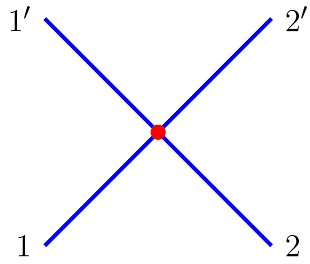
Calculating a One-Loop Amplitude

In these notes I explain the basic tools for calculating loop diagrams. For simplicity, let us stick to the $\lambda\Phi^4$ theory with a single real scalar and focus on the elastic 2-particle amplitude $\mathcal{M}(1+2\to 1'+2')$. At the tree level, only one diagram

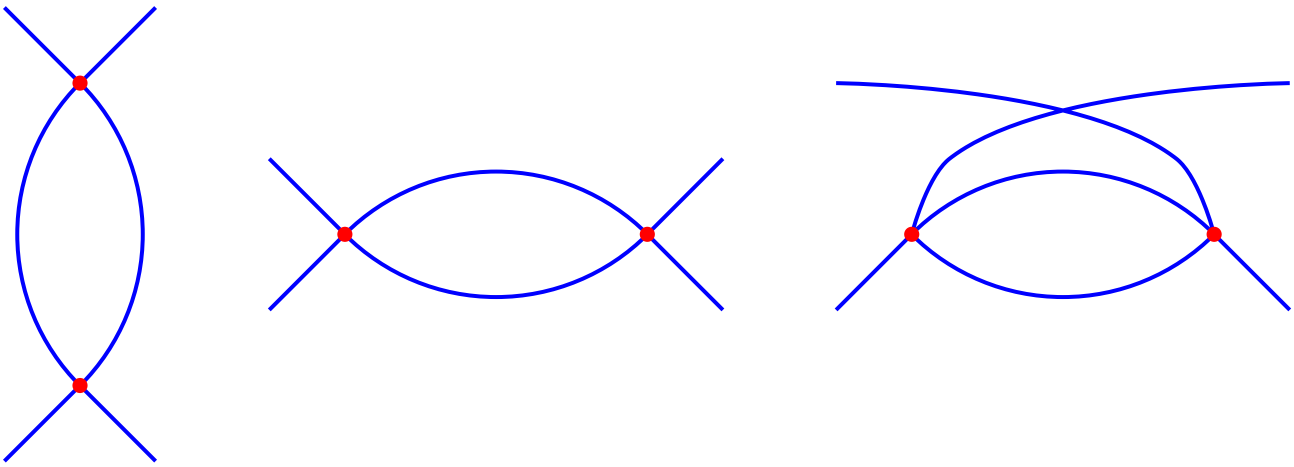


(1)

contributes to this scattering, hence

$$\mathcal{M}_{\text{tree}} \;=\; -\lambda \quad \forall s,t,u. \tag{2}$$

At the one-loop level, there are 3 diagrams — or rather, 3 connected diagrams without external leg bubbles, — namely



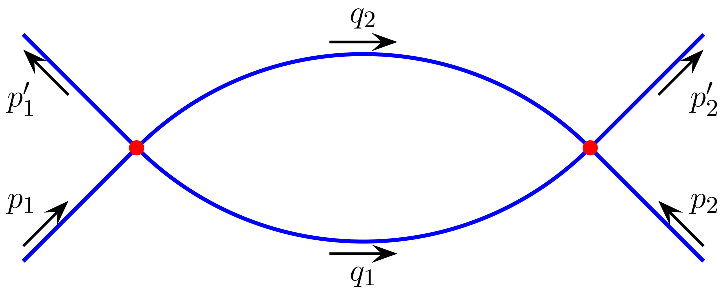
(3)

These 3 diagrams are related to each other by crossing symmetries, and each one diagram depends on only one Mandelstam's invariant — s, t, or u — of the external momenta, so altogether we have

$$\mathcal{M}_{1\,\text{loop}}(s,t,u) \;=\; \mathcal{F}(s) \;+\; \mathcal{F}(t) \;+\; \mathcal{F}(u) \tag{4}$$

for the same analytic function $\mathcal{F}$.

To calculate this function, let us focus on the t-channel diagram



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By momentum conservation in each vertex, we have

$$q_1 + q_2 = q_{\rm net} = p_1 - p'_1 = p'_2 - p_2 , \qquad q^2_{\rm net} = t. \tag{6}$$

Hence, evaluating the diagram (5), we have

$$i\mathcal{F}(t) = \frac{1}{2} \times (-i\lambda)^2 \times \int \frac{d^4q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \frac{i}{(q_2 = q_{\rm net} - q_1)^2 - m^2 + i\epsilon} \tag{7}$$

where $\frac{1}{2}$ is the combinatorial factor due to permutation symmetry of the 2 propagators. Note: at very large loop momenta, the integral in eq. (7) behaves as

$$\int \frac{d^4q_1}{(2\pi)^4} \frac{1}{(q_1^2)^2} , \tag{8}$$

which diverges for $q_1^\mu \to \infty$. Therefore, the momentum integral must be *regulated* to avoid this divergence, and I shall the regulation in detail later in these notes. For the moment, let us simply write down

$$\mathcal{F}(t) = \frac{-i\lambda^2}{2} \int_{\rm reg} \frac{d^4q_1}{(2\pi)^4} \frac{1}{q_1^2 - m^2 + i\epsilon} \times \frac{1}{(q_2 = q_{\rm net} - q_1)^2 - m^2 + i\epsilon} \tag{9}$$

where reg denotes that the integral is somehow regulated without going into the specifics.

