1. Strictly speaking, the Hamiltonian of a harmonic oscillator is not $\hbar\omega\hat{a}^\dagger\hat{a}$ but rather $\hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$. A quantum field comprises an infinite family of such oscillators, whose zero-point energies total to an infinite vacuum energy

$$E_{\text{vac}} = \sum_\alpha \tfrac{1}{2}\hbar\omega_\alpha\,. \tag{1}$$

For a free field in empty infinite space, this vacuum energy is just a constant and has no physical consequences (except for the quantum gravity theory, whatever that might be). However, when a quantum field 'lives' in a non-trivial background of other fields, the background dependence of the zero-point energy (1) is rather important as it constitutes a non-trivial quantum correction to background fields' effective Hamiltonian. Likewise, when a quantum field is confined to a small cavity with reflective walls, the resonant frequencies $\omega_\alpha$ depend on the cavity's geometry and so the total zero-point energy also becomes geometry dependent. The geometry-dependent part of the vacuum energy — or rather vacuum energy density — is finite and leads to finite, experimentally detectable consequences such as negative pressure on the cavity walls.

The Casimir effect is the finite, experimentally measurable difference between the infinite vacuum energy density of quantum fields — *e.g.*, electromagnetic radiation — in a resonant cavity and the similarly infinite vacuum energy density in the empty space. To calculate this finite difference, one first regularizes both infinities, then evaluates the difference between the two regularized sums over the resonant modes $\alpha$, and only then removes the regularization. The regularization should suppress the contribution of the high-frequency (ultraviolet) modes without affecting those of the geometry-dependent low-frequency modes. Moreover, the distinction between between the low-frequency and the high-frequency modes should not depend on the geometric properties of the mode (*e.g.*, mode number) but only on its frequency; the 'cutoff' frequency (beyond which the ultraviolet modes' contributions are suppressed) should be a free parameter that can be made as high as one wishes; in the limit of the infinite cutoff frequency, the regularization goes away. A good example of such a regularization is

$$E(\tau) \;=\sum_{\text{all}\,\alpha} \tfrac{1}{2}\hbar\omega_\alpha\, e^{-\tau\omega_\alpha} \;\xrightarrow[\tau\to+0]{}\; E_{\text{vac}} \tag{2}$$

where the $1/\tau$ serves as the ultraviolet frequency cutoff: Modes with frequencies $\omega_\alpha \ll \tau^{-1}$ contribute $\frac{1}{2}\hbar\omega_\alpha$, as they should, while the contributions of the utraviolet modes are suppressed by a factor $e^{-\tau\omega_\alpha} \ll 1$ for $\omega_\alpha > \tau^{-1}$. Using such a regularization, one calculates the Casimir energy density according to

$$\frac{E_{\text{Casimir}}}{\text{Volume}} \;=\; \lim_{\tau\to+0}\left[\frac{E(\tau)}{\text{Volume}}\bigg|_{\text{cavity}} \;-\; \frac{E(\tau)}{\text{Volume}}\bigg|_{\substack{\text{empty}\\ \text{space}}}\right]. \tag{3}$$

The purpose of this exercise is to actually compute the Casimir effect for the EM fields in a cavity of a particularly simple geometry: Two parallel reflecting walls (perfectly conducting plates) a short distance $b$ apart from each other. The walls themselves are large; for simplicity, we take them to be squares of large area $L^2$ and impose periodic ('Pacman') boundary conditions in the $x$ and $y$ directions.

(a) Write down the boundary conditions for the electromagnetic fields, solve the eigenvalue problem and show that the frequencies of the free oscillations are $\omega = c\sqrt{k_x^2+k_y^2+k_z^2}$ where $k_z = \pi n/b$ ($n = 0, 1, 2, \ldots$) and that there are two polarization modes for $n > 0$ but only one for $n = 0$. The $k_x$ and the $k_y$ run almost continuously from $-\infty$ to $+\infty$, with the combined spectral density $(L/2\pi)^2 dk_x dk_y$.

(b) Show that for this cavity,

$$\frac{E(\tau)}{\text{Volume}} \;=\; \frac{\hbar c}{(2\pi)^3}\int d^3\mathbf{k}\,|\mathbf{k}|e^{-c\tau|\mathbf{k}|} \sum_{n=-\infty}^{+\infty} \delta\Big(n \,-\, \frac{b\,k_z}{\pi}\Big). \tag{4}$$

Note the range of integration — the entire $\mathbf{k}$ space — and the summation over both positive and negative $n$; on the other hand, there is no sum over polarizations.

(c) Now we need a mathematical lemma: For real $x$,

$$\sum_{m=-\infty}^{+\infty} e^{2\pi imx} = \sum_{n=-\infty}^{+\infty} \delta(n-x). \tag{5}$$

To prove the lemma, first calculate $\sum_m e^{2\pi imx}e^{-\epsilon|m|}$ (this is easy), then take the limit $\epsilon \to +0$.

Using this lemma, we can rewrite eq. (4) as

$$\frac{E(\tau)}{\text{Volume}} = \frac{\hbar c}{(2\pi)^3} \sum_{m=-\infty}^{+\infty} \int d^3\mathbf{k}\, |\mathbf{k}| e^{-c\tau|\mathbf{k}|}\, e^{2imbk_z}. \tag{6}$$

(d) Show that the $m = 0$ term in this sum is precisely the (regularized) vacuum energy density of the empty space (*i.e.*, in the absence of any walls and associated boundary conditions). Consequently,

$$\frac{E_{\text{Casimir}}}{\text{Volume}} = \frac{\hbar c}{(2\pi)^3} \sum_{m\neq 0} \lim_{\tau\to+0} \int d^3\mathbf{k}\, |\mathbf{k}| e^{-c\tau|\mathbf{k}|}\, e^{2imbk_z}. \tag{7}$$

(e) At this point, a straightforward calculation is in order: Evaluate the integral in eq. (7), take the $\tau \to +0$ limit, sum over $m \neq 0$ and thus obtain the actual Casimir energy density of the cavity.

(f) Finally, calculate the negative pressure on the cavity's walls as a function of the distance $b$ between them and explain why it was important to subtract the energy densities rather than just energies in eq. (3). Hint: Think of the open space outside the cavity.