha\nu}F_{\alpha\nu}=2\mathbf{B}^2-2\mathbf{E}^2$, the combination (71) is invariant under *continuous* Lorentz symmetries, but it is a pseudoscalar rather than a scalar — it changes sign under the space reflection $\mathbf{x}\to-\mathbf{x}$. It is also C-even and therefore CP-odd and T-odd. Likewise, the axial anomaly $\partial_\mu J^{\mu 5}$ is a pseudoscalar — since $J^{\mu 5}$ is an axial Lorentz vector while ∂_μ is a true Lorentz vector, — and it's also C-even (*cf.* homework set#7) and hence CP-odd and T-odd.

Also, the ϵFF combination changes sign under electric-magnetic duality

$$F_{\alpha\nu}\;\to\;\tilde{F}_{\alpha\nu}\;\stackrel{\rm def}{=}\;\tfrac{1}{2}\epsilon_{\alpha\nu\beta\mu}F^{\beta\mu},\tag{72}$$

or in the non-relativistic notations

$$\mathbf{E}\;\to\;\tilde{\mathbf{E}}\;=\;-\mathbf{B},\qquad \mathbf{B}\;\to\;\tilde{\mathbf{B}}\;=\;+\mathbf{E}.\tag{73}$$

In terms of the dual EM fields, we may rewrite ϵFF as

$$\epsilon^{\alpha\nu\beta\mu}F_{\alpha\nu}F_{\beta\mu}\;=\;2F_{\alpha\nu}\tilde{F}^{\alpha\nu},\tag{74}$$

so the axial anomaly is often written as

$$\partial_\mu J^{\mu 5}\;=\;-\frac{e^2}{8\pi^2}\,F_{\alpha\nu}\tilde{F}^{\alpha\nu}.\tag{75}$$

Anomaly of the Fermionic Integral's Measure.

In the functional integral formulation of QED, the axial anomaly stems from the *measure* $\mathcal{D}[\overline{\Psi}]\mathcal{D}[\Psi]$ of the fermionic functional integral not being invariant under the axial symmetry. To see how this works, let's go to the Euclidean spacetime, pick a fixed but a non-trivial EM field background $A^\mu(x)$, and take a close look at the functional integral over the fermionic fields in that background,

$$Z[A^\mu] \;=\; \iiint\!\mathcal{D}[\overline{\Psi}] \iiint\!\mathcal{D}[\Psi]\,\exp\left(-S_E[\overline{\Psi},\Psi,A^\mu] \;=\; -\!\int\!\overline{\Psi}\,\not{D}\Psi\,d^4x_E\right) \tag{76}$$

Now, let's change the integration variables here from $\Psi(x)$ and $\overline{\Psi}(x)$ to

$$\Psi'(x) \;=\; \big(i\theta(x)\gamma^5\big)\Psi(x) \quad\text{and}\quad \overline{\Psi}'(x) \;=\; \overline{\Psi}(x)\exp\big(i\theta(x)\gamma^5\big) \tag{77}$$

for some x-dependent phase $\theta(x)$. On one hand, this is just a variable change, so the functional integral (76) in terms of the new variables should yield exactly the same partition function as the integral over the old variables,

$$Z'[A^\mu] \;=\; \iiint\!\mathcal{D}[\overline{\Psi}'] \iiint\!\mathcal{D}[\Psi']\,\exp\Big(-S_E[\overline{\Psi}',\Psi',A^\mu]\Big) \;=\; \text{original } Z[A^\mu]. \tag{78}$$

On the other hand, the transform (77) changes both the Euclidean action — since a *local* axial $U(1)$ is not a symmetry of the theory — and the measure of the functional integral, thus

$$S_E[\overline{\Psi}',\Psi',A^\mu] \;=\; S_E[\overline{\Psi},\Psi,A^\mu] \;+\; \Delta S_E \tag{79}$$

and the measure of the functional integral,

$$\mathcal{D}[\overline{\Psi}']\mathcal{D}[\Psi'] \;=\; \mathcal{D}[\overline{\Psi}]\mathcal{D}[\Psi]\times\mathcal{J} \tag{80}$$

where $\mathcal{J}$ is the functional Jacobian of the axial transform (77). Consequently,

$$Z'[A^\mu] \;=\; \iiint\!\mathcal{D}[\overline{\Psi}] \iiint\!\mathcal{D}[\Psi]\times\mathcal{J}\times\exp\Big(-S_E[\overline{\Psi},\Psi,A^\mu] \;-\; \Delta S_E\Big), \tag{81}$$

so to keep this integral exactly the same as the original integral (76), the effects of the

Jacobian and of the ΔS must cancel each other,

$$\mathcal{J} \times \exp(-\Delta S_E) \;=\; 1. \tag{82}$$

Or equivalently,

$$\log\mathcal{J} \;=\; \Delta S_E\,. \qquad (83).$$

Now let's take a closer look at the Jacobian $\mathcal{J}$ and the action change ΔS_E. The derivative $D_\mu = \partial_\mu - ieA_\mu$ is covariant WRT to the vector phase symmetry of the electron field accompanied by the gauge transform of the A_μ potential, but it is not covariant WRT to a local axial transform (77). Instead, we have

$$\begin{aligned} D_\mu\Psi'(x) \;&=\; D_\mu\bigl(e^{i\theta(x)\gamma^5}\Psi(x)\bigr) \;=\; \bigl(\partial_\mu e^{i\theta(x)\gamma^5}\bigr)\Psi(x) \;+\; e^{i\theta(x)\gamma^5}\bigl(D_\mu\Psi(x)\bigr) \\ &=\; e^{i\theta(x)\gamma^5}\Bigl(i(\partial_\mu\theta(x))\gamma^5\Psi(x) \;+\; D_\mu\Psi(x)\Bigr), \end{aligned} \tag{84}$$

hence

$$\begin{aligned} \mathcal{L}'_E \;&=\; \overline{\Psi}'\gamma^\mu D_\mu\Psi' \;=\; \overline{\Psi}e^{i\theta\Gamma^5}\gamma^\mu e^{i\theta\gamma^5}\Bigl((\partial_\mu\theta)\gamma^5\Psi \;+\; D_\mu\Psi\Bigr) \\ &=\; \overline{\Psi}\gamma^\mu\Bigl(i(\partial_\mu\theta)\gamma^5\Psi \;+\; D_\mu\Psi\Bigr) \;=\; i(\partial_\mu\theta)\times\overline{\Psi}\gamma^\mu\gamma^5\Psi \;+\; \overline{\Psi}\gamma^\mu D_\mu\Psi \\ &=\; i(\partial_\mu\theta)\times J^{\mu5} \;+\; \mathcal{L}_E^{\rm orig}, \end{aligned} \tag{85}$$

and therefore

$$\Delta S_E \;=\; \int\! d^4x_e\, i(\partial_\mu\theta)\times J^{\mu5} \;=\; -i\!\int\! d^4x_e\,\theta(x)\times\partial_\mu J^{\mu5}(x). \tag{86}$$

As to the Jacobian, *formally*

$$\frac{\mathcal{D}[\Psi'(x)]}{\mathcal{D}[\Psi(x)]} \;=\; \mathrm{Det}\bigl(\exp(i\hat{\Theta}\gamma^5)\bigr) \tag{87}$$

where Det is the functional determinant of the operator

$$\exp(i\hat{\Theta}\gamma^5) : \psi_\alpha(x) \;\mapsto\; \exp(i\theta(x)\gamma^5)_{\alpha\beta}\psi_\beta(x) \tag{88}$$

acting in the Hilbert space of spinor-valued functions $\psi_\alpha(x_1,x_2,x_3,x_4)$. Likewise,

$$\frac{\mathcal{D}[\overline{\Psi}'(x)]}{\mathcal{D}[\overline{\Psi}(x)]} \;=\; \mathrm{Det}\big(\exp(i\hat{\Theta}\gamma^5)\big), \tag{89}$$

so altogether

$$\mathcal{J} \;=\; \mathrm{Det}^2\big(\exp(i\hat{\Theta}\gamma^5)\big) \;=\; \mathrm{Det}\big(\exp(2i\hat{\Theta}\gamma^5)\big) \;=\; \exp\big(\mathrm{Tr}(2i\hat{\Theta}\gamma^5)\big). \tag{90}$$

Therefore, eq. (83) amounts to

$$-i\int d^4x_e\,\theta(x)\times\partial_\mu J^{\mu5}(x) \;=\; \mathrm{Tr}(2i\hat{\Theta}\gamma^5) \;=\; 2i\,\mathrm{Tr}(\hat{\Theta}\gamma^5). \tag{91}$$

Now let's evaluate the functional trace in this formula in the coordinate basis. Formally

$$\mathrm{Tr}(\Theta\gamma^5) \;=\; \int d^4x_e\ \mathrm{tr}_{\mathrm{Dirac}}\big(\langle x|\,\hat{\Theta}\gamma^5\,|x\rangle\big) \;=\; \int d^4x_e\,\theta(x)\times\mathrm{tr}_{\mathrm{Dirac}}\big(\langle x|\,\gamma^5\,|x\rangle\big), \tag{92}$$

but the diagonal matrix element here is UV-divergent as $\delta^{(4)}(x-x)$ so it needs to be UV-regulated. I shall discuss a suitable regulator momentarily, for the moment let's simply rewrite this formula as

$$\mathrm{Tr}(\Theta\gamma^5) \;=\; \int d^4x_e\,\theta(x)\times\mathrm{tr}_{\mathrm{Dirac}}\big(\langle x|\,\gamma^5\,|x\rangle_{\mathrm{reg}}\big). \tag{93}$$

In the context of eq. (91), it gives us

$$-i\int d^4x_e\,\theta(x)\times\partial_\mu J^{\mu5}(x) \;=\; +2i\int d^4x_e\,\theta(x)\times\mathrm{tr}_{\mathrm{Dirac}}\big(\langle x|\,\gamma^5\,|x\rangle_{\mathrm{reg}}\big). \tag{94}$$

Moreover, the consistency of the fermionic functional integral requires this cancellation to work for any x-dependent axial phase $\theta(x)$, and this calls for a local equation:

$$\forall x:\quad \partial_\mu J^{\mu5}(x) \;=\; -2\,\mathrm{tr}_{\mathrm{Dirac}}\big(\langle x|\,\gamma^5\,|x\rangle_{\mathrm{reg}}\big). \tag{95}$$

Next, consider the UV regulation of the matrix element here. In general,

$$\langle x|\,\gamma^5\,|x\rangle_{\text{reg}} \;=\; \langle x|\,\gamma^5\hat{G}\,|x\rangle \tag{96}$$

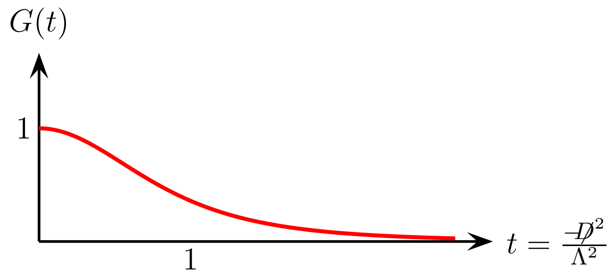
for some operator $\hat{G}$ such that its coordinate-space matrix elements $\langle x|\,\hat{G}\,|y\rangle$ are finite for $x = y$ but approximate the $\delta^{(4)}(x-y)$ for distances$\gg 1/\Lambda_{\text{UV}}$. In the momentum space, this means

$$G(p_e) \;\approx\; 1 \quad \text{for} \quad p_e^2 \ll \Lambda^2 \quad \text{but} \quad G(p_e) \;\approx\; 0 \quad \text{for} \quad p_e^2 \gg \Lambda^2. \tag{97}$$

However, back in the coordinate space, the regulating operator $\hat{G}$ should respect the gauge symmetry of the QED, so it should involve the covariant derivatives D_μ rather than the ordinary derivatives ∂_μ. Moreover, since the functional trace in eq. (91) is in the Hilbert space of spinor-valued wave functions with finite Euclidean actions $S_E = \int d^4x_e\,\overline{\psi}\,\not{D}\psi$, the regulating operator $\hat{G}$ should be a function of the $\not{D}_E = \gamma_E^\mu D_E^\mu$ combination rather that of the four D^μ. (This point is explained in detail in §22.2–3 of the Weinberg's textbook. Please read these sections as a part of your next homework#22.) Also, the UV regulating operator $\hat{G}$ should commute with the γ^5 matrix so it would not screw up the Dirac trace in eq. (95). Together, all these conditions require $\hat{G}$ to be a function of the positive-definite operator $-\not{D}_E^2$; specifically

$$\hat{G} \;=\; G(-\not{D}_E^2/\Lambda^2) \tag{98}$$

for some analytic function $G(t)$ which behaves like



$$\begin{aligned} G(t \ll 1) \;&\approx\; 1, \\ G(t \gg 1) \;&\approx\; 0, \end{aligned} \tag{99}$$

In Euclidean spacetime

$$\not{D}^2 \;=\; D^2 \;+\; \frac{e}{2}F^{\mu\nu}\sigma^{\mu\nu}, \tag{100}$$

and this is what gives the $\hat{G}$ operator it's Dirac indices. Assuming $F^{\mu\nu} \ll \Lambda^2$, we may expand $\hat{G}$ in powers of the tension fields, thus

$$\hat{G} \;=\; G(-D^2/\Lambda^2) - \frac{e}{2\Lambda^2}\,G'(-D^2/\Lambda^2)\times F^{\mu\nu}\sigma^{\mu\nu} + \frac{e^2}{8\Lambda^2}\,G''(-D^2/\Lambda^4)\times F^{\kappa\lambda}F^{\mu\nu}\sigma^{\kappa\lambda}\sigma^{\mu\nu} + \cdots, \tag{101}$$

where $G'(t) = dG/dt$, $G''(t) = d^2G/dt^2$, *etc.* In the Dirac trace $\mathrm{tr}(\hat{G}\gamma^5)$, we have

$$\mathrm{tr}(\gamma^5) \;=\; 0, \quad \mathrm{tr}(\gamma^5\sigma^{\mu\nu}) \;=\; 0, \quad \text{but} \quad \mathrm{tr}(\gamma^5\sigma^{\kappa\lambda}\sigma^{\mu\nu}) \;=\; 4\epsilon_E^{\kappa\lambda\mu\nu}, \tag{102}$$

hence to the leading order in $F^{\mu\nu}/\Lambda^2$,

$$\mathrm{tr}_{\mathrm{Dirac}}(\gamma^5\hat{G}) \;=\; \frac{e^2}{2\Lambda^4}\,G''(-D^2/\Lambda^4)\times\epsilon^{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu}. \qquad \approx$$

Altogether, we have

$$\begin{aligned}\partial_\mu J^{\mu 5}(x) \;&=\; -2\,\mathrm{tr}_{\mathrm{Dirac}}\Big(\langle x|\,\gamma^5\,|x\rangle_{\mathrm{reg}} \;=\; \langle x|\,\gamma^5\hat{G}\,|x\rangle\Big) \;=\; -2\,\langle x|\,\mathrm{tr}_{\mathrm{Dirac}}(\gamma^5\hat{G})\,|x\rangle \\ &\approx\; -\frac{e^2}{\Lambda^4}\,\langle x|\,G''(-\hat{D}^2/\Lambda^2)\,|x\rangle\times\epsilon^{\kappa\lambda\mu\nu}F^{\kappa\lambda}(x)F^{\mu\nu}(x),\end{aligned} \tag{103}$$

so let us calculate the remaining matrix element $\langle x|\,G''(-\hat{D}^2/\Lambda^2)\,|x\rangle$. Fourier transforming to the momentum space, we have

$$\begin{aligned}\langle x|\,G''(-\hat{D}^2/\Lambda^2)\,|x\rangle \;&=\; \int\frac{d^4p_e}{(2\pi)^4}\,e^{-ipx}G''(-\hat{D}^2/\Lambda^2)e^{+ipx} \\ &=\; \int\frac{d^4p_e}{(2\pi)^4}\,G''\big((p^\rho - iD^\rho)_E^2/\Lambda^2\big),\end{aligned} \tag{104}$$

where on the last line the derivatives D^ρ act on the $F^{\kappa\lambda}F^{\mu\nu}$ fields to the right of the matrix element on the bottom line of eq. (103). (And also on the A^π fields in the other derivatives D^π in the expansion of the G''.) However, all such derivatives produce effects

O(external momenta) $\ll \Lambda$, so to the leading order in $1/\Lambda$

$$\int\frac{d^4p_e}{(2\pi)^4}\,G''\big((p^\rho-iD^\rho)^2_E/\Lambda^2\big)\ \approx\int\frac{d^4p_e}{(2\pi)^4}\,G''\big(p^2_E/\Lambda^2\big), \tag{105}$$

specifically

$$\frac{1}{\Lambda^4}\int\frac{d^4p_E}{(2\pi)^4}\,G''\big((p^\rho-iD^\rho)^2_E/\Lambda^2\big)\ =\ \frac{1}{\Lambda^4}\int\frac{d^4p_E}{(2\pi)^4}\,G''\big(p^2_E/\Lambda^2\big)\ +\ O\left(\frac{k^2_{\rm ext}}{\Lambda^2}\right) \tag{106}$$

where the first term on the RHS is an $O(1)$ constant. Indeed,

$$\begin{aligned}
\frac{1}{\Lambda^4}\int\frac{d^4p_E}{(2\pi)^4}\,G''\big(p^2_E/\Lambda^2\big)\ &=\ \frac{1}{\Lambda^4}\int\limits_0^\infty\frac{dp_e^2\,p_e^2}{16\pi^2}\,G''(p_e^2/\Lambda^2)\\
\langle\!\langle\,\text{changing variable from } p_e^2 \text{ to } t=p_e^2/\Lambda^2\,\rangle\!\rangle\quad &=\ \int\limits_0^\infty\frac{dt\,t}{16\pi^2}\,G''(t)\\
\langle\!\langle\,\text{integrating by parts twice}\,\rangle\!\rangle\quad &=\ \frac{1}{16\pi^2}\Big[tG'(t)-G(t)\Big]_{t=0}^{t=\infty}\\
&=\ \frac{1}{16\pi^2}\times 1,
\end{aligned} \tag{107}$$

because for $t\to\infty$ we have $G(t)\to 0$ and hence also $tG'(t)\to 0$, while for $t=0$ we have $tG'(t)=0$ while $G(t)=1$. Thus, regardless of the details of the $G(t)$ function, we have

$$\frac{1}{\Lambda^4}\int\frac{d^4p_E}{(2\pi)^4}\,G''\big((p^\rho-iD^\rho)^2_E/\Lambda^2\big)\ =\ \frac{1}{16\pi^2}\ +\ O\left(\frac{k^2_{\rm ext}}{\Lambda^2}\right). \tag{108}$$

Consequently, in the $\Lambda\to\infty$ limit, we end up with

$$\partial_\mu J^{\mu 5}(x)\ =\ -\frac{e^2}{16\pi^2}\,\epsilon^{\kappa\lambda\mu\nu}F^{\kappa\lambda}(x)F^{\mu\nu}(x). \tag{109}$$

This completes my explanation of the axial anomaly stemming from the non-invariant measure of the fermionic functional integral.