Feynman's parameter trick

Our calculation of the integral (9) begins with the Feynman's parameter trick: Any product of 2 complex numbers — or rather, their inverses $1/A$ and $1/B$ can be written as the integral

$$\frac{1}{A}\times\frac{1}{B} \;=\; \int\limits_0^1 d\xi\,\frac{1}{[(1-\xi)A+\xi B]^2} \tag{10}$$

where the $(1-\xi)A+\xi B$ factor in the integral linearly interpolates between A and B. The identity (10) — and many similar identities involving more complicated products — can be easily verified by taking the integral on the RHS, so let me leave it for your homework (problem 1 of the current set#13). Applying the identity (10) to the product of the two propagators in the integrand of eq. (9), we have

$$\begin{aligned}\frac{1}{q_1^2-m^2+i\epsilon}\times\frac{1}{q_2^2-m^2+i\epsilon} \;&=\; \int\limits_0^1 \frac{d\xi}{[(1-\xi)(q_1^2-m^2+i\epsilon)+\xi(q_2^2-m^2+i\epsilon)]^2}\\ &=\; \int\limits_0^1 \frac{d\xi}{[(1-\xi)q_1^2+\xi q_2^2-m^2+i\epsilon]^2}\end{aligned} \tag{11}$$

and hence

$$\begin{aligned}\mathcal{F}(t) \;&=\; \frac{-i\lambda^2}{2}\int\limits_{\text{reg}}\frac{d^4q_1^2}{(2\pi)^4}\int\limits_0^1 \frac{d\xi}{[(1-\xi)q_1^2+\xi q_2^2-m^2+i\epsilon]^2}\\ &=\; \frac{-i\lambda^2}{2}\int\limits_0^1 d\xi\int\limits_{\text{reg}}\frac{d^4q_1^2}{(2\pi)^4}\,\frac{1}{[(1-\xi)q_1^2+\xi q_2^2-m^2+i\epsilon]^2}\,.\end{aligned} \tag{12}$$

In the denominator of the integrand here

$$\begin{aligned}(1-\xi)q_1^2 \;+\; \xi q_2^2 \;&=\; (1-\xi)q_1^2 \;+\; \xi(q_{\text{net}}-q_1)^2\\ &=\; (1-\xi)q_1^2 \;+\; \xi q_1^2 \;-\; 2\xi(q_1q_{\text{net}}) \;+\; \xi q_{\text{net}}^2\\ &=\; (q_1-\xi q_{\text{net}})^2 \;+\; (\xi-\xi^2)q_{\text{net}}^2,\end{aligned} \tag{13}$$

hence

$$(1-\xi)q_1^2 \;+\; \xi q_2^2 \;-\; m^2 \;+\; i\epsilon \;=\; (q_1-\xi q_{\rm net})^2 \;-\; \Delta(\xi) \;+\; i\epsilon \tag{14}$$

where

$$\Delta(\xi) \;\stackrel{\rm def}{=}\; m^2 \;-\; (\xi-\xi^2)q_{\rm net}^2 \;=\; m^2 \;-\; \xi(1-\xi)t, \tag{15}$$

and therefore

$$\mathcal{F}(t) \;=\; \frac{-i\lambda^2}{2}\int_0^1 d\xi \int_{\rm reg} \frac{d^4q_1^2}{(2\pi)^4}\,\frac{1}{[(q_1-\xi q_{\rm net})^2-\Delta(\xi)+i\epsilon]^2}\,. \tag{16}$$

In the inner integral here, ξ is fixed while we integrate over the q_1^μ, so we may shift the integration variable from the q_1^μ to

$$k^\mu \;=\; q_1^\mu \;-\; \xi q_{\rm net}^\mu\,. \tag{17}$$

Obviously, this variable shift has unit Jacobian, thus

$$\int\frac{d^4q_1^2}{(2\pi)^4}\,\frac{1}{[(q_1-\xi q_{\rm net})^2-\Delta(\xi)+i\epsilon]^2} \;=\; \int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-\Delta(\xi)+i\epsilon]^2}\,. \tag{18}$$

Or rather,

$$\int_{\rm reg}\frac{d^4q_1^2}{(2\pi)^4}\,\frac{1}{[(q_1-\xi q_{\rm net})^2-\Delta(\xi)+i\epsilon]^2} \;=\; \int_{\rm reg}\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-\Delta(\xi)+i\epsilon]^2}\,, \tag{19}$$

provided such variable shift does not screw up the regulator which makes the integral finite. But let us assume our regulator does allow such variable shift, then we may write the amplitude in question as

$$\mathcal{F}(t) \;=\; \frac{-i\lambda^2}{2}\int_0^1 d\xi \int_{\rm reg} \frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-\Delta(\xi)+i\epsilon]^2}\,. \tag{20}$$

Wick rotation

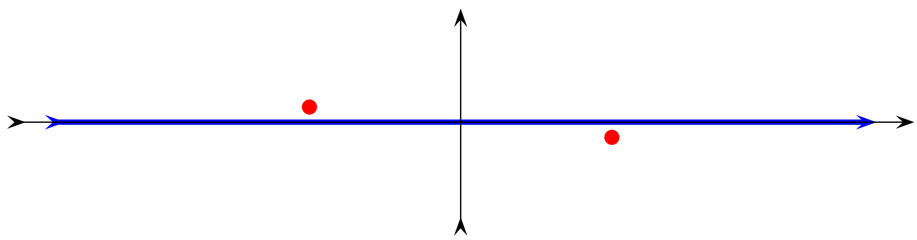
The loop momentum k^μ has four components, so let's integrate over the time component k^0 before integrating over the space components $\mathbf{k}$. Thus

$$\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-\Delta(\xi)+i\epsilon]^2}\ =\ \int\frac{d^3\mathbf{k}}{(2\pi)^3}\int\frac{dk^0}{2\pi}\,\frac{1}{[(k^0)^2-\mathbf{k}^2-\Delta(\xi)+i\epsilon]^2}\,. \tag{21}$$

In the k^0 complex plane, the integrand has 2 double poles at

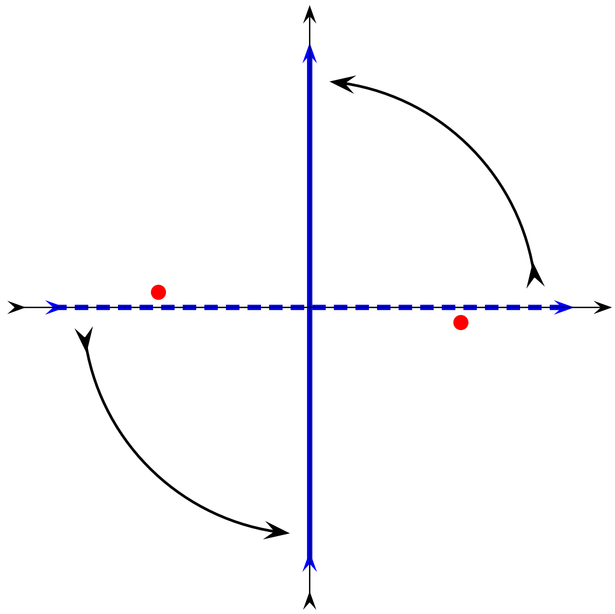
$$k^0\ =\ \pm\bigl(\sqrt{\mathbf{k}^2+\Delta(\xi)}-i\epsilon\bigr) \tag{22}$$

where (for the t channel) $\Delta>0\ \forall\xi$ because $t<0$. Graphically



(23)

where red dots denote the poles and the blue line along the real axis is the integration contour. Apart from the two poles, the integrand is regular everywhere else in the complex sphere, including the complex infinity. Indeed, for $k^0\to\infty$, the integrand decreases faster than $1/(k^0)^2$, so the directions in which the two ends of the contour approach the complex infinity does not affect the integral. This allows us to deform not only the middle of the integration contour but also the directions of its ends: as long as the deformation does not crosses the poles of the integrand, the integral would remain the same. In particular, we may rotate the integration contour 90° counterclockwise until it runs along the imaginary axis,



(24)

Note: the rotation must be in the counterclockwise direction because a clockwise rotation would make the contour cross the two poles — which is not allowed!

The rotation (24) of the k^0 integration contour is called Wick rotation after Gian Carlo Wick, an Italian physicist. It amounts to

$$k^0 \ = \ ik^4 \tag{25}$$

for a real k^4 running from $-\infty$ to $+\infty$. In terms of the k^4,

$$k^2 \ = \ (k^0)^2 \ - \ \mathbf{k}^2 \ = \ -(k^4)^2 \ - \ \mathbf{k}^2 \ = \ -\sum_{i=1}^{4}(k^i)^2. \tag{26}$$

In other words, we may combine the real 3-space vector $\mathbf{k} = (k^1, k^2, k^3)$ and the real k^4 into a real 4-vector $k_E \ = \ (k^1, k^2, k^3, k^4)$ in the Euclidean 4D momentum space with Euclidean positive-definite metric

$$k_E^2 \ = \ (k^1)^2 \ + \ (k^2)^2 \ + \ (k^3)^2 \ + \ (k^4)^2. \tag{27}$$

This Euclidean space has a 4D rotational symmetry $SO(4)$, which is related by analytic continuation (from real k^4 to real $k^0 = ik^4$) to the Lorentz $SO^+(3,1)$ symmetry of the Minkowski space. Also, under this analytic continuation

$$k_{\text{Mink}}^2 \;=\; g_{\mu\nu}k^\mu k^\nu \;=\; -k_{\text{Eucl}}^2 \;=\; -\sum_{i=1}^{4}(k^i)^2 \tag{28}$$

and

$$d^4k_{\text{Mink}} \;=\; d^3\mathbf{k}\times dk^0 \;=\; d^3\mathbf{k}\times idk^4 \;=\; id^4k_{\text{Eucl}}\,. \tag{29}$$