Axial Anomaly in QCD.

Axial anomalies exist in all kinds of gauge theories — abelian or non abelian — with massless fermions. As a non-abelian example, consider QCD with N_f flavors of exactly massless quark flavors. Or more generally, an $SU(N_c)$ gauge theory with $N_f \times N_c$ massless Dirac fermions $\Psi^{i,f}(x)$ in N_f copies of the fundamental $\mathbf{N_c}$ multiplet of the $SU(N_c)$. Let's start with an axial symmetry which acts in the same manner on quarks of all colors and flavors,

$$\Psi^{i,f}(x) \;\to\; e^{i\theta\gamma^5}\Psi^{i,f}(x), \qquad \overline{\Psi}_{i,f}(x) \;\to\; \overline{\Psi}_{i,f}(x)e^{i\theta\gamma^5}, \qquad \text{same } \theta\ \forall i\,\forall f. \tag{110}$$

The classical QCD action is invariant under this symmetry, so naively one expects a conserved axial current

$$J^{\mu 5} \;=\; \sum_{i,f}\overline{\Psi}_{i,f}\gamma^\mu\gamma^5\Psi^{i,f}, \qquad \partial_\mu J^{\mu 5} \;=\text{classically}=\;0, \tag{111}$$

but the measure of the fermionic functional integral is not invariant, and this leads to the anomalous non-conservation of the axial current.

We may derive the anomaly of the fermionic integral exactly as we did in the previous section for the QED, and we end up with a similar formula

$$\partial_\mu J^{\mu 5}(x) \;=\; -2\,\langle x|\,\mathrm{tr}(\gamma^5\hat{G})\,|x\rangle \tag{112}$$

where $\hat{G} = G(-\not{D}^2/\Lambda^2)$ is the UV-regulating operator, but this time the trace in eq. (112) is over all the indices of the quark fields — Dirac, color, and flavor. Also, for the non-abelian covariant derivatives $D_\mu = \partial_\mu + igt^aA^a_\mu(x)$ — where $t^a = \frac{1}{2}\lambda^a$ are the $SU(N_c)$ generators in the fundamental multiplet, — we have

$$\not{D}^2_E \;=\; D^2_E \;-\; \frac{g}{2}\,F^a_{\mu\nu}t^a\sigma^{\mu\nu} \tag{113}$$

where $F^a_{\mu\nu}$ are the non-abelian tension fields

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu \;-\; \partial_\nu A^a_\mu \;-\; gf^{abc}A^b_\mu A^c_\nu\,. \tag{114}$$

Consequently,

$$\mathrm{tr}(\gamma^5\hat{G}) \;=\; \frac{g^2}{2\Lambda^4}\,\mathrm{tr}_{\mathrm{colors\&flavors}}\Big(G''(-D_E^2/\Lambda^2)\epsilon^{\kappa\lambda\mu\nu}F^a_{\kappa\lambda}F^b_{\mu\nu}t^at^b\Big) \;+\; \begin{pmatrix}\text{subleading}\\ \text{powers of } 1/\Lambda^2\end{pmatrix}, \tag{115}$$

and therefore

$$\partial_\mu J^{\mu5}(x) \;=\; -\frac{g^2}{16\pi^2}\,\epsilon^{\kappa\lambda\mu\nu}F^a_{\kappa\lambda}(x)F^b_{\mu\nu}(x)\times\mathrm{tr}_{\mathrm{colors\&flavors}}\big(t^at^b\big). \tag{116}$$

Since the t^a and t^b matrices act only on the colors, the trace over the flavors is trivially equal to N_f, while the trace over the colors is

$$\mathrm{tr}_{\mathrm{colors}}\big(t^at^b\big) \;=\; R\begin{pmatrix}\text{fundamental}\\ \text{multiplet}\end{pmatrix}\times\delta^{ab} \;=\; \frac{1}{2}\times\delta^{ab}. \tag{117}$$

Thus, the net anomaly of the axial $U(1)$ current in QCD is

$$\partial_\mu J^{\mu5}(x) \;=\; -\frac{g^2N_f}{32\pi^2}\,\epsilon^{\kappa\lambda\mu\nu}F^a_{\kappa\lambda}(x)F^a_{\mu\nu}(x), \tag{118}$$

which is often written down as

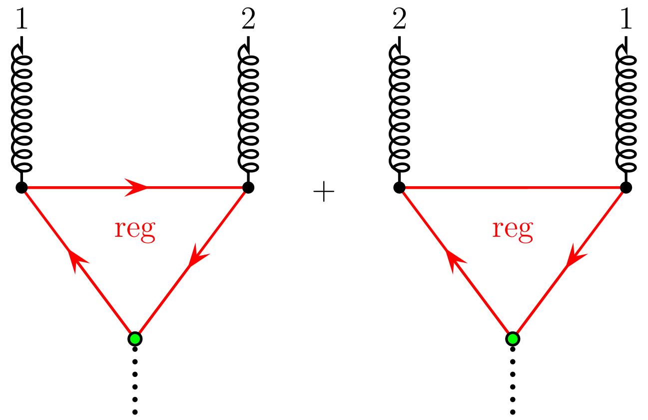
$$\partial_\mu J^{\mu5} \;=\; -\frac{g^2N_f}{16\pi^2}\,\mathrm{tr}\big(\epsilon^{\kappa\lambda\mu\nu}F_{\kappa\lambda}F_{\mu\nu}\big) \;=\; -\frac{g^2N_f}{8\pi^2}\,\mathrm{tr}\big(\tilde{F}_{\mu\nu}F^{\mu\nu}\big), \tag{119}$$

where the trace is over the color indices of the non-abelian tension fields represented by matrices in the fundamental multiplet of the $SU(N_c)$.

The main difference between axial anomalies in QED and in QCD is that QCD has non-abelian tension fields. In terms of the Feynman diagram and the amplitudes

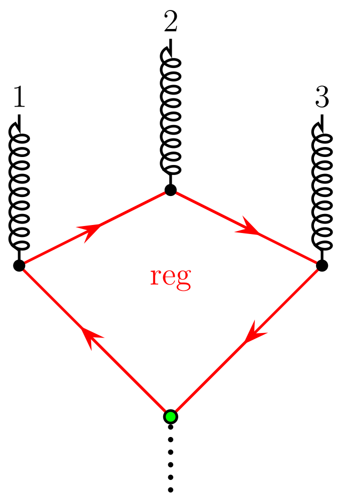
$$\mathcal{M}(\partial_\mu J^{\mu5} \to \text{gluons}), \tag{120}$$

this means that QCD has not only the two-gluon amplitudes similar to the two-photon amplitudes of QED but also the three-gluon and the four-gluon amplitudes. Actually, the four-gluon amplitude from the anomaly (118) happens to vanish — proving this is a part of your homework set#22, namely problem 2(a). But the two-gluon and the three-gluon amplitudes do not vanish; instead, they obtain from the 1-loop triangle and quadrangle diagrams



(121)

and



$+$ gluon permutations. (122)

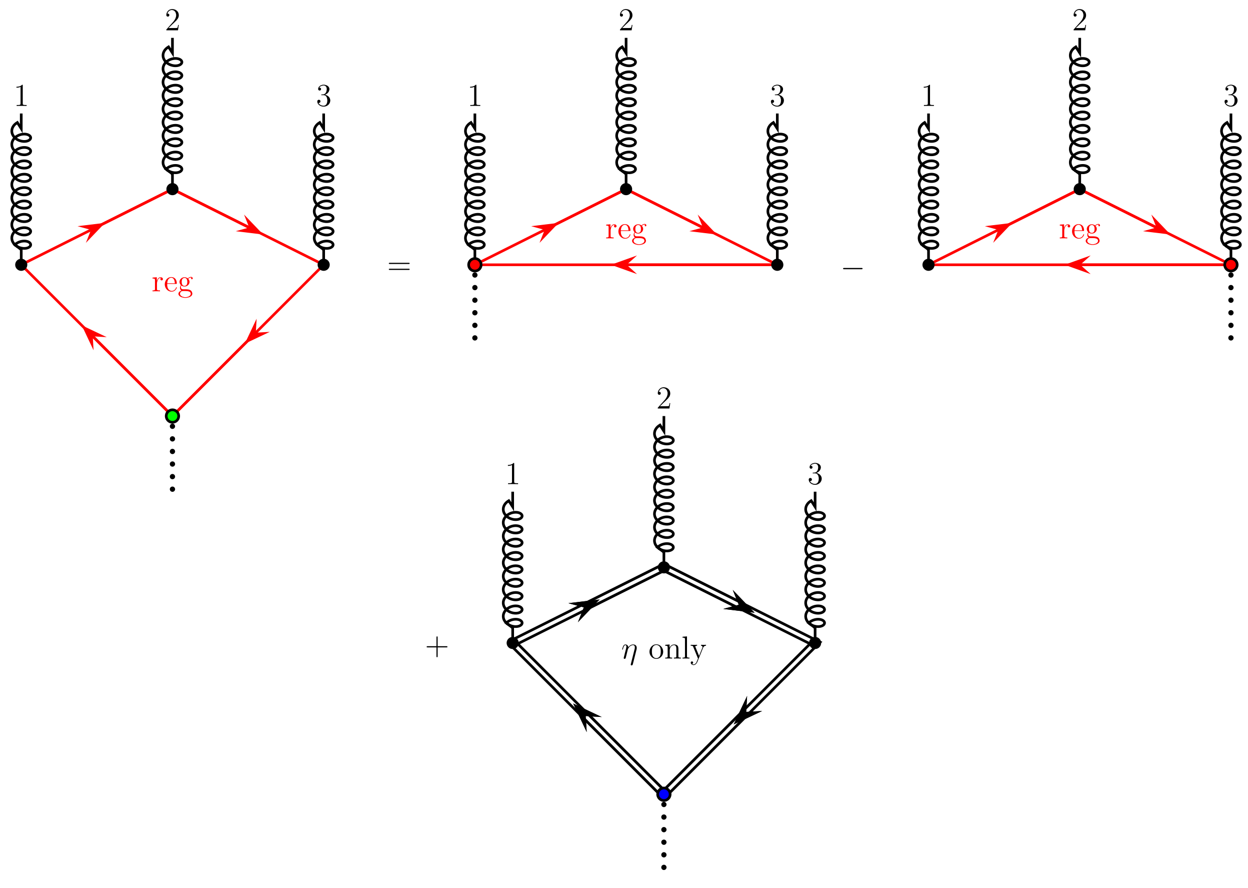
Similar to QED, the axial anomaly formally cancels between the diagrams related by the gluon permutations, but this cancellation does not quite work unless the diagrams in question are either UV finite (or only logarithmically divergent), or else the UV divergence can be regulated without destroying the cancellation. All the multi-loop diagrams can be UV-regulated by the covariant higher derivative method, so they do not contribute to the axial anomaly. Instead, the anomaly comes solely from the on-loop diagrams with $n \leq 3$ gluon vertices (121) or (122), since the diagrams with $n \geq 4$ vertices are finite or log-divergent.

In QED, the three-photon amplitudes similar to (122) must vanish due to the charge-conjugation symmetry of QED. But in the non-abelian gauge theories like QCD, the charge conjugation works in a more complicated manner, namely

$$\mathbf{C} : \Psi^{i,f} \ \mapsto\ \gamma^2\Psi^*_{i,f} \quad \text{and also} \quad A^a_\mu t^a \ \mapsto\ -A^a_\mu (t^a)^*, \tag{123}$$

which means the gluon field A^a_μ can be C-odd or C-even, depending on whether the corresponding t^a matrix is real or imaginary. Consequently, unlike QED in which the 3-photon states are always C-odd, in QCD the three-gluon states can be either C-odd or C-even, depending on the gluon's colors; in particular, the $f^{abc}A^a_\lambda A^b_\mu A^a_\nu$ combination of the 3 gluons is C-even. And that's why QCD — unlike QED — does have 3-gluon contributions to the axial anomaly stemming from the diagrams (122).

I leave the actual evaluation of the quadrangle diagrams (122)for your homework set#22, problem 2(b–c). I suggest you do it similarly to how I handles the triangle diagrams in an earlier section of these notes: First, you use the Pauli–Villars UV regulator for the quarks and show that



(124)

Second, you sum over the gluon permutations and show that all the regulated diagrams on the RHS of eq. (124) cancel each other. Third, you evaluate the compensator-only loop diagram. Hint: Bring the 3-gluon amplitude to the form

$$\mathcal{M}^{\lambda\mu\nu} \;=\; (\text{pre-factor})\times\int d\begin{pmatrix}\text{Feynman}\\ \text{parameters}\end{pmatrix}\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^{\lambda\mu\nu}}{[\ell^2-M^2-O(k_{\text{ext}}^2)+i0]^4}\,, \tag{125}$$

expand the numerator $\mathcal{N}^{\lambda\mu\nu}$ here into powers of the external momenta,

$$\mathcal{N}^{\lambda\mu\nu} \;=\; \mathcal{N}_0^{\lambda\mu\nu}(\ell,M)\;+\sum_{i=1,2,3}k_{i\alpha}\times\mathcal{N}_1^{\alpha,\lambda\mu\nu}(\ell,M)\;+\;\cdots, \tag{126}$$

and argue that the integrals of terms carrying the higher-than-first powers of $k_{1,2,3}$ are proportional to the negative powers of $M=\Lambda_{\text{UV}}$ so they can be neglected in the $M\to\infty$ limit. Likewise, argue that $O(k_{\text{ext}}^2)$ term in the denominator can be neglected in the $M\to\infty$ limit.

Finally, once you complete evaluating a particular compensator-loop diagram, do not forget to sum once again over the gluon permutations.