Thus, analytically continuing the integral (20) over the Minkowski loop momentum k to the integral over the Euclidean loop momentum k_E, we get

$$\begin{aligned}\mathcal{F}(t) \;&=\; \frac{-i\lambda^2}{2}\int\limits_0^1 d\xi\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-\Delta(\xi)+i\epsilon]^2}\\ &\to\; \frac{-i\lambda^2}{2}\int\limits_0^1 d\xi\int\limits_{\text{reg}}\frac{i\,d^4k_E}{(2\pi)^4}\,\frac{1}{[-k_E^2-\Delta(\xi)+i\epsilon]^2}\\ &=\; +\frac{\lambda^2}{2}\int\limits_0^1 d\xi\int\limits_{\text{reg}}\frac{d^4k_E}{(2\pi)^4}\,\frac{1}{[k_E^2+\Delta(\xi)-i\epsilon]^2}\,.\end{aligned} \tag{30}$$

Moreover, in the Euclidean momentum space, the integrand does not have any poles at real k_E, so we do not need the $-i\epsilon$ term in the denominator to regulate them, thus

$$\mathcal{F}(t) \;=\; +\frac{\lambda^2}{2}\int\limits_0^1 d\xi\int\limits_{\text{reg}}\frac{d^4k_E}{(2\pi)^4}\,\frac{1}{[k_E^2+\Delta(\xi)]^2}\,. \tag{31}$$

Next, let's make use of the $SO(4)$ symmetry of the Euclidean momentum space. The integrand of eq. (31) is $SO(4)$ invariant, so let's use the 4D analog of spherical coordinates in the Euclidean momentum space. In such coordinates (k_e,ψ,θ,ϕ),

$$d^4k_E \;=\; k_e^3\,dk_e\times d^3\Omega(\psi,\theta,\phi) \tag{32}$$

where the 3-volume of the unit 3-sphere in 4D is

$$\int d^3\Omega(\psi,\theta,\phi) \;=\; 2\pi^2, \tag{33}$$

thus

$$\frac{d^4k_E}{(2\pi)^4} \;=\; \frac{2\pi^2}{(2\pi)^4}\times k_e^3\,dk_e \tag{34}$$

and hence

$$\mathcal{F}(t) \;=\; +\frac{\lambda^2}{16\pi^2}\int\limits_0^1 d\xi \;\int\limits_0^\infty{}_{\text{reg}} \frac{k_e^3\,dk_e}{[k_E^2+\Delta(\xi)]^2}\,. \tag{35}$$

Divergence and Cutoff

Note that the remaining momentum integral (35) must be regulated, because without a regulator it diverges logarithmically as $k_e\to\infty$. By *logarithmic* divergence, I mean that for a very large $\Lambda^2\gg\Delta$,

$$\int\limits_0^\Lambda \frac{k_e^3\,dk_e}{[k_e^2+\Delta]^2} \;=\; \log\Lambda \;+\; \text{finite}, \tag{36}$$

so for $\Lambda\to\infty$ the integral diverges as $\log\Lambda$. Indeed,

$$\begin{aligned}
\int\limits_0^\Lambda \frac{k_e^3\,dk_e}{[k_e^2+\Delta]^2} &= \int\limits_0^{\Lambda^2} \frac{k_e^2\times\frac12 d(k_e^2)}{[k_e^2+\Delta]^2}\\
&= \frac12\int\limits_\Delta^{\Lambda^2+\Delta}\frac{(x-\Delta)dx}{x^2} \qquad \langle\!\langle\,\text{where } x=k_e^2+\Delta\,\rangle\!\rangle\\
&= \frac12\left(\log x\;+\;\frac{\Delta}{x}\right)\Bigg|_\Delta^{\Lambda^2+\Delta}\\
&= \frac12\left(\log\frac{\Lambda^2+\Delta}{\Delta}\;+\;\frac{\Delta}{\Lambda^2+\Delta}\;-\;\frac{\Delta}{\Delta}\right)\\
&= \frac12\log\frac{\Lambda^2}{\Delta}\;-\;\frac12\;+\;O\!\left(\frac{\Delta}{\Lambda^2}\right)\\
&\xrightarrow[\Lambda\to\infty]{}\;\log\Lambda\;+\;\text{finite}\,.
\end{aligned} \tag{37}$$

The divergences like this are called the *ultraviolet divergences* because they stem from the 'ultraviolet limit' of infinitely large loop momenta. Such UV divergences are all over the place in quantum field theory, and the physicists have learned how to regulate them from the beginning of modern QED in late 1940's, with more ways being developed over the following decades. Basically, each regulator somehow suppresses or cancels the effects of very large loop momenta $k_e \gtrsim \Lambda$ for some *UV cutoff scale* Λ.