⋆ ⋆ ⋆

QCD with N_f flavors of massless quarks has $2N_FN_c$ Weyl fermions — N_fN_c left-handed ψ_L^{fi} and N_fN_c right-handed ψ_R^{fi}, —

$$\mathcal{L}_{\text{phys}} \;=\; -\tfrac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\;+\;i\sum_{f,i}\psi^\dagger_{Lfi}\bar\sigma^\mu D_\mu\psi_L^{fi}\;+\;i\sum_{f,i}\psi^\dagger_{Rfi}\sigma^\mu D_\mu\psi_R^{fi}\,, \tag{127}$$

but the fermions of the same chirality and flavor but different colors are inter-related by the gauge-covariant derivatives D_μ. Consequently, the continuous fermionic symmetries of QCD may act on the quark's flavors but not on their colors. For the infinitesimal symmetries, this means

$$\delta\psi_L^{fi} \;=\; i\epsilon\sum_{f'}T^f_{Lf'}\psi_L^{f'i},\qquad \delta\psi_R^{fi}\;=\;i\epsilon\sum_{f'}T^f_{Rf'}\psi_R^{f'i}, \tag{128}$$

for some matrices T_L and T_R acting only on the flavor indices of the LH and RH quarks. Specifically, T_L and T_R are two independent $N_f\times N_f$ matrices, so the continuous symmetry group of the QCD's fermions is $U(N_f)_L \times U(N_f)_R$. In terms of the Dirac quarks, the infinitesimal symmetries (128) become

$$\delta\Psi^{fi} = i\sum_{f'}(T^f_{Vf'} + T^f_{Af'}\gamma^5)\Psi^{f'i} \tag{129}$$

for arbitrary Hermitian $N_f \times N_f$ matrices $T_V = \frac{1}{2}(T_R + T_L)$ and $T_A = \frac{1}{2}(T_R - T_L)$, hence classically conserved vector and axial currents

$$\begin{aligned} J^\mu[T_V] &= \sum_{i,f,f'} \overline{\Psi}_{fi}\gamma^\mu T^f_{Vf'}\Psi^{f'i}, \\ J^{\mu 5}[T_A] &= \sum_{i,f,f'} \overline{\Psi}_{fi}\gamma^\mu\gamma^5 T^f_{Af'}\Psi^{f'i}, \\ \partial_\mu J^\mu[T_V] &= \text{classically} = \ 0 \quad \forall\ T_V\,, \\ \partial_\mu J^{\mu 5}[T_A] &= \text{classically} = \ 0 \quad \forall\ T_A\,. \end{aligned} \tag{130}$$

In the quantum theory, all the vector currents $J^\mu[T_V]$ remain exactly conserved thanks to the non-anomalous Ward–Takahashi identities. Indeed, all the diagrams which could lead to these currents' non-conservation formally cancel each other, and there is a UV regulator — the dimensional regularization — which respects both QCD Feynman rules and the vector flavor currents and hence makes sure the formal cancellation leads to the actual cancellation of the UV-regulated diagrams.

But the axial flavor currents are subject to the anomalous mis-cancellation of the triangle and quadrangle diagrams. Formally,

$$\partial_\mu J^{\mu 5}[T_A](x) \ = \ -2\,\langle x|\,\mathrm{tr}(T_A\gamma^5\hat{G})\,|x\rangle \tag{131}$$

where $\hat{G}$ is the UV regulating operator and the trace is over Dirac, color, and flavor indices. Proceeding similarly to the flavor-blind axial symmetry, we end up with

$$\begin{aligned} \partial_\mu J^{\mu 5}[T_A](x) &= -\frac{g^2}{16\pi^2}\,\mathrm{tr}_{\text{colors\&flavors}}\big(T_A \times \epsilon^{\kappa\lambda\mu\nu}F_{\kappa\lambda}F_{\mu\nu}\big) \\ &= -\frac{g^2}{16\pi^2}\,\mathrm{tr}_{\text{flavors}}(T_A) \times \mathrm{tr}_{\text{colors}}\big(\epsilon^{\kappa\lambda\mu\nu}F_{\kappa\lambda}F_{\mu\nu}\big). \end{aligned} \tag{132}$$

Thus, an axial flavor symmetry is anomalous if and only if its generator T_A has a non-zero trace; in $U(N_f) = SU(N_f) \times U(1)$ terms this means that all the $SU(N_f)$ axial symmetries are anomaly-free but the $U(1)$ axial symmetry is anomalous. Or in terms of the net chiral symmetry of QCD with massless quarks,

$$U(N_f)_L \times U(N_f)_R \;=\; SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A\,, \tag{133}$$

where the $U(1)_A$ factor is anomalous while the other factors $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ are anomaly-free.

In QCD, the chiral symmetry (133) is *spontaneously* broken down to the vector $U(N_f)_V$ symmetry by the expectation value of the quark-antiquark condensate $\left\langle\overline{\Psi}\Psi\right\rangle \neq 0$. I shall address this condensate and its consequences in a separate set of notes on non-linear sigma-models, but for the moment let us simply stick to the Goldstone–Nambu theorem. Without the axial anomaly, we would have spontaneous symmetry breaking

$$U(N_f)_L \times U(N_f)_R \;\xrightarrow[\text{SSB}]{}\; U(N_f)_V\,, \tag{134}$$

thus N_f^2 broken generators in the adjoint multiplet of the unbroken $U(N_f)$, hence an adjoint multiplet of N_f^2 massless Goldstone bosons. And since the broken symmetry currents are axial rather than polar vectors, the Goldstone bosons have negative parity, *i.e.* are pseudoscalar rather than true scalar particles.

From the SSB point of view, the axial anomaly (132) is an *explicit* breaking of the $U(1)_A$ axial symmetry, so despite the further spontaneous breaking of this symmetry by the quark-antiquark condensate, the corresponding would-be Goldstone pseudoscalar turns out to be massive rather than massive. On the other hands, the remaining $N_f^2 - 1$ Goldstone pseudoscalars due to SSB of the anomaly-free chiral symmetries

$$SU(N_f)_L \times SU(N_f)_R \;\xrightarrow[\text{SSB}]{}\; SU(N_f)_V \tag{135}$$

remain massless.

In real-life QCD, two quark flavors u and d are particularly light. If we approximate them as massless, we get chiral flavor symmetry

$$U(2)_L \times U(2)_R \;=\; \Big(SU(2)_L \times SU(2)_R \times U(1)_V\Big)^{\text{anomaly}}_{\text{free}} \;+\; \Big(U(1)_A\Big)^{\text{anomalous}}. \tag{136}$$

When this symmetry is spontaneously broken down to the $U(2)_V = SU(2)_V^{\text{isospin}} \times U(1)_V$, we get 4 would-be Goldstone pseudoscalars made from the u and d quarks and antiquarks, namely the isotriplet π^+, π^0, π^- of pions and the isosinglet eta-meson η. Without the quark masses and the axial anomaly, all 4 of these mesons would be massless Goldstone bosons. The small but not quite zero masses of the real-life u and d quarks provide a small explicit breaking of the chiral symmetry, so the pions get a small non-zero mass $m_\pi \approx 140$ MeV, and without the anomaly the eta meson would have the same small mass. However, in real life the eta meson is much heavier, $m_\eta \approx 550$ MeV, thus $m_\eta^2/m_\pi^2 \approx 16$; historically, this large mass difference was called the *pi/eta puzzle.* The solution to this puzzle is the axial anomaly of the $U(1)_A$ axial current but not of the $SU(2)_A$ axial currents: This anomaly provides a must stronger explicit breaking of the $U(1)_A$ symmetry than the quark masses' breaking of the whole $U(2)_A = U(1)_A \times SU(2)_A$. Altogether, the net $U(A)_A$ breaking turns out to be about 16 times stronger that the $SU(2)_A$ breaking, and that's why $m_\eta^2 \approx 16 m_\pi^2$. Or rather, that's the reason for the $m_\eta^2 \gg m_\pi^2$ in the 2-light-flavor approximation.

In real life, there is a third relatively light quark flavor s, albeit heavier than u or d, and the η meson is only about 55% the isosinglet $\sqrt{\frac{1}{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$ state while the remaining 45% it's $|s\bar{s}\rangle$. Consequently, its mass^2 gets contributions from both the axial anomaly of QCD and from the s quark's mass m_s (as well as smaller contributions from the m_u and m_d). Also, there is another pseudoscalar meson η' that's also a mixture of the isosinglet $\sqrt{\frac{1}{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$ state and the $|s\bar{s}\rangle$ state, and its larger $(m_{\eta'} \approx 950\text{ MeV})^2$ also gets contributions from both m_s and from the axial anomaly of QCD. I explain how this works at the end of my notes on non-linear sigma models in QCD context.

⋆ ⋆ ⋆

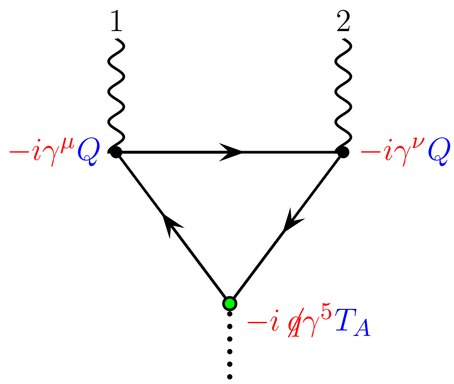
Besides the QCD anomaly, some axial symmetries of the quarks — and hence of the hadrons — are subject to the electromagnetic anomalies. To see how this works, let's forget about the QCD for a moment and consider the EM field $A^\mu(x)$ coupled to a bunch of massless Dirac fermions $\Psi^1(x),\ldots,\Psi^N(x)$ of different electric charges $q_1,\ldots,q_N$. Or rather some charges may be similar while other charges are different. The fermions with similar charges may be mixed with each other by the global vector or axial symmetries; infinitesimally,

$$\delta\Psi^i(x) \;=\; i\epsilon\sum_j (T^i_{Vj}+T^i_{Aj}\gamma^5)\Psi^j(x), \tag{137}$$

where the hermitian matrices T_V and T_A must commute with the electric charge matrix $Q=\mathrm{diag}(q_1,\ldots,q_N)$. Classically, there are conserved currents corresponding to these symmetries

$$J^\mu(T_V) \;=\; \sum_{ij}\overline{\Psi}_i T^i_{Vj}\gamma^\mu\Psi^j, \qquad J^{\mu5}(T_A) \;=\; \sum_{ij}\overline{\Psi}_i T^i_{Aj}\gamma^\mu\gamma^5\Psi^j, \tag{138}$$

where the vector currents remain conserved in the fully quantum theory while the axial currents are subject to the anomalies stemming from the triangular graphs



$+$ photon permutation. (139)

In these diagrams we include all fermion species, hence the charge matrices Q in the photon vertices and the T_A matrix in the axial current vertex. Apart from these matrices, the diagrams work exactly as in QED with a single massless electron, hence

$$\mathcal{M} \;=\; \frac{1}{e^2}\,\mathrm{tr}_{\text{species of }\Psi}\big(Q^2T_A\big)\times\mathcal{M}[\text{single electron}] \tag{140}$$

and therefore

$$\partial_\mu J^{\mu5}[T_A] \;=\; -\frac{1}{16\pi^2}\,\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}\times\mathrm{tr}_{\text{species of }\Psi}\big(Q^2T_A\big). \tag{141}$$

In particular, for the axial symmetries of the quarks' flavors,

$$\mathrm{tr}_{\text{species of }\Psi}\big(Q^2T_A\big)\ =\ 3_{\text{colors}}\times\mathrm{tr}_{\text{flavors}}\big(Q^2T_A\big),\tag{142}$$

hence

$$\partial_\mu J^{\mu 5}[T_A]\ ==\ -\frac{1}{16\pi^2}\left(\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}\right)_{\text{EM}}\times 3\,\mathrm{tr}_{\text{flavors}}\big(Q^2T_A\big).\tag{143}$$

Phenomenologically, this EM axial anomaly is particularly important for the neutral pion's decay to two photons, $\pi^0\to\gamma\gamma$. To see how this works, note that the neutral pion is the pseudo-Goldstone boson of the axial isospin current, specifically of the T^3 component of the isospin which commutes with the electric charge. The T^3 generator is diagonal in the flavor basis: it has eigenvalue $+\frac{1}{2}$ for the u quark, $-\frac{1}{2}$ for the d quark, and zero for all other flavors, hence

$$\mathrm{tr}_{\text{flavors}}\big(Q^2T_A\big)\ =\ +\tfrac{1}{2}Q^2(u)\ -\ \tfrac{1}{2}Q^2(d)\ =\ +\tfrac{1}{2}(+2e/3)^2\ -\ \tfrac{1}{2}(-e/3)^2\ =\ \frac{4-1}{18}e^2\ =\ +\frac{e^2}{6}\,.\tag{144}$$

Consequently, the corresponding axial current has EM anomaly

$$\partial_\mu J^{\mu 5}[T^3]\ =\ -\frac{1}{16\pi^2}\times 3\times\frac{e^2}{6}\times\left(\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}\right)_{\text{EM}}.\tag{145}$$

But from the neutral pion's point of view

$$J^{\mu 5}[T^3]\ =\ -f_\pi\partial^\mu\pi^0\ +\ \text{multi-pion terms},\tag{146}$$

so the EM anomaly translates to

$$f_\pi\partial^2\pi_0\ +\ \text{multi-pion terms}\ =\ \frac{e^2}{32\pi^2}\,(\epsilon FF)_{\text{EM}}\,.\tag{147}$$

Besides the anomaly, there is further non-conservation of the axial current due to the quark masses, but we may account for this by adding the pion's mass^2 term to eq. (147),

$$f_\pi(\partial^2 + m_\pi^2)\pi_0 \;+\; \text{multi-pion terms} \;=\; \frac{e^2}{32\pi^2}\,(\epsilon FF)_{\rm EM}\,. \tag{148}$$

This anomalous equation of motion for the neutral pion field can be accounted by an effective Lagrangian

$$\mathcal{L}_{\rm eff} \;=\; \frac{1}{2}(\partial_\mu\pi^0)^2 \;-\; \frac{m_\pi^2}{2}(\pi^0)^2 \;+\; \frac{e^2}{32\pi^2 f_\pi}\,\pi^0\times(\epsilon FF)_{\rm EM} \;+\; \text{multi-pion terms.} \tag{149}$$

The neutral pion decays mostly into two photons (branching ratio $B\approx 99\%$), and the amplitude for this decay follows directly from the red interaction term in the effective Lagrangian (149):

$$\mathcal{M}(\pi^0\to\gamma\gamma) \;=\; -8\,\frac{e^2}{32\pi^2 f_\pi}\times\epsilon^{\alpha\beta\mu\nu}(k_\alpha e^*_\beta)_1(k_\mu e^*_\nu)_2\,. \tag{150}$$

I'll leave explaining the -8 factor and the following calculation of the net decay rate as an exercise for the students; it's going to be problem 4 of your next homework#23. For the moment, let me simply note that the π^0 decay amplitude (150) is *inversely* proportional to the pion decay constant f_π. This is quite different from the weak decay amplitude of the charged pion that is directly proportional to the f_π.

Anomalies in Chiral Gauge Theories.

Thus far, we have focused on axial symmetries of Dirac fermions in gauge theories. But some gauge theories are *chiral* — that is, the left-handed Weyl fermions and the right-handed Weyl fermions have different abelian charges or belong to different multiplet types of a non-abelian gauge symmetry. For example, in the Standard Model, the left-handed quarks and leptons belong to different multiplets of the electroweak $SU(2)\times U(1)$ than the right-handed quarks and leptons.

In this section, we shall learn about anomalies of various fermionic symmetries of such chiral theories. For simplicity, let's start with an abelian $U(1)$ gauge field A^μ coupled to

N_L left-handed Weyl fermions ψ_L^i of respective charges q_{Li} and to N_R right-handed Weyl fermion ψ_R^i of respected charges q_{Ri},

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; i\sum_{i=1}^{N_L}\psi_{Li}^\dagger\bar{\sigma}^\mu(\partial_\mu + iq_{Li}A_\mu)\psi_L^i \;+\; i\sum_{i=1}^{N_R}\psi_{Ri}^\dagger\sigma^\mu(\partial_\mu + iq_{Ri}A_\mu)\psi_R^i\,, \quad (151)$$

or in matrix notations

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; i\psi_L^\dagger\bar{\sigma}^\mu(\partial_\mu + iQ_LA_\mu)\psi_L \;+\; i\psi_R^\dagger\sigma^\mu(\partial_\mu + iQ_RA_\mu)\psi_R\,. \quad (152)$$

An infinitesimal global symmetry of the Weyl fermions here has general form

$$\delta\psi_L(x) \;=\; i\epsilon T_L\psi_L(x), \qquad \delta\psi_R(x) \;=\; i\epsilon T_R\psi_R(x), \quad (153)$$

for some independent hermitian matrices T_L and T_R commuting with the respective electric charge matrices Q_L and Q_R, and the classically conserved currents of such symmetries are

$$J_L^\mu \;=\; \psi_L^\dagger T_L\bar{\sigma}^\mu\Psi_L \quad \text{and} \quad J_R^\mu \;=\; \psi_R^\dagger T_R\sigma^\mu\psi_R\,. \quad (154)$$

When the Weyl fermions here happen to be the LH and the RH components of some Dirac fermions, the LH and the RH current can be written as

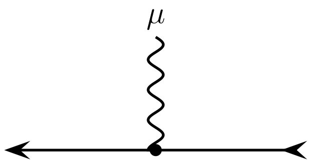
$$J_L^\mu(T) \;=\; \tfrac{1}{2}J_V^\mu(T) \;-\; \tfrac{1}{2}J_A^\mu(T), \qquad J_R^\mu(T) \;=\; \tfrac{1}{2}J_V^\mu(T) \;+\; \tfrac{1}{2}J_A^\mu(T); \quad (155)$$

but more generally, the J_L^μ and the J_R^μ are independent currents of unrelated Weyl fermions, and these currents are not related to each other by parity.