At first blush, such regulation makes the perturbative expansion of various scattering amplitudes dependent on the cutoff scale, for example

$$\mathcal{M}(s,t,u) \;=\; -\lambda \;+\; \lambda^2 \times F_{1\,\mathrm{loop}}(s,t,u;\Lambda) \;+\; \lambda^3 \times F_{2\,\mathrm{loops}}(s,t,u;\Lambda) \;+\; \cdots. \tag{38}$$

However, this cutoff-dependence can be canceled by a suitable renormalization of the λ coupling. Basically, we distinguish between the *bare coupling* λ_b in the theory' Lagrangian and hence the Feynman rules, and the physical coupling λ_{ph} defined in terms of a scattering amplitude, for example

$$\lambda_{\mathrm{ph}} \;\stackrel{\mathrm{def}}{=}\; -\mathcal{M}(\text{elastic, at threshold}). \tag{39}$$

The λ in the series (38) is the bare coupling λ_b, and when we re-express the same amplitude $\mathcal{M}(s,t,u)$ as a power series in the physical coupling λ_{ph}, the result turns out to be 'miraculously' independent on the cutoff scale Λ. We shall see how this works at the one-loop level later in these notes.

The techniques for eliminating the Λ dependence from the relations between the physical amplitudes were developed back in 1950. But for a a couple of decades these techniques seemed to be mere 'sweeping infinities under the carpet' without a clear understanding why they work. The understanding came from the concept of *effective low-energy / long-distance quantum field theory* which was developed in late 1960s through early 1970s by Leo Kadanoff and Kenneth Wilson in condensed matter and by Steven Weinberg *et al* in relativistic QFT.

Kadanoff and Wilson were focused on critical phenomena in condensed matter. They noticed that near a critical point, the long-distance behavior does not depend on the short-distance structure of the condensed matter in question. Instead, all the short-distance effects can be summarized in a **few** parameters of an effective QFT governing the long-distance physics.

For example, in a magnetic material near a Curie point, the magnetization (along a preferred axis) is governed by a 3D scalar field theory with Hamiltonian

$$H \;=\; \int d^3\mathbf{x} \left(\frac{A(T)}{2} (\nabla\Phi)^2 \;+\; \frac{B(T)}{2}\Phi^2 \;+\; \frac{C(T)}{24}\Phi^4 \right), \tag{40}$$

and all the short-distance detailed of the material in question affect this effective long-distance theory only via the temperature-dependent parameters $A(T)$, $B(T)$, and $C(T)$.

At the same time, many high-energy physicists were investigating the consequences of the spontaneously broken chiral symmetry of the strong interaction. At first, this was done in the non-QFT language of the scattering amplitudes — especially the amplitudes involving 'soft' pions of energies $E \lesssim 400$ MeV, — and then led to the development of the *current algebra*, *i.e.* the algebra of the current operators of the chiral symmetry. But eventually Steven Weinberg *et al* showed that the simplest way to implement this current algebra is in terms of the effective low-energy QFT for the soft pions, for example a linear or a non-linear sigma-model. Such a sigma model summarizes the effects of the spontaneous chiral symmetry breaking on the soft pions and has sigma model has only a couple of parameters — the pion's decay constant f_π and the pion's mass m_π, — and it does not depend on any high-energy features of the ultimate strong-interaction theory except via these parameters.

As a simple example of a long-distance effective theory in condensed matter, consider the Debye theory of the heat capacity of a solid. A single harmonic oscillator has heat capacity

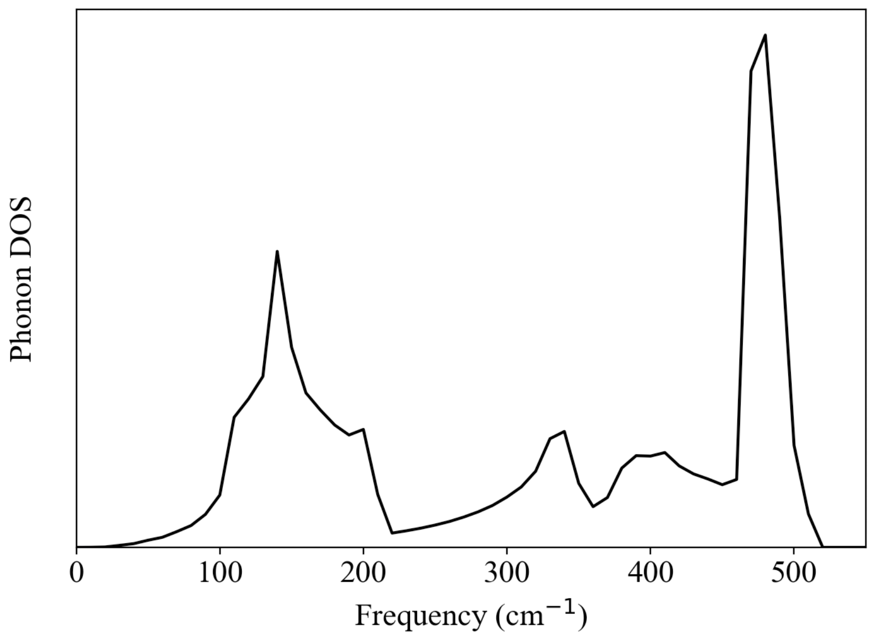
$$C(T) \;=\; \left(\frac{(\omega/2T)}{\sinh(\omega/2T)} \right)^2 \tag{41}$$