To work out the possible anomalies of such classically conserved chiral currents we need the Feynman rules for the charged Weyl fermions. The simplest way to derive these Feynman rule is to promote each LH Weyl fermion to the left half of a Dirac fermion whose right half is neutral and does not couple to anything. Likewise, each RH Weyl fermion is promoted to the right half of a Dirac fermion whose left half is neutral and does not couple to anything. Consequently, we get massless Dirac propagators

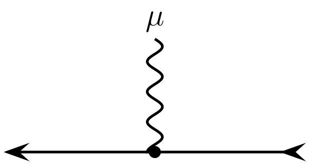
$$= \frac{i}{\not{p} + i0} = \frac{i\,\not{p}}{p^2 + i0} \tag{156}$$

for all the Weyl fermions, but the vertices have the $\frac{1}{2}(1 \mp \gamma^5)$ projection factors onto the appropriate chirality:



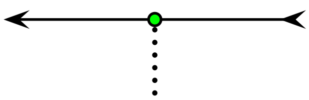
$$= -iQ_L \gamma^\mu \frac{1-\gamma^5}{2} \tag{157}$$

for the LH Weyl fermions, and



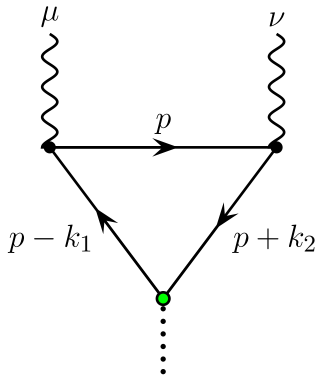
$$= -iQ_R \gamma^\mu \frac{1+\gamma^5}{2} \tag{158}$$

for the RH Weyl fermions. Likewise, the green vertices for measuring the current divergences $\partial_\mu J^\mu_L$ or $\partial_\mu J^\mu_R$ become



$$= -iT_L \not{q} \frac{1-\gamma^5}{2} \quad \text{or} \quad -iT_R \not{q} \frac{1+\gamma^5}{2}\,. \tag{159}$$

Now let's focus on a single Weyl fermion — either left-handed or right-handed — or electric charge $Q = +e$ and global charge $T = 1$, and consider its contribution to the anomaly $\partial_\mu J^\mu_{L\,\mathrm{or}\,R}$. The triangle graph



(160)

evaluates to

$$-\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(-i\,\slashed{a}\frac{1\mp\gamma^5}{2}\,\frac{i}{\slashed{p}+\slashed{k}_2}\,(-ie)\gamma^\nu\frac{1\mp\gamma^5}{2}\,\frac{i}{\slashed{p}}\,(-ie)\gamma^\mu\frac{1\mp\gamma^5}{2}\,\frac{i}{\slashed{p}-\slashed{k}_1}\right). \tag{161}$$

To simplify the Dirac trace here, we use the fact that γ^5 anticommutes with all the massless propagators here as well as with all the γ^μ factors in the vertices, hence

$$\frac{i}{\slashed{p}}\,(-ie)\gamma^\mu\frac{1\mp\gamma^5}{2} \;=\; \frac{1\mp\gamma^5}{2}\times\frac{i}{\slashed{p}}\,(-ie)\gamma^\mu \tag{162}$$

and therefore

$$\begin{aligned}
\mathrm{tr}(\cdots) \;&=\; \mathrm{tr}\left(-i\,\slashed{a}\left(\frac{1\mp\gamma^5}{2}\right)^3\frac{i}{\slashed{p}+\slashed{k}_2}\,(-ie)\gamma^\nu\,\frac{i}{\slashed{p}}\,(-ie)\gamma^\mu\,\frac{i}{\slashed{p}-\slashed{k}_1}\right)\\
&\qquad\langle\!\langle\, \text{using } (1\mp\gamma^5)^3=4(1\mp\gamma^5)\,\rangle\!\rangle\\
&=\; \mathrm{tr}\left(-i\,\slashed{a}\,\frac{1\mp\gamma^5}{2}\,\frac{i}{\slashed{p}+\slashed{k}_2}\,(-ie)\gamma^\nu\,\frac{i}{\slashed{p}}\,(-ie)\gamma^\mu\,\frac{i}{\slashed{p}-\slashed{k}_1}\right)\\
&=\; \frac{1}{2}\,\mathrm{tr}\left(-i\,\slashed{a}\,\frac{i}{\slashed{p}+\slashed{k}_2}\,(-ie)\gamma^\nu\,\frac{i}{\slashed{p}}\,(-ie)\gamma^\mu\,\frac{i}{\slashed{p}-\slashed{k}_1}\right)\\
&\qquad\mp\;\frac{1}{2}\,\mathrm{tr}\left(-i\,\slashed{a}\gamma^5\,\frac{i}{\slashed{p}+\slashed{k}_2}\,(-ie)\gamma^\nu\,\frac{i}{\slashed{p}}\,(-ie)\gamma^\mu\,\frac{i}{\slashed{p}-\slashed{k}_1}\right).
\end{aligned} \tag{163}$$

In Dirac fermion terms, the trace on the penultimate line here is the divergence of the vector current while the trace on the bottom line is the divergence of the axial current. And when we UV-regulate the diagrams and sum over permutations of the two photons, the vector current divergence cancels out while the axial current divergence yields the axial anomaly we have seen earlier in these notes. Thus altogether, for a single Weyl fermion — left-handed or right-handed —

$$\begin{aligned} \partial_\mu J^\mu_L &= -\frac{1}{2}\times\frac{-e^2}{16\pi^2}\,(\epsilon FF), \\ \partial_\mu J^\mu_R &= +\frac{1}{2}\times\frac{-e^2}{16\pi^2}\,(\epsilon FF). \end{aligned} \tag{164}$$

For multiple Weyl fermions, the triangle graphs work in a similar way, except each photon vertex comes with the matrix Q factor (instead of e), the current vertex has a matrix factor T, and we should trace the product of these factors over the species of the Weyl fermions. Thus, we end up with net anomalies

$$\begin{aligned} \partial_\mu J^\mu_L[T] &= +\frac{1}{32\pi^2}\,(\epsilon FF)\times\mathrm{tr}_{\mathrm{LHWF}}(Q_L^2 T_L), \\ \partial_\mu J^\mu_R[T] &= -\frac{1}{32\pi^2}\,(\epsilon FF)\times\mathrm{tr}_{\mathrm{RHWF}}(Q_R^2 T_R), \end{aligned} \tag{165}$$

and if a global chiral symmetry involves both LH and RH Weyl fermions, then

$$\partial_\mu J^\mu[T] \;=\; \frac{1}{32\pi^2}\,(\epsilon FF)\times\Big(\mathrm{tr}_{\mathrm{LHWF}}(T_L Q_L^2) - \mathrm{tr}_{\mathrm{RHWF}}(T_R Q_R^2)\Big). \tag{166}$$

Now consider a non-abelian chiral gauge theory, with gauge group G, the LH Weyl fermions ψ^i_L in some multiplet $(m)_L$ of G, and the RH Weyl fermions ψ^i_R in some other multiplet $(m)_R$ of G. In general, the multiplets $(m)_L$ and $(m)_R$ are reducible and may contain multiple copies of the same irreducible multiplet, so we need some kind of “flavor” indices to distinguish between them, and there are going to be all kinds of chiral flavor symmetries acting on such indices. Specifically, for

$$(m)_L \;=\; n_1(m_1) \;+\; n_2(m_2) \;+\; \cdots, \qquad (m)_R \;=\; n'_1(m'_1) \;+\; n'_2(m'_2) \;+\; \cdots,$$

the chiral flavor symmetry is

$$\big[U(n_1)\times U(n_2)\times\cdots\big]_L\times\big[U(n'_1)\times U(n'_2)\times\cdots\big]_R\,. \tag{167}$$

But let’s skip the messy indexology of a general case and use the index-less matrix notations in which

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) \;+\; i\psi_L^\dagger\bar\sigma^\mu D_\mu\psi_L \;+\; i\psi_R^\dagger\sigma^\mu D_\mu\psi_R \tag{168}$$

for the non-abelian covariant derivatives D_μ acting on the appropriate multiplets $(m)_L$ and $(m)_R$, and a general global chiral symmetry acts as

$$\delta\psi_L(x) \;=\; i\epsilon T_L\psi(x), \qquad \delta\psi_R(x) \;=\; i\epsilon T_R\psi_R(x) \tag{169}$$

where the hermitian matrices T_L and T_R commute with all the generators t^a of the gauge symmetry G. The current of the symmetry (169) is

$$J^\mu(T) \;=\; \psi_L^\dagger T_L\bar\sigma^\mu\psi_L \;+\; \psi_R^\dagger T_R\sigma^\mu\psi_R\,, \tag{170}$$

it's classically conserved by the equations of motion but may suffer from the anomalies due to triangle and quadrangle Feynman diagrams.

The triangle diagrams (160) for the Weyl fermion loops in a non-abelian gauge theory works exactly as in the abelian case, except that instead of the Q matrices in the gauge-boson vertices we now have gt^a and gt^b, so the trace over the species becomes $g^2\,\mathrm{tr}(Tt^at^b)$. But since the global symmetry generator T commutes with the gauge generators t^a and t^b, this trace-over-the-species is not affected by the permutation of the two gauge bosons,

$$\mathrm{tr}(Tt^at^b) \;=\; \mathrm{tr}(t^aTt^b) \;=\; \mathrm{tr}(Tt^bt^a), \tag{171}$$

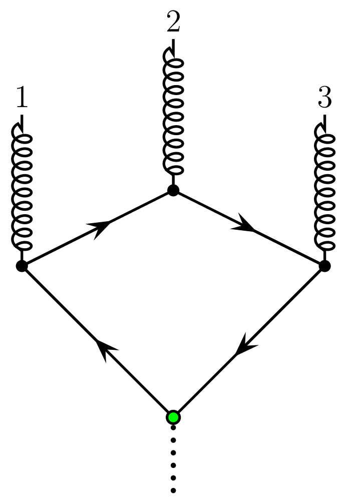
so it does not affect the cancellations between the two triangle diagrams. As to the trace over the Dirac indices, similar to the abelian case we have

$$\mathrm{tr}_{\mathrm{Weyl}} \;=\; \tfrac{1}{2}\,\mathrm{tr}^{\mathrm{Dirac}}_{\mathrm{vector}} \mp \tfrac{1}{2}\,\mathrm{tr}^{\mathrm{Dirac}}_{\mathrm{axial}}\,, \tag{172}$$

so after we UV-regulate the diagrams and sum over the gauge boson permutations, the vector terms cancel each other while the axial parts do not quite cancel but add up to the axial anomaly,

$$\partial_\mu J^\mu[T]_{\text{triangle}} \;=\; \frac{g^2}{32\pi^2}\Big(\text{tr}_{\text{LHWF}}(T_L t^a t^b) - \text{tr}_{\text{RHWF}}(T_R t^a t^b)\Big) \times \epsilon^{\alpha\beta\mu\nu}\big(F^a_{\alpha\beta}F^b_{\mu\nu}\big)^{2\,\text{gluon part}} \tag{173}$$

The non-abelian part of the ϵFF combination comes from the quadrangle diagrams



$$+\;\text{permutations of gauge bosons.} \tag{174}$$

Similar to the triangle diagrams, the Dirac trace here amounts to

$$\text{tr}_{\text{Weyl}} \;=\; \tfrac{1}{2}\,\text{tr}^{\text{Dirac}}_{\text{vector}} \;\mp\; \tfrac{1}{2}\,\text{tr}^{\text{Dirac}}_{\text{axial}}\,, \tag{175}$$

while the species trace has one more generator of the gauge symmetry, $\text{tr}(Tt^at^bt^c)$. But since the global symmetry generator T commute with all the gauge generators, this species trace depends only on the cyclical order of the t^a, t^b, t^c relative to each other,

$$\begin{aligned}\text{tr}(Tt^at^bt^c) \;&=\; \text{tr}(Tt^bt^ct^a) \;=\; \text{tr}(Tt^ct^at^b),\\ \text{tr}(Tt^ct^bt^a) \;&=\; \text{tr}(Tt^bt^at^c) \;=\; \text{tr}(Tt^at^ct^b).\end{aligned} \tag{176}$$

Consequently, when we UV-regulate the quadrangle diagrams and some over the cyclical permutations of the gauge bosons, the vector parts of the diagrams cancel each other, while the axial parts do not quite cancel but add up to the quadrangle parts of the axial anomaly, *cf.* homework set#22 for the details. Finally, when we sum over the two cyclic orders of the gauge bosons we end up with

$$\mathcal{M} \;\propto\; \operatorname{tr}(Tt^at^bt^c) \;-\; \operatorname{tr}(Tt^ct^bt^a) \;=\; if^{abc} \times C \tag{177}$$

for exactly the same coefficient C as in

$$\operatorname{tr}(Tt^at^b) \;=\; \delta^{ab} \times C. \tag{178}$$

Consequently, the quadrangle diagrams yield the three-gauge-boson part of the anomaly with the same overall coefficient as the two-gauge-boson anomaly (173),

$$\partial_\mu J^\mu[T]_{\text{quadrangle}} \;=\; \frac{g^2}{32\pi^2}\Big(\operatorname{tr}_{\text{LHWF}}(T_Lt^at^b) - \operatorname{tr}_{\text{RHWF}}(T_Rt^at^b)\Big) \times \epsilon^{\alpha\beta\mu\nu}\big(F^a_{\alpha\beta}F^b_{\mu\nu}\big)^{\text{3 gluon part}}. \tag{179}$$

And altogether,

$$\partial_\mu J^\mu[T]_{\text{net}} \;=\; \frac{g^2}{32\pi^2}\Big(\operatorname{tr}_{\text{LHWF}}(T_Lt^at^b) - \operatorname{tr}_{\text{RHWF}}(T_Rt^at^b)\Big) \times \epsilon^{\alpha\beta\mu\nu}\big(F^a_{\alpha\beta}F^b_{\mu\nu}\big)^{\text{complete}}_{\text{non-abelian}}. \tag{180}$$

For example, consider QCD with massless quarks. It's a non-chiral gauge theory where ψ_L and ψ_R form similar multiplets — namely N_f copies of the same fundamental multiplet $\mathbf{N_c}$ — but it has a chiral flavor symmetry $U(N_f)_L \times U(N_f)_R$. In this theory, trace over species means trace over the colors and the flavors, thus for a flavor generator T and two color generators $t^a = \frac{1}{2}\lambda^a$ and $t^b = \frac{1}{2}\lambda^b$ we have

$$\operatorname{tr}_{\text{species}}\big(Tt^at^b\big) \;=\; \operatorname{tr}_{\text{flavors}}(T) \times \operatorname{tr}_{\text{colors}}(t^at^b) \;=\; \operatorname{tr}_{\text{flavors}}(T) \times \tfrac{1}{2}\delta^{ab}, \tag{181}$$

hence the net anomaly of the chiral symmetry generated by $T = (T_L, T_R)$ is

$$\partial_\mu J^\mu[T]_{\text{net}} \;=\; \frac{g^2}{64\pi^2}\big(\epsilon^{\alpha\beta\mu\nu}F^a_{\alpha\beta}F^a_{\mu\nu}\big)_{\text{QCD}} \times \big(\operatorname{tr}_{\text{flavors}}(T_L) \;-\; \operatorname{tr}_{\text{flavors}}(T_R)\big). \tag{182}$$

Thus, out of the $2N_f^2$ generators of the chiral flavor symmetry $U(N_f)_L \times U(N_f)_R$, there is only one anomalous generator, namely the $U(1)_A$.