(in $\hbar = 1$, $k_{\text{Boltzmann}} = 1$ units), so approximating the solid crystal's degrees of freedom as a bunch of independent vibrational modes, we have

$$C(T) \;=\; \int\limits_0^{\omega_{\max}} d\omega\, \frac{dN}{d\omega} \times \left(\frac{(\omega/2T)}{\sinh(\omega/2T)} \right)^2 \tag{42}$$

where $dN/d\omega$ is the *phonon density of states.* In real-life crystals, this density of states can be a rather complicated function of the frequency, depending on the phonon's dispersion relation

$\omega(\mathbf{k})$; for example, the silicon crystal has



However, the low-frequency phonons always have

$$\omega \;=\; c_s|\mathbf{k}| \tag{43}$$

where c_s is the speed of sound in the crystal, while

$$dN \;=\; 3_{\text{polarizations}} \times \left(\frac{L}{2\pi}\right)^3 d^3\mathbf{k} \;\longrightarrow\; \frac{3L^3}{2\pi^2}\,|\mathbf{k}|^2\,d|\mathbf{k}|, \tag{44}$$

hence

$$\text{for low frequencies,}\quad \frac{dN}{d\omega} \;=\; \frac{3L^3}{2\pi^2 c_s^3} \times \omega^2. \tag{45}$$

At higher frequencies, the real-life crystals have a much messier phonon densities of states, but the Debye theory cuts off all this high-frequency mess. Instead, it extends the low-frequency formula (45) up to some maximal frequency Ω_D (now called the Debye frequency), and then abruptly cuts off to zero, thus

$$\left(\frac{dN}{d\Omega}\right)_{\text{Debye}} = \begin{cases} (3L^3/2\pi^2c_s^3)\times\omega^2 & \text{for } \omega<\Omega_D, \\ 0 & \text{for } \omega>\Omega_D. \end{cases} \tag{46}$$

Thus, in the Debye theory

$$C(T) = \frac{3L^3}{2\pi^2c_s^3}\int\limits_0^{\Omega_D}d\omega\,\omega^2\times\left(\frac{(\omega/2T)}{\sinh(\omega/2T)}\right)^2, \tag{47}$$

which depends on the short-distance structure of the crystal only via its speed of sound c_s and the Debye frequency Ω_D.

Back to the $\lambda\Phi^4$ theory

In the momentum terms, the Debye theory corresponds to the hard-edge cutoff of the phonon's momenta to

$$|\mathbf{k}| \le \frac{\omega_D}{c_s}. \tag{48}$$

The relativistic QFT analogy of the Debye cutoff is a similar hard-edge cutoff of the Euclidean loop momentum k_E,

$$|k_E| \le \Lambda. \tag{49}$$

In my my next set of notes I shall introduce several other cutoff types commonly used in relativistic QFTs, but for the moment let me calculate the one-loop scattering amplitude in the $\lambda\Phi^4$ theory using the Debye-like hard-edge cutoff (49).

Applying this cutoff as a regulator of the Euclidean momentum integral (35), we get

$$\begin{aligned}\overset{\text{reg}}{\int\limits_0^\infty}\frac{k_e^3\,dk_e}{[k_E^2+\Delta]^2} &= \int\limits_0^\Lambda\frac{k_e^3\,dk_e}{[k_E^2+\Delta(\xi)]^2} \\ &= \frac{1}{2}\left(\log\frac{\Lambda^2+\Delta}{\Delta}+\frac{\Delta}{\Lambda^2+\Delta}-\frac{\Delta}{\Delta}\right) \\ &= \frac{1}{2}\log\frac{\Lambda^2}{\Delta}-\frac{1}{2}+O\left(\frac{\Delta}{\Lambda^2}\right)\end{aligned} \tag{50}$$

(*cf.* eq. (37)). Moreover, we assume that the cutoff scale Λ is much larger than the particle's mass m or any of the incoming or outgoing particles' energies. Consequently,

$$\Delta(\xi) \;=\; m^2 \;-\; t\times\xi(1-\xi) \;\ll\; \Lambda^2 \quad (\forall\xi), \tag{51}$$

which allows us to neglect all positive powers of Δ/Λ^2, thus

$$\overset{\text{reg}}{\int_0^\infty} \frac{k_e^3\,dk_e}{[k_E^2+\Delta(\xi)]^2} \;\longrightarrow\; \frac{1}{2}\log\frac{\Lambda^2}{\Delta(\xi)} \;-\; \frac{1}{2} \tag{52}$$

and therefore

$$\mathcal{F}(t) \;=\; \frac{\lambda^2}{32\pi^2}\int_0^1 d\xi\left(\log\frac{\Lambda^2}{\Delta(\xi)} \;-\; 1\right). \tag{53}$$