For a more interesting example, consider the electroweak $SU(2)_W \times U(1)_Y$ gauge symmetry, or rather its $SU(2)_W$ subgroup. Unlike QCD, the $SU(2)_W$ is chiral: All the LH quarks and leptons form a bunch of $SU(2)_W$ doublets, while all the RH quarks and leptons are $SU(2)_W$ singlets. Consequently, for any two $SU(2)_W$ generators t^a and t^b and any chiral global symmetry of the quarks and leptons,

$$\mathrm{tr}_{\mathrm{RHWF}}\big(Tt^a t^b\big) \;=\; 0, \tag{183}$$

while

$$\mathrm{tr}_{\mathrm{LHWF}}\big(Tt^a t^b\big) \;=\; \mathrm{tr}_{\mathbf{2}}\big(t^a t^b\big) \times \mathrm{tr}_{\mathrm{LH\,doublets}}(T_L), \tag{184}$$

where the first trace is over the gauge indices of a single $SU(2)_W$ doublet and the second trace is over the remaining species indices distinguishing different doublets of LH quarks and leptons from each other. Moreover,

$$\mathrm{tr}_{\mathbf{2}}\big(t^a t^b)\big) \;=\; \mathrm{tr}\left(\begin{matrix}\text{Pauli}\\ \text{matrices}\end{matrix}\; \frac{\tau^a}{2}\,\frac{\tau^b}{2}\right) \;=\; \tfrac{1}{2}\delta^{ab}, \tag{185}$$

thus

$$\Big(\mathrm{tr}_{\mathrm{LHWF}}(T_L t^a t^b) - \mathrm{tr}_{\mathrm{RHWF}}(T_R t^a t^b)\Big) \;=\; \tfrac{1}{2}\delta^{ab} \times \mathrm{tr}_{\mathrm{LH\,doublets}}(T_L) \tag{186}$$

and therefore the $SU(2)_W$ anomaly

$$\partial_\mu J^\mu[T] \;=\; \frac{g_2^2}{64\pi^2}\left(\epsilon^{\alpha\beta\mu\nu} F^a_{\alpha\beta} F^a_{\mu\nu}\right)_{\mathrm{SU(2)_W}} \times \mathrm{tr}_{\mathrm{LH\,doublets}}(T_L). \tag{187}$$

Note that this anomaly depends only on the global symmetry's action on the LH quarks and leptons but it does not care how it acts on the RH fermions. This global symmetry may act in a chiral, axial, or even vector fashion, but we may still have a chiral anomaly because the $SU(2)_W$ gauge symmetry itself is acting chirally. For example, consider the lepton number symmetry: a vector $U(1)$ symmetry under which all leptons — charged or neutral, LH or RH, — have charge $L = +1$ while all the quarks have $L = 0$. Counting the

LH Weyl fermion doublets, this gives us 3 doublets (ν_e, e^-), (ν_μ, μ^-), (ν_τ, τ^-) of $L = +1$ and $3 \times 3_{\text{colors}}$ quark doublets of $L = 0$, thus

$$\operatorname{tr}_{\text{LH doublets}}(L) \;=\; 3 \times 1 + \; 9 \times 0 \;=\; 3 \tag{188}$$

and hence non-zero $SU(2)_W$ anomaly

$$\partial_\mu J^\mu[L] \;=\; \frac{3g_2^2}{64\pi^2} \left(\epsilon^{\alpha\beta\mu\nu} F^a_{\alpha\beta} F^a_{\mu\nu}\right)_{\text{SU(2)w}} \;\neq\; 0. \tag{189}$$

Thus, in the Standard Model the lepton number is not exactly conserved!

Likewise, the baryon number is not exactly conserved. Indeed, the $U(1)_B$ is a vector-like symmetry under which all the quarks have charge $B = +\frac{1}{3}$ while all the leptons have charge $B = 0$, so if we count the LH doublets we get 3 lepton doublets of $B = 0$ and 9 quark doublets — (u, d'), (c, s'), (t, b'), each coming in 3 colors — of $B = \frac{1}{3}$, thus

$$\operatorname{tr}_{\text{LH doublets}}(B) \;=\; 3 \times 0 \;+\; 3 \times 3 \times \tfrac{1}{3} \;=\; 3 \tag{190}$$

and hence non-zero $SU(2)_W$ anomaly

$$\partial_\mu J^\mu[B] \;=\; \frac{3g_2^2}{64\pi^2} \left(\epsilon^{\alpha\beta\mu\nu} F^a_{\alpha\beta} F^a_{\mu\nu}\right)_{\text{SU(2)w}} \;\neq\; 0. \tag{191}$$

Curiously, the $B - L$ combination of the baryon and lepton numbers happens to be conserved, since eqs. (189) and (191) give exactly similar formulae for the $\partial_\mu J^\mu[L]$ and the $\partial_\mu J^\mu[B]$. But the baryon or lepton numbers themselves are not conserved, although the non-conservation is non-perturbative and rather weak.

To see how this works, consider the net baryon or lepton number non-conservation

$$\Delta B \;=\; \Delta L \;=\; \int d^4x \, \partial_\mu J^\mu[B \text{ or } L] \;=\; \frac{3g_2^2}{32\pi^2} \int d^4x \, \operatorname{tr}_{\mathbf{2}}\big(\epsilon F F\big)_{SU(2)_W}. \tag{192}$$

As you shall see in your next homework#23 (problem 2), the integrand here is a total derivative,

$$\operatorname{tr}(\epsilon F F) \;=\; \partial^\alpha W_\alpha \tag{193}$$

for some function W^α of the $SU(2)$ gauge fields. However, the integral of this total derivative does not vanish; instead, it depends on the gauge fields topology at $x \to \infty$. Indeed, as I explained in the extra lecture Last Friday (April 7),

$$I = \frac{g_2^2}{32\pi^2}\int d^4x\ \mathrm{tr}_{\mathbf{2}}\bigl(\epsilon FF\bigr)_{SU(2)} \tag{194}$$

is the topological index of the $SU(2)$ gauge fields, also known as the instanton number. Thus,

$$\Delta B = \Delta L = 3\times I[SU(2)_W \text{ gauge field configuration}], \tag{195}$$

which gives rise to 2 kinds on non-perturbative effects: *instantons* and *sphalerons*.

The instantons are tunneling events between topologically different vacuum states of the non-abelian gauge fields, *cf.* 't Hooft's lecture notes (chapter 4) for details. In Euclidean spacetime, they become compact configurations of the gauge fields which are self-dual, $\frac{1}{2}\epsilon_E^{\alpha\beta\mu\nu}F_E^{\mu\nu} = +F_E^{\alpha\beta}$, and having the topological index $I=+1$. Consequently, because of the $SU(2)_W$ anomaly of the baryon and lepton numbers, and instanton tunneling between the vacuum states of the $SU(2)$ gauge fields is accompanied by creation of quarks and leptons and/or annihilation of antiquarks and antileptons, with the net effect of $\Delta B = \Delta L = +3$. Likewise, there are inverse tunneling processes — the anti-instantons — which have anti-self-dual gauge fields in Euclidean spacetime, $\frac{1}{2}\epsilon_E^{\alpha\beta\mu\nu}F_E^{\mu\nu} = -F_E^{\alpha\beta}$, and have topological index $I=-1$; these events are accompanied by the creation of antiquarks and antileptons and/or annihilation of quarks and leptons, with the net effect of $\Delta B = \Delta L = -3$.

The instantons and the anti-instantons of the $SU(2)_W$ have finite but large Euclidean actions

$$S_E = \frac{8\pi^2}{g_2^2} = \frac{2\pi}{\alpha_2} \approx 186. \tag{196}$$

Consequently, the amplitude of any (anti)instanton mediated process like baryon-number-violating decay

$$\text{deuteron} \to \text{antiproton} + 3\,\text{positrons}, \qquad \mathit{\Delta}B = \mathit{\Delta}L = -3, \tag{197}$$

is suppressed by a very small factor

$$\exp(-S_E) \sim 10^{-81}. \tag{198}$$

This factor — and especially its square in the decay rate — is so small that we shall probably never observe this process experimentally. (Although we might see a baryon decay induced by some other mechanism.)

The sphalerons exist in real (Minkowski) rather than Euclidean time. Or rather, a sphaleron is a static but unstable semi-classical configurations of the $SU(2)_W$ gauge fields and Higgs fields of large but finite energy

$$E \sim \frac{2\pi}{\alpha_2} M_W \sim 15 \text{ TeV}. \tag{199}$$

The sphaleron configuration is unstable and decays to waves of W and Higgs fields on top of the vacuum — which in the quantum theory become large numbers ~ 100 of W and Higgs particles. Importantly, there are two topologically distinct ways for a sphaleron to decay, one with $I = +\frac{1}{2}$ and the other with $I = -\frac{1}{2}$. Consequently, when waves of Higgs and W fields converge and assemble a sphaleron in a reverse process to one of the decay channels, and then the sphaleron decays via the other channel, the net gauge field configuration in Minkowski spacetime has $I = \pm 1$, and the anomaly leads to the net change of the baryon and lepton numbers by $\Delta B = \Delta L = \pm 3$.

The sphalerons are too heavy to be made at the LHS, end even when the future accelerators will reach higher energies, it would be very hard to create a non-trivial bound state of so many quanta in a 2-particle collision. However, in the Early Universe the space was filled with a hot plasma, so the sphalerons would have been regularly made and unmade in the multi-particle collisions, and their abundance would have been governed by the Boltzmann factor $\exp(-E/T)$. In particular, soon after the electroweak transition — especially if it was second-order or weakly first order — we would have a smaller Higgs VEV than today, which would lead to $M_W \ll T$, hence $\exp(-E_{\text{sphaleron}}/T)$ being not too small, and therefore plenty of sphalerons.

In thermal equilibrium, assembly and decays of sphaleron configurations via either of the $I = \pm\frac{1}{2}$ channels would be in detailed balance, so there would be no net change of baryon or lepton numbers. However, if the hot plasma is out of chemical equilibrium and has non-zero net baryon and/or lepton number densities, the sphaleron processes would be out of detailed balance, and their net effect would bring B^{net} and L^{net} into equilibrium with each other while keeping the difference $(B-L)^{\text{net}}$ unchanged. Working through the chemical potentials of various lepton and quark species, one finds that the plasma ends up with the equilibrium having

$$B^{\text{eq}} \;=\; \tfrac{28}{79}(B-L) \;=\; \tfrac{28}{79}(B-L)^{\text{init}}, \qquad L^{\text{eq}} \;=\; -\tfrac{51}{79}(B-L) \;=\; -\tfrac{51}{79}(B-L)^{\text{init}}. \quad (200)$$

Thus, the sphalerons at the electroweak transition time allow for the Leptogenesis route to the baryon-antibaryon asymmetry of the present-day Universe. The leptogenesis works in several stages:

- First, in the very early Universe some out-of-equilibrium CP-violating process creates more anti-leptons than leptons. For example, suppose there exist right-handed neutrinos with very large Majorana masses. In the see-saw mechanism for the ordinary LH neutrino's masses, these RH neutrinos have Yukawa couplings to LH leptons and the Higgs doublet, so they can decay into a lepton and a Higgs or an anti-lepton and an anti-Higgs; for complex Yukawa couplings violating the CP symmetry, these decay channels may have different branching ratios. Hence, if these RH neutrinos happen to decay out of thermal equilibrium (because the Universe is cooling down due to Hubble expansion faster than they decay), their decay products would contain more antileptons than leptons (or the other way around). And once they all decay, the resulting lepton number excess would remain frozen till the next stage of the leptogenesis.

- Second, the less-early Universe cools down to the electroweak transition, and for a while the sphaleron energy is not much higher than the temperature. During this time, the sphalerons are rapidly made and unmade, and in the process they bring the baryon and the lepton chemical potentials to equilibrium with each other. Thus, part of the initial antilepton-over-lepton excess becomes converted to the baryon-over-antibaryon excess.

- When the temperature fall significantly below the electroweak transition, the sphaleron energy becomes much larger than the temperature, the sphalerons become very rare, and the sphaleron-mediated B and L violating processes stop. After this point, the baryon asymmetry — a larger number of quarks than antiquarks — remains fixed to this day. However, the *relative value* of this asymmetry is initially rather small,

$$\frac{n_{\text{quarks}} - n_{\text{antiquarks}}}{n_{\text{quarks}}} \sim 10^{-9}. \tag{201}$$

- At much later later times when the temperature drops to about 170 MeV, the quarks and the antiquarks become confined to mesons, baryons, and antibaryons, and then the baryons and the anti-baryons annihilate each other. The annihilation products — mostly mesons — decay to muons, electrons, and neutrinos, with $\mu^+\mu^-$ and e^+e^- pairs annihilating to photons. By the time the universe cools down to 100 keV, about half of its net entropy is converted to photons (which today comprise the Cosmic Background radiation) and the other half to the neutrinos. As to the baryons and the antibaryons, they all annihilate each other except for the small baryon excess, so today

$$\frac{n_{\text{baryons}}}{\text{entropy density}} \sim 10^{-9} \tag{202}$$

while

$$n_{\text{antibaryons}} < 10^{-20} \times n_{\text{baryons}} \,. \tag{203}$$

Gauge Anomalies

Thus far, we have focused on the anomalies of various global symmetries of the gauge theories. But in a chiral gauge theory, the gauge symmetry itself could be anomalous, which would then destroy the quantum consistency of the theory. For example, consider the abelian EM gauge field A_μ coupled to massless chiral fermions ψ_L^i and ψ_R^j with different electric charges $q_L^i \neq q_R^j$. As we have learned back in February, quantum consistency of QED — and likewise of any chiral $U(1)$ gauge theory — requires Ward–Takahashi identities stemming from the electric current conservation,

$$\partial_\mu J^\mu_{\text{el}} = 0 \quad \text{for} \quad J^\mu_{\text{el}} = \sum_i q_L^i \psi^\dagger_{Li} \bar\sigma^\mu \psi_L^i + \sum_j q_R^j \psi^\dagger_{Rj} \sigma^\mu \psi_R^j, \tag{204}$$

or in matrix notations

$$J^\mu_{\text{el}} = \psi_L^\dagger Q_L \bar\sigma^\mu \psi_L + \psi_R^\dagger Q_R \sigma^\mu \psi_R \,. \tag{205}$$

This electric current has the same form as a global symmetry current with a chiral generator

$T = Q$, so its anomaly has the same form:

$$\partial_\mu J^\mu_{\rm el} \;=\; \frac{\mathcal{A}}{32\pi^2} \times (\epsilon F F)_{\rm EM} \tag{206}$$

where

$$\mathcal{A} \;=\; {\rm tr}_{\rm LHWF}\bigl(Q_L^2 \times (T_L = Q_L)\bigr) - {\rm tr}_{\rm RHWF}\bigl(Q_R^2 \times (T_R = Q_R)\bigr) \;=\; {\rm tr}_{\rm LHWF}(Q_L^3) - tr_{\rm RHWF}(Q_R^3), \tag{207}$$

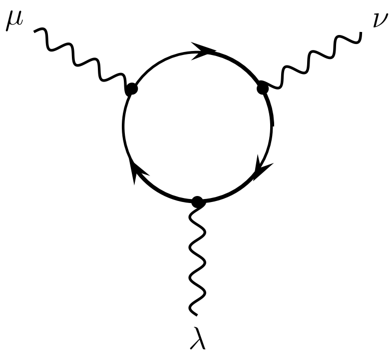
or in terms of the individual Weyl fermions and their electric charges

$$\mathcal{A} \;=\; \sum_{\psi_L} Q^3 \;-\; \sum_{\psi_R} Q^3. \tag{208}$$

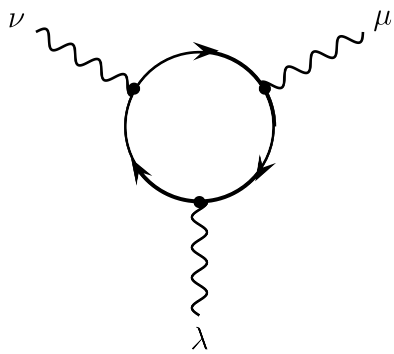
Thus, *unless a chiral gauge theory happens to have* $\mathcal{A} = 0$*, its electric current is not conserved and the theory is inconsistent!*

Diagrammatically, the inconsistency manifests itself as a failure of the Ward–Takahashi identity of three-photon one-loop amplitude:

$$\mathcal{M}^{\lambda\mu\nu} \;=$$



$+$



(209)

In a chiral theory with $\mathcal{A} \neq 0$, instead of $k_{1\lambda}\mathcal{M}^{\lambda\mu\nu} = 0$ we get

$$k_{1\lambda}\mathcal{M}^{\lambda\mu\nu} \;=\; \frac{i\mathcal{A}}{4\pi^2}\,\epsilon^{\alpha\mu\beta\nu} k_{2\alpha} k_{3\beta} \;\neq\; 0. \tag{210}$$

In the functional integral formulation, a non-zero $\mathcal{A}$ breaks the gauge invariance of the fermionic integral's measure and hence the effective action for the gauge field due to the fermionic integral:

$$\begin{aligned}
\hat{Z}[A_\mu(x)] &= \iiint \mathcal{D}[\overline{\Psi}(x)] \iiint \mathcal{D}[\Psi(x)] \, \exp\left(-\int d^4x_e \, \overline{\Psi} \not{D} \Psi\right) \\
&= \mathrm{Det}(\not{D}), \qquad (211) \\
S_E^{\mathrm{eff}}[A_\mu(x)] &= S_E^{\mathrm{classical}}[A_\mu(x)] \;-\; \log \hat{Z}[A_\mu(x)] \;=\; S_E^{\mathrm{classical}} - \mathrm{Tr}\big(\log(\not{D})\big). \qquad (212)
\end{aligned}$$

While the classical Euclidean action of the theory is invariant under gauge transforms

$$\begin{aligned}
\psi_L(x) &\to \exp(iQ_L\Lambda(x))\psi_L(x), \\
\psi_R(x) &\to \exp(iQ_R\Lambda(x))\psi_R(x), \qquad (213) \\
A_\mu(x) &\to A_\mu(x) \;-\; \partial_\mu\Lambda(x),
\end{aligned}$$

the measure of the fermionic integral (211) has a non-trivial Jacobian

$$\mathcal{J} \;=\; \mathrm{Det}^2_{\psi_L}\big(iQ_L\hat{\Lambda}\big) \times \mathrm{Det}^2_{\psi_R}\big(iQ_R\hat{\Lambda}\big) \;\neq\; 1. \qquad (214)$$

The UV regulation and consequent evaluation of this formal Jacobian (or rather its log) works similarly to what we had for the axial symmetry of the massless electron, except for the factor $\mp\frac{1}{2}$ for the LH/RH Weyl fermions. Thus, for a single charged Weyl fermion we get

$$\Delta_{\mathrm{gauge}} S_E^{\mathrm{eff}} \;=\; \mp\frac{q^3}{32\pi^2}\int d^4x_e \, \Lambda(x) \times \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}(x) F_{\mu\nu}(x), \qquad (215)$$

so for the whole set of both LH and RH Weyl fermions we get

$$\begin{aligned}
\Delta_{\mathrm{gauge}} S_E^{\mathrm{eff}} &= \frac{-\mathcal{A}}{32\pi^2}\int d^4x_e \, \Lambda(x) \times \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}(x) F_{\mu\nu}(x) \qquad (216) \\
\text{for } \mathcal{A} &= \mathrm{tr}_{\mathrm{LHWF}}(Q_L^3) \;-\; tr_{\mathrm{RHWF}}(Q_R^3). \qquad (217)
\end{aligned}$$

⋆ ⋆ ⋆

Now consider a non-abelian gauge theory with chiral fermions: ψ^i_L in some multiplet $(m)_L$ of the gauge group G and ψ^j_R in some other multiplet $(m)_R$. Classically, *the gauge symmetry currents*

$$J^a_\mu \;=\; \psi^\dagger_L t^a_L \bar\sigma^\mu \psi_L \;+\; \psi^\dagger_R t^a_R \sigma^\mu \psi_R \tag{218}$$

— where t^a_L and t^a_R are matrices representing the gauge group generators in the $(m)_L$ and $(m)_R$ multiplets — are covariantly conserved,

$$D^\mu J^a_\mu \;\equiv\; \partial^\mu J^a_\mu \;-\; f^{abc} A^{\mu b} J^c_\mu \;=\; 0. \tag{219}$$

But in the quantum theory, the measure of the fermionic functional integral

$$\hat Z[A^a_\mu] \;=\; \iiiint\mathcal{D}[\overline\Psi] \iiiint\mathcal{D}[\Psi]\,\exp\left(-\int d^4x_e\,\overline\Psi\,\not{D}\Psi\right) \;=\; \mathrm{Det}(\not{D}) \tag{220}$$

— and hence the effective action

$$S^{\rm eff}_E[A^a_\mu] \;=\; S^{\rm classical}_E[A_\mu] \;-\; \log\mathrm{Det}(\not{D}) \tag{221}$$

for the gauge fields — are not invariant under the non-abelian gauge theories

$$\begin{aligned} \delta\psi_L(x) &= i\Lambda^a(x) t^a_L \psi_L(x), \\ \delta\psi_R(x) &= i\Lambda^a(x) t^a_R \psi_R(x), \\ \delta A^a_\mu(x) &= \frac{-1}{g}\, D_\mu \Lambda^a(x). \end{aligned} \tag{222}$$

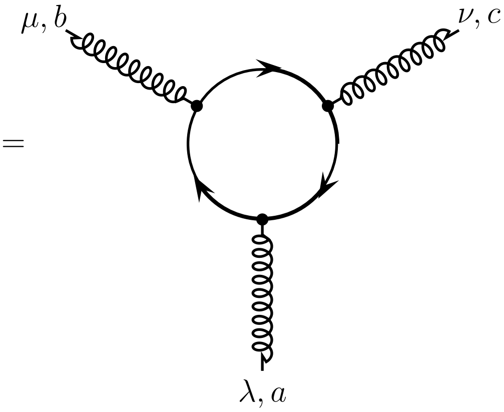
Formally,

$$\Delta_{\rm gauge} S^{\rm eff}_E \;=\; -2\,\mathrm{Tr}\big(\hat\Lambda\gamma^5\big) \;=\; -2\int d^4x_e\,\Lambda^a(x)\times \mathrm{tr}\big(\langle x|\, t^a\gamma^5\,|x\rangle\big), \tag{223}$$

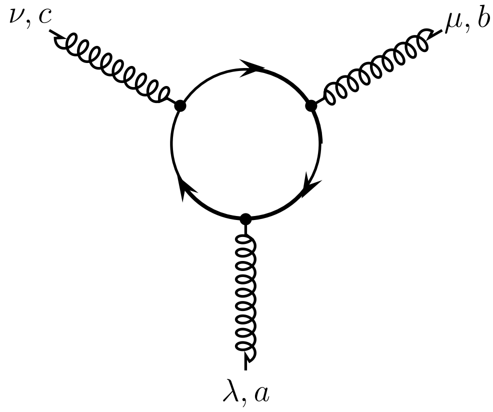
but the matrix element $\langle x|\cdots|x\rangle$ needs to be UV regulated. And the UV regulation here is more complicated than for a global chiral symmetry because the gauge symmetry generators t^a do not commute with the $\not{D}$.

Diagrammatically, the chiral anomaly of the non-abelian gauge currents stem from the one-loop 3-gauge-boson and 4-gauge-boson diagrams,

$$\mathcal{M}^{abc}_{\lambda\mu\nu} \;=$$



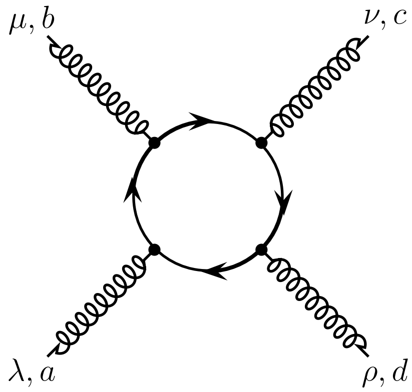
$$+$$



(224)

and

$$\mathcal{M}^{abcd}_{\lambda\mu\nu\rho} \;=$$



$+$ gauge boson permutations. (225)

The 3-gauge-boson diagrams (122) work similarly to the three-photon diagrams: Apart for gauge group factors, both diagrams yield the same amplitude

$$k_1^\lambda \mathcal{M}_{\lambda\mu\nu} \;=\; \pm\frac{ig^3}{8\pi^2}\,\epsilon_{\alpha\mu\beta\nu}k_2^\alpha k_3^\beta\,, \tag{226}$$

with the overall sign being + for the LH fermions and − for the RH fermions. But the group factors are different for the two 3-gauge-boson diagrams:

$$\begin{aligned} &\mathrm{tr}\big(t^c t^b t^a\big) \quad \text{for the left diagram (122)},\\ &\mathrm{tr}\big(t^b t^c t^a\big) \quad \text{for the right diagram (122)}. \end{aligned} \tag{227}$$

Consequently, summing up the two diagrams for each fermion chirality and then summing over the two chiralities, we end up with

$$k_1^\lambda \mathcal{M}_{\lambda\mu\nu}^{abc} = \frac{ig^3}{8\pi^2} \times \epsilon_{\alpha\mu\beta\nu} k_2^\alpha k_3^\beta \times \mathcal{A}^{abc} \tag{228}$$

where the net group factor $\mathcal{A}^{abc}$ amounts to

$$\mathcal{A}^{abc} = \mathrm{tr}_{\mathrm{LHWF}}\big(t^a\{t^b,t^c\}\big) - \mathrm{tr}_{\mathrm{RHWF}}\big(t^a\{t^b,t^c\}\big). \tag{229}$$

Note that by the cyclic symmetry of the trace, $\mathcal{A}^{abc}$ is totally symmetric in its 3 indices.

The 4-gauge-boson diagrams (225) are more complicated. For the Weyl fermions of specific LH or RH chirality, the anomalous amplitude $k_1^\lambda \mathcal{M}_{\lambda\mu\nu\rho}^{abcd}$ is $\mp\frac{1}{2}$ of the axial amplitude of the quadrangle diagram you should have calculated in homework#22 (problem 2), except for a different group factor $\mathrm{tr}(t^at^bt^ct^d)$ instead of $\mathrm{tr}(t^at^bt^d)$. Thus,

$$\begin{aligned}
k_1^\lambda \mathcal{M}_{\lambda\mu\nu\rho}^{abcd} &= \pm\frac{ig^4}{8\pi^2}\,(k_2+k_4)^\alpha \epsilon_{\alpha\mu\nu\rho} \times \mathrm{tr}\big(t^at^bt^ct^d\big) \\
&\qquad \text{non-cyclic permutations of the 4 gauge bosons} \\
&= \pm\frac{ig^4}{8\pi^2}\,\epsilon_{\alpha\mu\nu\rho} \times \begin{pmatrix} (k_2+k_4)^\alpha \times \big(t^at^bt^ct^d - t^at^dt^ct^b\big) \\ +\,(k_3+k_2)^\alpha \times \mathrm{tr}\big(t^at^ct^dt^b - t^at^bt^dt^c\big) \\ +\,(k_4+k_3)^\alpha \times \mathrm{tr}\big(t^at^dt^bt^c - t^at^ct^bt^d\big) \end{pmatrix} \\
&= \pm\frac{ig^4}{8\pi^2}\,\epsilon_{\alpha\mu\nu\rho} \times \begin{pmatrix} k_2^\alpha \times \mathrm{tr}\big(t^at^bt^ct^d - t^at^dt^ct^b + t^at^ct^dt^b - t^at^bt^dt^c\big) \\ +k_3^\alpha \times \mathrm{tr}\big(t^at^ct^dt^b - t^at^bt^dt^c + t^at^dt^bt^c - t^at^ct^bt^d\big) \\ +k_4^\alpha \times \mathrm{tr}\big(t^at^bt^ct^d - t^at^dt^ct^b + t^at^dt^bt^c - t^at^ct^bt^d\big) \end{pmatrix}.
\end{aligned} \tag{230}$$

In this formula

$$\begin{aligned}
\mathrm{tr}\big(t^at^bt^ct^d - t^at^dt^ct^b + t^at^ct^dt^b - t^at^bt^dt^c\big) &= \mathrm{tr}\big(t^at^b[t^c,t^d] + t^a[t^c,t^d]t^b\big) \\
&= if^{cde}\, tr\big(t^at^bt^e + t^at^et^c\big) \\
&= if^{cde} \times \mathrm{tr}\big(\{t^a,t^b\}t^e\big),
\end{aligned} \tag{231}$$

and likewise

$$\begin{aligned}
\mathrm{tr}\big(t^at^ct^dt^b - t^at^bt^dt^c + t^at^dt^bt^c - t^at^ct^bt^d\big) \ &= \ if^{dbe}\times\mathrm{tr}\big(\{t^a,t^c\}t^e\big),\\
\mathrm{tr}\big(t^at^bt^ct^d - t^at^dt^ct^b + t^at^dt^bt^c - t^at^ct^bt^d\big) \ &= \ if^{bce}\times\mathrm{tr}\big(\{t^a,t^d\}t^e\big).
\end{aligned} \tag{232}$$

Thus, for the Weyl fermions of one particular chirality

$$k_1^\lambda \mathcal{M}^{abcd}_{\lambda\mu\nu\rho} \ = \ \mp\frac{g^4}{8\pi^2}\,\epsilon_{\alpha\mu\nu\rho}\times\begin{pmatrix} \ \ k_2^\alpha\times f^{cde}\,\mathrm{tr}\big(\{t^a,t^b\}t^e\big)\\ +\,k_3^\alpha\times f^{dbe}\,\mathrm{tr}\big(\{t^a,t^c\}t^e\big)\\ +\,k_4^\alpha\times f^{bce}\,\mathrm{tr}\big(\{t^a,t^d\}t^e\big)\end{pmatrix}, \tag{233}$$

and when we sum up the contributions of both chiralities — and mind the opposite $\mp$ signs for the two chiralities, — each remaining trace in this formula becomes

$$\mathrm{tr}\big(\{t^a,t^b\}t^e\big) \ \mapsto \ -\mathrm{tr}_{\mathrm{LHWF}}\big(\{t^a,t^b\}t^e\big) \ + \ \mathrm{tr}_{RHWF}\big(\{t^a,t^b\}t^e\big) \ = \ -\mathcal{A}^{abe} \tag{234}$$

for exactly the same anomaly coefficients $\mathcal{A}^{abe}$ as for the 3-gauge-boson anomalies! Altogether,

$$k_1^\lambda \mathcal{M}^{abcd}_{\lambda\mu\nu\rho} = \ -\frac{g^4}{8\pi^2}\,\epsilon_{\alpha\mu\nu\rho}\times\begin{pmatrix} \ \ k_2^\alpha\times\mathcal{A}^{abe}f^{acd}\\ +\,k_3^\alpha\times\mathcal{A}^{ace}f^{edb}\\ +\,k_4^\alpha\times\mathcal{A}^{ade}f^{ebc}\end{pmatrix}. \tag{235}$$

In terms of the gauge currents J^a_μ and their covariant divergences $D^\mu J^a_\mu$, the 3-gauge-boson and the 4-gauge boson amplitudes (228) and (235) add up to

$$\begin{aligned}
D^\mu J^a_\mu(x) \ &= \ -\frac{g^4}{316\pi^2}\,\epsilon^{\lambda\mu\nu\rho}\,\mathcal{A}^{abe}\,\partial_\lambda\left(A^b_\mu\left(\partial_\nu A^e_\rho \ - \ \frac{g}{4}f^{ecd}A^c_\nu A^d_\rho\right)\right)\\
&= \ -\frac{g^4}{32\pi^2}\,\epsilon^{\lambda\mu\nu\rho}\,\mathcal{A}^{abe}\,\partial_\lambda\left(A^b_\mu\left(F^e_{\nu\rho} \ + \ \frac{g}{2}A^e_\nu A^d_\rho\right)\right).
\end{aligned} \tag{236}$$

Note the $g/4$ coefficient in the top formula here is different from both

$$\tfrac12\epsilon^{\lambda\mu\nu\rho}F^e_{\nu\rho} \ = \ \epsilon^{\lambda\mu\nu\rho}\left(\partial_\nu A^e_\rho \ - \ \frac{g}{2}f^{ecd}A^c_\nu A^d_\rho\right) \tag{237}$$

and from the Chern–Simons form proportional to the

$$\epsilon^{\lambda\mu\nu\rho}\left(A^b_\mu\,\partial_\nu A^e_\rho \ - \ \frac{g}{3}f^{ecd}A^b_\mu A^c_\nu A^d_\rho\right). \tag{238}$$

But regardless of such details, the most important aspect of eq. (236) is that *all gauge* *currents are covariantly conserved if and only if all the anomaly coefficients $\mathcal{A}^{abe}$ happen to vanish,* or in other words, if all the symmetrized traces $\mathrm{tr}(\{t^a,t^b\}t^e)$ happen to cancel out between the LH and the RH Weyl fermions.

The same condition governs the gauge invariant of the effective action for the gauge fields

$$S_E^{\mathrm{eff}}[A_\mu^a] \;=\; S_E^{\mathrm{classical}}[A_\mu] \;-\; \log\mathrm{Det}(\not{D}). \tag{239}$$

Indeed, eqs. (236) means that under an infinitesimal gauge transform parametrized by $\Lambda^a(x)$, the effective action varies by

$$\begin{aligned}\Delta_{\mathrm{gauge}}S_E^{\mathrm{eff}}[A_\mu] \;&=\; -\frac{\mathcal{A}^{abe}g^3}{16\pi^2}\int d^4x_e\,\Lambda^a\times\epsilon^{\lambda\mu\nu\rho}\,\partial_\lambda\left(A^b_\mu\partial_\nu A^e_\rho \;-\; \frac{g}{4}f^{ecd}A^b_\mu A^c_\nu A^d_\rho\right)\\ &=\; +\frac{\mathcal{A}^{abe}g^3}{32\pi^2}\int d^4x_e\,\partial_\lambda\Lambda^a\times\epsilon^{\lambda\mu\nu\rho}\left(A^b_\mu\left(F^e_{\nu\rho} \;+\; \frac{g}{2}A^e_\nu A^d_\rho\right)\right).\end{aligned} \tag{240}$$

Thus, *the effective action is gauge invariant if and only if* $\mathcal{A}^{abe}=0$.

Anomaly Cancellation

This far, we have learned that the chiral gauge theories are consistent as quantum theories only if all the anomaly coefficients $\mathcal{A}^{abc}$ happen to cancel out between the LH and the RH Weyl fermions. In this section, we shall see a few non-trivial examples of such cancellation and learn some general rules for calculating the $\mathcal{A}^{abc}$ coefficients.