To evaluate the remaining integral over the Feynman parameter ξ, we use

$$\log\frac{\Lambda^2}{\Delta(\xi)} \;=\; \log\frac{\Lambda^2}{m^2} \;-\; \log\frac{\Delta(\xi)}{m^2} \;=\; \log\frac{\Lambda^2}{m^2} \;-\; \log\left(1\;-\;\frac{t}{m^2}\times\xi(1-\xi)\right). \tag{54}$$

Consequently,

$$\int_0^1 d\xi\left(\log\frac{\Lambda^2}{\Delta(\xi)} \;-\; 1\right) \;=\; \log\frac{\Lambda^2}{m^2} \;-\; J(t/m^2) \;-\; 1 \tag{55}$$

where

$$J(t/m^2) \;\stackrel{\text{def}}{=}\; \int_0^1 d\xi\,\log\left(1\;-\;\frac{t}{m^2}\times\xi(1-\xi)\right), \tag{56}$$

and therefore

$$\mathcal{F}(t) \;=\; \frac{\lambda^2}{32\pi^2}\left(\log\frac{\Lambda^2}{m^2} \;-\; 1 \;-\; J(t/m^2)\right). \tag{57}$$

Finally, the net one-loop contribution to the elastic scattering amplitude comes from 3 diagrams (3) related by the crossing symmetries, thus

$$\begin{aligned} \mathcal{M}_{1\,\text{loop}}(s,t,u) \;&=\; \mathcal{F}(t) \;+\; \mathcal{F}(u) \;+\; \mathcal{F}(s) \\ &=\; \frac{\lambda^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} \;-\; 3 \;-\; J(t/m^2) \;-\; J(u/m^2) \;-\; J(s/m^2)\right). \end{aligned} \tag{58}$$

Physical and Bare couplings

Altogether, the elastic scattering amplitude is

$$\begin{aligned}
\mathcal{M}(s,t,u) &= \mathcal{M}_{\text{tree}} + \mathcal{M}_{1\,\text{loop}}(s,t,u) + \mathcal{M}_{\text{higher loops}}(s,t,u) \\
&= -\lambda_b + \frac{\lambda_b^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 3 - J(t/m^2) - J(u/m^2) - J(s/m^2)\right) + O(\lambda_b^3)
\end{aligned} \tag{59}$$

where λ_b is the bare coupling of the Feynman rules. But in order to compare this amplitude to the experiment, we need to re-express it in terms of some directly measurable physical coupling λ_{ph} rather than the bare coupling. There are many slightly different ways to define this physical coupling, so let me pick a particularly simple choice for this class: $-\lambda_{\text{ph}}$ is the elastic scattering amplitude at the threshold, $s = 4m^2$ while $t = u = 0$. Thus, in light of out one-loop result (59),

$$\begin{aligned}
\lambda_{\text{ph}} &= -\mathcal{M}(s = 4m^2, t = u = 0) \\
&= +\lambda_b - \frac{\lambda_b^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 3 - 2J(0) - J(4)\right) + O(\lambda_b^3) \\
&= +\lambda_b - \frac{\lambda_b^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 2\right) + O(\lambda_b^3),
\end{aligned} \tag{60}$$

where the last equality follows from $J(0) = 0$ and $J(4) = -1$, *cf.* eq. (56). Reversing this series, we have

$$\begin{aligned}
\lambda_b &= \lambda_{\text{ph}} + \frac{1}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 2\right) \times \left(\lambda_b = \lambda_{\text{ph}} + O(\lambda_{\text{ph}}^2)\right)^2 + O\big((\lambda_b = \lambda_{\text{ph}} + O(\lambda_{\text{ph}}^2))^3\big) \\
&= \lambda_{\text{ph}} + \frac{1}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 2\right) \times \lambda_{\text{ph}}^2 + O(\lambda_{\text{ph}}^3),
\end{aligned} \tag{61}$$

and when we plug this bare coupling into the scattering amplitude (59) for general (s,t,u), we end up with

$$
\begin{aligned}
\mathcal{M}(s,t,u) \;&=\; -\left(\lambda_b \,=\, \lambda_{\rm ph} \,+\, \frac{\lambda_{\rm ph}^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} \,-\, 2\right) \,+\, O(\lambda_{\rm ph}^3)\right)\\
&\qquad +\; \frac{1}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} \,-\, 3 \,-\, J(t/m^2) \,-\, J(u/m^2) \,-\, J(s/m^2)\right) \times\\
&\qquad\qquad\quad \times \left(\lambda_b = \lambda_{\rm ph} + O(\lambda_{\rm ph}^2)\right)^2\\
&\qquad +\; O\big((\lambda_b = \lambda_{\rm ph} + O(\lambda_{\rm ph}^2))^3\big)\\
&=\; -\lambda_{\rm ph} \;+\; \frac{\lambda_{\rm ph}^2}{32\pi^2} \times \begin{bmatrix} -3\log\dfrac{\Lambda^2}{m^2} \,+\, 2 & \\ +3\log\dfrac{\Lambda^2}{m^2} \,-\, 3 \,-\, J(t/m^2) \,-\, J(u/m^2) \,-\, J(s/m^2) \end{bmatrix}\\
&\qquad +\; O(\lambda_{\rm ph}^3)\\
&=\; -\lambda_{\rm ph} \;+\; \frac{\lambda_{\rm ph}^2}{32\pi^2} \times \left[-1 \,-\, J(t/m^2) \,-\, J(u/m^2) \,-\, J(s/m^2)\right] \;+\; O(\lambda_{\rm ph}^3).
\end{aligned}
\tag{62}
$$

Note how the cutoff scale Λ cancels out from this formula!

At higher loop orders we have similar behavior: The Feynman rules yield scattering amplitudes as power series in the bare coupling λ_b with cutoff-dependent coefficients, for example

$$
\mathcal{M}_{\rm elastic}(s,t,u) \;=\; -\lambda_b + \lambda_b^2\times A(s,t,u;\Lambda) + \lambda_b^3\times B(s,t,u;\Lambda) + \lambda_b^4\times C(s,t,u;\Lambda) + \cdots. \tag{63}
$$

Consequently, the physical coupling $\lambda_{\rm ph}$ obtains as a power series in the bare coupling,

$$
\lambda_{\rm ph} \;=\; \lambda_b \;-\; \lambda_b^2\times A_0(\Lambda) \;-\; \lambda_b^3\times B_0(\Lambda) \;-\; \lambda_b^4\times C_0(\Lambda) \;+\; \cdots \tag{64}
$$

where $A_0(\Lambda)$ is the value of the $A(s,t,u;\Lambda)$ at the threshold, and ditto for the B_0, C_0, *etc.* Reversing this expansion, we get λ_b as a power series in the physical coupling,

$$
\lambda_b \;=\; \lambda_{\rm ph} \;+\; \lambda_{\rm ph}^2\times A_0 \;+\; \lambda_{\rm ph}^3\times(2A_0^2+B_0) \;+\; \lambda_{\rm ph}^4\times(5A_0^3+5A_0B_0+C_0) \;+\; \cdots \tag{65}
$$

which allows us to re-express the scattering amplitudes such as (63) as power series in the physical coupling $\lambda_{\rm ph}$:

$$\begin{aligned}\mathcal{M}(s,t,u) \;=\; & -\lambda_{\rm ph} \;+\; \lambda_{\rm ph}^2 \times \bigl[A(s,t,u) - A_0\bigr] \\ & +\; \lambda_{\rm ph}^3 \times \bigl[\bigl(B(s,t,u) - B_0\bigr) + 2A_0\bigl(A(s,t,u) - A_0)\bigr)\bigr] \\ & +\; \lambda_{\rm ph}^4 \times \left[\begin{array}{c}\bigl(C(s,t,u) - C_0\bigr) \;+\; 3A_0\bigl(B(s,t,u) - B_0\bigr) \\ +\,(2B_0 + 5A_0^2) \times \bigl(A(s,t,u) - A_0\bigr)\end{array}\right] \\ & +\; \cdots.\end{aligned} \tag{66}$$

In eq. (62) we saw how the cutoff dependence cancels out from the one-loop coefficient

$$A(s,t,u;\Lambda) \;-\; A_0(\Lambda) \;=\; \frac{1}{32\pi^2} \times \bigl[-1 \;-\; J(t/m^2) \;-\; J(u/m^2) \;-\; J(s/m^2)\bigr] \tag{67}$$

in this series. It turns out that all the higher-loop coefficients in the series (66)

$$B(s,t,u;\Lambda) \;-\; B_0(\Lambda) \;+\; 2A_0(\Lambda) \times \bigl(A(s,t,u;\Lambda) - A_0(\Lambda)\bigr), \tag{68}$$

$$\begin{aligned}C(s,t,u;\Lambda) \;-\; C_0(\Lambda) \;&+\; 3A_0(\Lambda) \times \bigl(B(s,t,u;\Lambda) - B_0(\Lambda)\bigr) \\ &+\; \bigl(2B_0(\Lambda + 5A_0^2(\Lambda)\bigr) \times \bigl(A(s,t,u;\Lambda) - A_0(\Lambda)\bigr),\end{aligned} \tag{69}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

are also cutoff-independent. Thus, the low-energy scattering amplitudes not only remain finite in the $\Lambda \to \infty$ limit, but are completely independent of the very-high-energy aspects of the QFT such as its UV cutoff scale. This is an example of general rule of effective long-distance field theories: The long-distance physics depends on the UV aspects of the theory — such as its bare coupling λ_b, the bare mass m_b, the UV cutoff scale Λ, or the specific manner of the cutoff — only through a few parameters: In our case, the physical coupling $\lambda_{\rm ph}$ and the physical particle mass $m_{\rm ph}$ — both of which are directly measurable at long distances.

I shall explain the *mass renormalization* — the difference between the bare m_b^2 parameter in the QFT's Lagrangian and the physical mass^2 $m_{\rm ph}^2$ of the scalar particle — next week. And later in class we shall learn how to reorganize the perturbation theory in order to expand the scattering amplitudes in power of the physical coupling $\lambda_{\rm ph}$ without going through intermediate series like (63) in terms of the bare coupling.