Let us start with the simple non-abelian gauge groups. A very useful concept for calculating anomalies in such theories is the *cubic Casimir operator*: it's a cubic polynomial in the gauge group generators,

$$\hat{C}_3 \;=\; d_{abc}\hat{t}^a\hat{t}^b\hat{t}^c \tag{241}$$

with some totally symmetric coefficients

$$d_{abc} \;=\; d_{bca} \;=\; d_{cab} \;=\; d_{cba} \;=\; d_{acb} \;=\; d_{bac} \tag{242}$$

chosen such that $\hat{C}_3$ commutes with all the generators,

$$[\hat{C}_3, \hat{t}^d] \;=\; 0 \quad \forall \hat{t}^d. \tag{243}$$

Note: unlike the quadratic Casimir operator $\hat{C}_2 = g_{ab}\hat{t}^a\hat{t}^b$ which exists for any simple Lie algebra, the cubic Casimir operator exists in some simple Lie algebras but does not exist for others. Specifically, it exists for all the $SU(N)$ algebras with $N \geq 3$ (and also for the Spin(6) $= SU(4)$) but not for any other simple Lie algebras.

Theorem 1: If a simple Lie algebra G does not have a cubic Casimir operator, then for any complete multiplet (m) of G, for any 3 generators $\hat{t}^a$, $\hat{t}^b$, $\hat{t}^c$ of G, the matrices $t^a_{(m)}$, $t^b_{(m)}$, $t^c_{(m)}$ representing these generators in the multiplet (m) obey

$$\mathrm{tr}\bigl(t^a_{(m)}t^b_{(m)}t^c_{(m)} \;+\; t^b_{(m)}t^a_{(m)}t^c_{(m)}\bigr) \;=\; 0. \tag{244}$$

Or in more compact notations,

$$\mathrm{tr}_{(m)}\bigl(\{t^a, t^b\}t^c\bigr) \;=\; 0. \tag{245}$$

From the anomaly point of view, this means that if a simple gauge group G does not have a cubic Casimir operator, then all the gauge anomaly coefficients $\mathcal{A}^{abc}$ automatically vanish and the theory is anomaly-free regardless of the chiral fermion's quantum numbers. Indeed, as long as LH Weyl fermions belong to a complete multiplet $(m)_L$ of G and the RH Weyl fermions belong to another complete multiplet $(m)_R$, then regardless of what these multiplets happen to be

$$\mathcal{A}^{abc} \;=\; \mathrm{tr}_{(m)_L}\bigl(t^a\{t^b, t^c\}\bigr) \;-\; \mathrm{tr}_{(m)_R}\bigl(t^a\{t^b, t^c\}\bigr) \;=\; 0 \;-\; 0 \;=\; 0. \tag{246}$$

Thus, if a simple gauge group G is $SU(2)$, or any $SO(N)$ with $N \neq 6$, or any $USp(N)$, or any exceptional group (G_2, F_4, E_6, E_7, or E_8), we do not have to worry about the gauge anomaly regardless of the chiral fermion's quantum numbers.

For the remaining types of simple gauge groups — namely $SU(N)$ with $N \geq 3$ — we do need to worry about the gauge anomaly. But fortunately, the anomaly counting for such theories is drastically simplified by the **Theorem 2:** If a simple Lie algebra does have a cubic Casimir $d_{abc}\hat{t}^a\hat{t}^b\hat{t}^c$, then for any complete multiplet (m) of G and any 3 generators $\hat{t}^a$, $\hat{t}^b$, $\hat{t}^c$,

$$\operatorname{tr}_{(m)}\big(\{t^a,t^b\}t^c\big) \;=\; R_3(m)\times d^{abc} \tag{247}$$

where $R_3(m)$ is the cubic index depending only on the multiplet (m) but not on the particular generators $\hat{t}^a$, $\hat{t}^b$, $\hat{t}^c$, while d^{abc} depend on a,b,c but not on the multiplet. In fact, d^{abc} are the same totally-symmetric coefficients as the d_{abc} in the construction of the cubic Casimir.

Note: eq. (247) applies to any complete multiplet (m), reducible or irreducible. And just like the quadratic index R_2, the cubic index R_3 of a reducible multiplet is a sum of cubic indices of its irreducible components:

$$\text{for } (m) \;=\; (m_1) \;+\; (m_2) \;+\; \cdots, \qquad R_3(m) \;=\; R_3(m_1) \;+\; R_3(m_2) \;+\; \cdots. \tag{248}$$

Thanks to this Theorem, for any kinds of a chiral gauge theory with an $SU(N)$ gauge group,

$$\begin{aligned} \operatorname{tr}_{\text{LHWF}}\big(\{t^a,t^b\}t^c\big) \;&=\; R_3^{\text{net}}(\text{LHWF})\times d^{abc}, \\ \operatorname{tr}_{\text{RHWF}}\big(\{t^a,t^b\}t^c\big) \;&=\; R_3^{\text{net}}(\text{RHWF})\times d^{abc}, \end{aligned} \tag{249}$$

and hence

$$\mathcal{A}^{abc} \;=\; \big(R_3^{\text{net}}(\text{LHWF}) \;-\; R_3^{\text{net}}(\text{RHWF})\big)\times d^{abc}. \tag{250}$$

Consequently, we do not need to check the anomaly cancellation for all possible combinations of the 3 adjoint indices a,b,c. Instead, all we need is to check the net cubic indices of the LH and RH Weyl fermions and to check that

$$R_3^{\text{net}}(\text{LHWF}) \;-\; R_3^{\text{net}}(\text{RHWF})\big) \;=\; 0. \tag{251}$$

If this condition is satisfied, the $SU(N)$ gauge theory is anomaly free; otherwise, it's anomalous and would not work at the quantum level unless we change its spectrum of the chiral fermions.

In simplify counting the cubic indices of various multiplets of $SU(N)$ gauge groups, it's convenient to rescale them to the so-called *anomaly indices*

$$A(m) \stackrel{\text{def}}{=} \frac{R_3(m)}{R_3(\text{fundamental})} \tag{252}$$

in practice $A(m)$ is often easier to calculate then $R_3(m)$, and the anomaly cancellation condition (251) can just as well be stated in terms of the $A(m)$:

$$\mathcal{A}^{\text{net}} = A^{\text{net}}(\text{LHWF}) - A^{\text{net}}(\text{RHWF})\big) = 0. \tag{253}$$

Let me give you the values of A of some commonly used $SU(N)$ multiplets:

- $A(\text{fundamental}) = +1$, $A(\text{antifundamental}) = -1$, $A(\text{adjoint}) = 0$.
- The antisymmetric tensor multiplet $\psi^{ij} = -\psi^{ji}$ has $A = N - 4$, the symmetric tensor multiplet $\psi^{ij} = +\psi^{ji}$ has $A = N + 4$.

⋆ More generally, any real or pseudo-real multiplet (m) — meaning, its complex conjugate multiplet (m^*) is equivalent to (m), — has $A(m) = 0$.

⋆ Any complex multiplet (m) and its complex conjugate (m^*) have anomaly indices of equal magnitudes and opposite signs, $A(m^*) = -A(m)$.

Note that a bunch of LH Weyl fermions ψ^i_L in some multiplet (m) of the gauge symmetry — plus their Hermitian conjugate fields $\psi^\dagger Li$ which comprise the complex conjugate multiplet (m^*) — are physically equivalent to the RH Weyl fermions ψ_{Ri} in the (m^*) multiplet, plus their conjugate fields $(\psi^\dagger_R)^i$ in (m). Fortunately, both ways of representing these fermions give exactly the same contribution to the net gauge anomaly:

$$\begin{aligned} \psi_L \in (m) \text{ contribute } \Delta\mathcal{A}^{\text{net}} &= +A(m), \\ \psi_R \in (m^*) \text{ contribute } \Delta\mathcal{A}^{\text{net}} &= -A(m^*) = +A(m). \end{aligned} \tag{254}$$

With all of these rules in mind, let's consider a specific example: the Grand Unified Theory with the $SU(5)$ gauge group. All the fermions of this theory are usually described in terms of the LH Weyl fields ψ_L and their Hermitian conjugates $\psi^\dagger_L$, without any RH Weyl fields ψ_R or their conjugates ψ^R. Physically, the ψ_L account for the LH quarks and leptons as well as LH antiquarks and antileptons, while the $\psi_L^\dagger$ account for the RH antiquarks and antileptons as well as RH quarks and leptons.

From the $SU(5)$ point of view, all the fermions of the Standard Model belong to a reducible multiplet

$$(m)_{\text{net}} \;=\; 3\times(\mathbf{10}+\bar{\mathbf{5}}), \tag{255}$$

where $\bar{\mathbf{5}}$ is the antifundamental multiplet of $SU(5)$ while $\mathbf{10}$ is the antisymmetric tensor multiplet. By the above rules of anomaly counting,

$$\begin{aligned} A(\bar{\mathbf{5}}) \;&=\; -A(\mathbf{5}) \;=\; -1,\\ A(\mathbf{10}) \;&=\; (N=5)\;-\;4 \;=\; +1, \end{aligned} \tag{256}$$

hence

$$\mathcal{A}^{\text{net}} \;=\; 3\times\bigl(A(\mathbf{10})+A(\bar{\mathbf{5}})\bigr) \;=\; 3\times(+1-1) \;=\; 0. \tag{257}$$

Thus, the $SU(5)$ Grand Unified theory is anomaly free and is a perfectly good quantum field theory. Too bad its phenomenology has been ruled out experimentally.

⋆ ⋆ ⋆

Now let's turn our attention from the simple gauge groups to the product groups. In particular, consider the Standard Model with $G=SU(3)_C\times SU(2)_W\times U(1)_Y$.

For a product group, we can no longer reduce all the gauge anomaly to a single number $\mathcal{A}^{\text{net}}$; instead, we need to consider separate $\mathcal{A}^{abc}$ coefficients with different numbers of the 3 adjoint indices a,b,c belonging to specific factors of the gauge group. For a 3-factor group like the Standard Model, there are 10 different ways to allocate the 3 indices to different group factors, and we must make sure that the $\mathcal{A}^{abc}$ anomaly coefficients cancel for all these 10 allocations.

Fortunately, some allocations of the a,b,c indices to different factors lead to automatically vanishing $\mathcal{A}^{abc}$. For example, suppose one of the indices — say a — belongs to the $SU(3)_C$ while the other two indices b and c belong to the electroweak factors $SU(2)_W\times U(1)_Y$.

In this case, for any multiplet (m) of the combined gauge group G, the t^a generator acts on the color index of the multiplet members (assuming they do have non-trivial colors) while the t^b and t^c generators act on the other indices, whatever they might be. Thus,

$$\mathrm{tr}_{(m)}\bigl(t^a\{t^b,t^c\}\bigr) \;=\; \mathrm{tr}_{\text{colors of }(m)}(t^a)\times\mathrm{tr}^{\text{other indices}}_{\text{of }(m)}\bigl(\{t^b,t^c\}\bigr) \;=\; 0\times 2\,\mathrm{tr}^{\text{other indices}}(t^bt^c) \;=\; 0 \tag{258}$$

because in all multiplets of the $SU(3)$ group all the generators $\hat{t}^a$ are represented by traceless matrices, thus $\mathrm{tr}_{\text{colors}}(t^a) = 0$. Consequently, the trace (258) vanish for each and every complete multiplet of the Standard Model's gauge group, and therefore

$$\mathcal{A}^{abc} \;=\; 0 \quad\text{when}\quad a\in SU(3)_C \quad\text{but}\quad b,c\notin SU(3)_C\,. \tag{259}$$

Likewise, in all multiplets of the $SU(2)$ group, all the generators are represented by the traceless matrices, so a similar argument tells us that

$$\mathcal{A}^{abc} \;=\; 0 \quad\text{when}\quad a\in SU(2)_W \quad\text{but}\quad b,c\notin SU(2)_W\,. \tag{260}$$

Beyond the standard model, the same argument applies to all product gauge groups

$$G \;=\; G_1\times G_2\times\cdots \tag{261}$$

where at least some of the G_i groups are non-abelian. For any multiplet of a simple non-abelian group G_i, all matrices representing the G_i generators in that multiplet are traceless, hence for $a\in G_i$ while $b,c\notin G_i$, for any complete multiplet of the whole product group G,

$$\mathrm{tr}_{\text{allindices}}\bigl(t^a\{t^b,t^c\}\bigr) \;=\; \mathrm{tr}_{G_i\text{ indices}}(t^a)\times 2\,\mathrm{tr}_{\text{other indices}}(t^bt^c) \;=\; 0\times\text{whatever} \;=\; 0 \tag{262}$$

because $\mathrm{tr}(t^a)=0$. Therefore,

$$\mathcal{A}^{abc} \;=\; 0 \quad\text{whenever}\quad a\in(\text{some non-abelian }G_i) \quad\text{but}\quad b,c\notin(\text{same }G_i). \tag{263}$$

Corollary: When all of the gauge group factors G_i are non-abelian, $\mathcal{A}^{abc}=0$ unless all 3 indices a,b,c belong to the same simple non-abelian factor. Moreover, if that factor is not an $SU(N)$ with $N \geq 3$ we would also get an automatic $\mathcal{A}^{abc} = 0$; otherwise

$$\mathcal{A}^{abc} \;=\; d^{abc}[SU(N)] \times \mathcal{A}^{\rm net}[\text{WRT to that } SU(N) \text{factor}]. \tag{264}$$

Thus, to make sure all the anomalies cancel out, all we need to check are the net anomaly indices WRT to all the $SU(N \geq 3)$ factors of G: If all such indices happen to vanish, then the whole theory is anomaly free.

Now let's go back to the Standard Model with $G = SU(3)_C \times SU(2)_W \times U(1)_Y$. Since one of the gauge group factors is abelian, checking for anomaly cancellation $\mathcal{A}^{abc} = 0$ is a bit more complicated: Besides checking the anomalies for a, b, c indices belonging to the same factor — nonabelian or abelian — we should also check them for two indices belonging to the same nonabelian factor while the third index belongs to the $U(1)$. Altogether, we need to check the $\mathcal{A}^{abc}$ for 5 types of a, b, c indices:

(1) All 3 indices $a, b, c \in SU(3)$.

(2) All 3 indices $a, b, c \in SU(2)$.

(3) Two indices $a, b \in SU(3)$ but $c \in U(1)$.

(4) Two indices $a, b \in SU(2)$ but $c \in U(1)$.

(5) All 3 indices $a = b = c \in U(1)$.

For all other types of index combinations, the anomaly trivially vanishes and we do not need to check it.

Actually, type (2) is also trivial because the $SU(2)$ gauge group by itself is automatically anomaly free — it has no cubic Casimir, and therefore in any multiplet of $SU(2)$

$$\forall a, b, c \in SU(2): \quad \mathrm{tr}\big(t^a \{t^b, t^c\}\big) \;=\; 0. \tag{265}$$

Consequently,

$$\mathcal{A}^{abc} \;=\; 0 \quad \text{when} \quad a, b, c \in SU(2). \tag{266}$$

Type (1) is not so trivial, but simple enough. Although the $SU(3)_C$ factor is not inherently anomaly free, it happens to be vector-like when we focus on this factor by itself and disregard the electroweak $SU(2)\times U(1)$. That is, all the fermions of the Standard Model are either leptons — which are neutral WRT to the $SU(3)_C$ or — Dirac quarks for which both LH and RH Weyl component belong to the similar $SU(3)_C$ triplets. Consequently, QCD by itself is free from the gauge anomaly, thus

$$\mathcal{A}^{abc} = 0 \quad \text{when} \quad a,b,c\in SU(3). \tag{267}$$

Moreover, in any extension of the Standard Model in which all colored particles get masses — or could get masses without breaking the $SU(3)_C$ symmetry — the QCD viewed by itself would be vector-like and therefore free from the gauge anomaly.

Next, type (3) of t^a, t^b being $SU(3)_C$ generators while t^c is the $U(1)$ generator Y. Clearly, the leptons do not contribute to the traces involving the $SU(3)_C$ generators, while the quarks contribute

$$\mathrm{tr}_{\text{quarks}}\bigl(\{t^a,t^b\}Y\bigr) = 2\,\mathrm{tr}_{\text{colors}}(t^at^b)\times\mathrm{tr}_{\text{flavors}} = 2\times\tfrac{1}{2}\delta^{ab}\times\mathrm{tr}_{\text{flavors}}(Y) \tag{268}$$

(where the second equality follows from quarks being color-triplets, hence $\mathrm{tr}_{\mathbf{3}}(t^at^b)=\frac{1}{2}\delta^{ab}$), and therefore

$$\mathcal{A}^{abY} = \delta^{ab}\times\left(\sum_{\substack{\text{LH quark}\\ \text{flavors}}} Y - \sum_{\substack{\text{RH quark}\\ \text{flavors}}} Y\right). \tag{269}$$

In the Standard Model, each family has 2 LH quark flavors (in a doublet of $SU(2)_W$) with hypercharge $Y=+\frac{1}{6}$ and 2 RH quark flavors with hypercharges $Y=+\frac{2}{3}$ and $Y=-\frac{1}{3}$. Thus,

$$\begin{aligned}\sum_{\substack{\text{LH quark}\\ \text{flavors}}} Y &= 3_{\text{families}}\times 2\times\frac{+1}{6} = +1,\\ \sum_{\substack{\text{RH quark}\\ \text{flavors}}} Y &= 3_{\text{families}}\times\left(\frac{+2}{3}+\frac{-1}{3}\right) = +1,\end{aligned} \tag{270}$$

and this cancels all the case (3) anomalies,

$$\text{for} \quad a,b\in SU(3) \quad \text{and} \quad t^c=Y, \quad \mathcal{A}^{abc} = 0. \tag{271}$$

More generally, the same results obtains in all extensions of the Standard Model where the electroweak symmetry $SU(2)_W \times U(1)_Y$ is Higgsed down to the electromagnetic $U(1)$ where the unbroken electric charge (in units of e) is $Q = t^3[SU(2)]+Y$. Since the t^3 generator of the $SU(2)$ is traceless, this means

$$\sum_{\substack{\text{LH quark}\\ \text{flavors}}} Y \;=\; \sum_{\substack{\text{LH quark}\\ \text{flavors}}} Q, \qquad \sum_{\substack{\text{RH quark}\\ \text{flavors}}} Y \;=\; \sum_{\substack{\text{RH quark}\\ \text{flavors}}} Q, \tag{272}$$

and in any model where all quark flavors are massive — or could get mass without breaking the electromagnetic $U(1)$ symmetry — the electric charge Q must be vector-like, hence

$$\sum_{\substack{\text{LH quark}\\ \text{flavors}}} Y \;=\; \sum_{\substack{\text{LH quark}\\ \text{flavors}}} Q \;=\; \sum_{\substack{\text{RH quark}\\ \text{flavors}}} Q \;=\; \sum_{\substack{\text{RH quark}\\ \text{flavors}}} Y \tag{273}$$

and therefore vanishing case (3) anomalies,

$$\text{for}\quad a,b\in SU(3)\quad\text{and}\quad t^c=Y,\quad \mathcal{A}^{abc} \;=\; 0. \tag{274}$$

Unlike the anomaly types (1), (2), and (3) — which cancel automatically for vector-like $SU(3)_C \times U(1)_{\text{EM}}$, — the remaining anomaly types (4) and (5) cancel only when the electric charges of all quarks and leptons add up to zero,

$$\sum_{\substack{\text{quarks,}\\ \text{leptons}}} Q_{\text{el}} \;=\; 3_{\text{colors}} \times \sum_{\substack{\text{quark}\\ \text{flavors}}} Q_{\text{el}} \;+\; \sum_{\substack{\text{lepton}\\ \text{species}}} Q_{\text{el}} \;=\; 0. \tag{275}$$

Fortunately, this is indeed the case in real life:

$$3\times\left(3\times\frac{+2}{3}\,+\,3\times\frac{-1}{3}\right)\;+\;\Bigl(3\times -1\,+\,3\times 0\Bigr)\;=\;0. \tag{276}$$

Indeed, let's check type (4): $a,b\in SU(2)_W$ while $t^c=Y$. Since all the RH quark and leptons are $SU(2)$ singlets, we immediately have

$$\operatorname{tr}_{\text{RHWF}}\bigl(\{t^a,t^b\}Y\bigr) \;=\; 0. \tag{277}$$

As to the LH quarks and leptons, they all belong to the $SU(2)$ doublets, hence

$$\mathcal{A}^{abY} \;=\; \operatorname{tr}_{\text{LHWF}}\bigl(\{t^a,t^b\}Y\bigr) \;=\; 2\operatorname{tr}_{\mathbf{2}}(t^at^b)\times\operatorname{tr}_{\text{LH doublets}}(Y) \;=\; \delta^{ab}\times\operatorname{tr}_{\text{LH doublets}}(Y). \tag{278}$$

So the type (4) anomaly cancels if and only if the hypercharges of all the LH quark and lepton doublets add up to zero. And in real life, they do: all the LH quarks have $Y=+\frac{1}{6}$

while the LH leptons have $Y = -\frac{1}{2}$, and there are $3 \times 3_{\rm colors} = 9$ quark doublets and 3 lepton doublets, thus

$$\mathrm{tr}_{\mathrm{LH\,doublets}}(Y) \;=\; 9 \times \frac{+1}{6} \;+\; 3 \times \frac{-1}{2} \;=\; 0. \tag{279}$$

In terms of the electric charges,

$$\begin{aligned}
\mathrm{tr}_{\mathrm{LH\,doublets}}(Y) \;&=\; \tfrac{1}{2}\,\mathrm{tr}_{\mathrm{LHWF}}(Y) \\
&=\; \tfrac{1}{2}\,\mathrm{tr}_{\mathrm{LHWF}}(Q_{\mathrm{el}}) \;-\; \tfrac{1}{2}\,\mathrm{tr}_{\mathrm{LHWF}}(t^3[SU(2)_W]) \\
&=\; \tfrac{1}{2}\,\mathrm{tr}_{\mathrm{LHWF}}(Q_{\mathrm{el}}) \;-\; 0 \qquad \langle\!\langle\, \text{because } t^3[SU(2)] \text{ is traceless} \,\rangle\!\rangle \\
&=\; \tfrac{1}{2}\,\mathrm{tr}_{\mathrm{Dirac\,fermions}}(Q_{\mathrm{el}}) \qquad \langle\!\langle\, \text{because } Q_{\mathrm{el}} \text{ is vector-like} \,\rangle\!\rangle.
\end{aligned} \tag{280}$$

Thus, the type (4) anomalies cancel out if and only if the electric charges of the quarks and leptons are related to each other such that

$$\sum_{\substack{\text{quarks,}\\ \text{leptons}}} Q_{\mathrm{el}} \;=\; 0, \tag{281}$$

which is exactly the condition (275).

Finally, consider the type (5) anomaly

$$\mathcal{A}^{YYY} \;=\; 2\,\mathrm{tr}_{\mathrm{LHWF}}(Y^3) \;-\; 2\,\mathrm{tr}_{\mathrm{RHWF}}(Y^3). \tag{282}$$

Let's re-express this anomaly in terms of particles' electric charges. All the LH quarks and leptons belong to $SU(2)_W$ doublets, so for any such doublet of hypercharge $Y = y$, the two LH particles have electric charges

$$Q_{\mathrm{el}} \;=\; Y \;+\; t^3[SU(2)] \;=\; Y \pm \tfrac{1}{2}. \tag{283}$$

But the electric charge is non-chiral, so for any charged LH fermion there is a RH fermion with the same electric charge; and since the RH quarks and leptons are $SU(2)_W$ singlets, their hypercharges must be equal to their electric charges $Y = y \pm \frac{1}{2}$. Altogether, we have

2 LH Weyl fermions of hypercharge y and 2 RH Weyl fermions of hypercharges $y \pm \frac{1}{2}$, and between these 4 Weyl fermions

$$\mathrm{tr}_{\mathrm{LH}}(Y^3) \;-\; \mathrm{tr}_{\mathrm{RH}}(Y^3) \;=\; 2y^3 \;-\; (y+\tfrac{1}{2})^3 \;-\; (y-\tfrac{1}{2})^3 \;=\; -\tfrac{3}{2}y. \tag{284}$$

The net $\mathcal{A}^{YYY}$ anomaly obtains as a sum of such contributions (284) from all such sets of 2 LH and 2 RH Weyl fermions.[⋆] Thus,

$$\mathcal{A}^{YYY} \;=\; -3 \sum_{\text{LH doublets}} y \;=\; -\tfrac{3}{2} \sum_{\substack{\text{quarks,}\\ \text{leptons}}} Q_{\mathrm{el}}\,. \tag{285}$$

Similar to the type (4) anomalies, the $\mathcal{A}^{YYY}$ cancels out only when the electric charges of all the quarks and all the leptons add up to zero.

Conclusions

Let me conclude these notes with a few general statements about anomalies of both gauge and global symmetries. In 4D, all such anomalies are proportional to the

$$\mathcal{A}^{abc} \;=\; \mathrm{tr}_{\mathrm{LHWF}}\big(t^a\{t^b,t^c\}\big) \;-\; \mathrm{tr}_{\mathrm{RHWF}}\big(t^a\{t^b,t^c\}\big) : \tag{286}$$

For the anomalies of global symmetry t^a is its generator while t^b, t^c are generators of the gauge group, while for the anomalies of gauge symmetry all 3 t^a, t^b, t^c are gauge group generators. A point of terminology:

- Anomalies of global symmetries are called *abelian anomalies* because only an abelian factor of the net global symmetry can suffer from such anomaly. For example, in QCD with N_f massless flavors, only the abelian $U(1)_A$ factor of the global $U(N_f)_L \times U(N_f)_R$ chiral symmetry is anomalous, while the remaining $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ factors are anomaly-free. For another example, consider the Standard–Model-like theory without the Yukawa couplings (and hence having exactly massless quarks

⋆ For simplicity, I assume the RH neutrinos exist, so I count the leptons of each family as 2 LHWF and 2 RHWF. But since the RH neutrinos have $Y = 0$, it does not matter if they exist or not as their contribution to eq. (284) and hence the net $\mathcal{A}^{YYY}$ anomaly would be nil.

and leptons). Classically, this theory has a $[U(3)]^5$ global family symmetry which mixes Weyl fermions of similar gauge quantum numbers with each other. Some of the abelian $U(1)$ factors of this symmetry — for example the lepton number and the baryon number — suffer from the $SU(2)_W$ anomaly, but all of the $SU(3)$ factors are anomaly free.

To see why this should be true in a general case, suppose the net global symmetry of some gauge theory has a simple non-abelian factor H, and let t^a be a generator of this H. Then for any complete multiplet (m) of H, $\mathrm{tr}_{(m)}(t^a) = 0$. Consequently, for any complete multiplet of the combined for any multiplet of the combined gauge $\times$ global symmetry and any two generators t^b, t^c of the gauge group,

$$\mathrm{tr}\big(t^a\{t^b,t^c\}\big) \;=\; \mathrm{tr}^{\text{global}}_{\text{indices}}(t^a)\times 2\,\mathrm{tr}^{\text{gauge}}_{\text{indices}}(t^bt^c) \;=\; 0\times\text{whatever} \;=\; 0. \tag{287}$$

Applying this formula to the complete sets of the LH and RH Weyl fermions of the theory, we immediately obtain $\mathcal{A}^{abc} = 0$. Thus, all the non-abelian simple factors of the global symmetry group are always anomaly free.

- On the other hand, anomalies of the gauge symmetries themselves are called *nonabelian anomalies,* even when the gauge theory happens to be abelian.

Finally, a couple of very useful rules for calculating anomaly coefficients $\mathcal{A}^{abc}$ and checking for the anomaly cancellation. Let G be the relevant symmetry group — the gauge group for a nonabelian anomaly or the gauge $\times$ global group for the abelian anomaly. **Lemma:** *for any complete multiplet* (m) *of* G *and any 3* G *generators* t^a, t^b, t^c,

$$\mathrm{tr}_{(m^*)}\big(t^a\{t^b,t^c\}\big) \;=\; -\,\mathrm{tr}_{(m)}\big(t^a\{t^b,t^c\}\big). \tag{288}$$

In particular, *if the multiplet* (m) *happens to be real or pseudo-real,*

$$\forall(m)\cong(m^*):\quad \mathrm{tr}_{(m)}\big(t^a\{t^b,t^c\}\big) \;=\; 0. \tag{289}$$

Physically, this Lemma immediately tells us that *for the purpose of calculating a* G *anomaly* $\mathcal{A}^{abc}$, *a multiplet* (m) *of LH Weyl fermions* ψ^i_L *is equivalent to a complex-conjugate multiplet*

(m^*) *of RH Weyl fermions* $\psi_{Ri} = (\psi_L^i)^\dagger$. Indeed, either multiplet contributes

$$\Delta\mathcal{A}^{abc} \;=\; +\operatorname{tr}_{(m)}\bigl(t^a\{t^b,t^c\}\bigr) \;=\; -\operatorname{tr}_{(m^*)}\bigl(t^a\{t^b,t^c\}\bigr). \tag{290}$$

Moreover, the Lemma (288) leads to the following **Theorem:** *the massive fermions — or the fermions which can be given mass terms without breaking the relevant symmetry G — cancel out from the anomalies. Instead, the $\mathcal{A}^{abc}$ come solely from the fermions which cannot be given any G-invariant masses.*

To prove this theorem, let's treat all the fermions of the theory as a reducible multiplet of LH Weyl fermions. In matrix notations, the fermionic Lagrangian has general form

$$\mathcal{L}_\psi \;=\; i\psi_L^\dagger\bar\sigma^\mu D_\mu\psi_L \;-\; \tfrac12\psi_L^\top\sigma_2 M\psi_L \;-\; \tfrac12\psi_L^\dagger\sigma_2 M^*\psi_L^* \tag{291}$$

for some complex symmetric mass matrix M. By assumption, the mass terms are invariant under all the relevant symmetries of the theory, which restricts the non-zero matrix elements $M_{ij}\neq 0$ to the following Dirac and Majorana masses:

- A Dirac mass term $M_{ij}\neq 0$ connects LH Weyl fermions ψ_L^i in some complex multiplet (m) to ψ_L^j in a complex-conjugate multiplet (m^*) and vice verse. By the lemma (288), the net contribution of the two multiplets $(m)+(m^*)$ to the anomaly cancels out, $\Delta\mathcal{A}^{abc}=0$.
- A Majorana mass term $M_{ij}\neq 0$ connects two LH Weyl fermions ψ_L^i and ψ_L^j in the same real multiplet $(m)\cong(m^*)$. By the lemma (289), such real multiplets do not contribute to the anomaly, $\Delta\mathcal{A}^{abc}=0$.

Altogether, the massive fermions — or any fermions which can be given Dirac or Majorana masses without breaking the relevant symmetries — cancel out from all the anomalies, $\Delta_{\text{massive}}\mathcal{A}^{abc}=0$. Instead, the entire anomaly comes from the massless fermions which can be given G-invariant masses,

$$\mathcal{A}^{abc}_{\text{net}} \;=\; \mathcal{A}^{abc}_{\text{massless}}\,. \tag{292}$$

Thanks to this Theorem, the gauge anomaly cancellation in the Standard Model would not be affected by any additional superheavy fermions we have not yet discovered experimentally. Indeed, suppose there are some superheavy fermions with non-trivial $SU(3)\times U(2)\times$

$U(1)$ quantum numbers. In light of the LHC unsuccessful searches for such particles, their masses should be larger than about 2 TeV for colored particles and 200 GeV for particles without colors. Now consider 3 possible origins for such large masses:

1. Yukawa couplings to the Standard Model's Higgs VEV. In this case, the Yukawa couplings should be rather large, and they would affect the precision electroweak measurements at LEP, Fermilab, and LHC. Since no such effects were observed experimentally, we may rule out this scenario.

2. Bare mass terms in the Standard Model's Lagrangian. In this case, the mass terms must be invariant under SM gauge symmetry, so by the Theorem, the contribution of all such massive fermions to the Standard Model's anomalies $\mathcal{A}^{abc}$ cancel out.

3. At very high energies there is a bigger gauge symmetry then the $SU(3)\times SU(2)\times U(1)$, but at some energy scale $M_H \gg 100$ GeV it's Higgsed down to the $SU(3) \times SU(2) \times U(1)$. The Higgs VEV involved in this process may have Yukawa couplings to some fermions, and that would give them $O(M_H)$ masses. In this scenario, the masses of the heavy fermions would be invariant under the un-broken $SU(3) \times SU(2) \times U(1)$ gauge symmetry group, so by the Theorem, the contributions of these heavy fermions to the $SU(3) \times SU(2) \times U(1)$ anomalies would cancel out,

$$\Delta_{\text{heavy}}\mathcal{A}^{abc} = 0 \quad \text{for} \quad a,b,c \in SU(3) \times SU(2) \times U(1). \tag{293}$$

   Note: for t^a, t^b, t^c belonging to the Higgsed down symmetry generators outside the Standard Model, the anomaly contribution from the heavy fermions might not cancel out. But that would be a problem for our attempt at the BSM model building, not for the Standard Model itself.

A similar argument applies to the abelian anomalies of the Standard Model's global symmetries such as lepton or baryon numbers. For any superheavy BSM fermions whose masses do not break either the SM gauge symmetries or the global symmetries in question, their contributions to the abelian anomalies in question would cancel out. For example, as long as the superheavy fermions' masses do not break baryon or lepton numbers, they would not affects the anomalous violation of these numbers by

$$\Delta B \;=\; \Delta L \;=\; 3 \times \mathrm{Index}[\mathcal{F}_{\mathrm{SU}(2)}]. \tag{294}$$

Note: this rule does not apply to the sterile neutrinos $N_{1,2,3}$ involved in the see-saw mechanism because their large Majorana masses break the lepton number symmetry.

This concludes my notes on the anomalies. If I have any time left in this class to discuss other anomaly-related issues, I shall use the blackboard or rather the document camera.