Relativistic Causality

Relativistic Causality forbids any matter or any signal to travel outside of the future light cone of its origin. As I have explained last lecture, for a quantum field theory this means that any two measurable local operators located at points separated by a spacelike interval must commute with each other,

$$\text{for } (x-y)^2 < 0 \text{ and any } \hat{\mathcal{O}}_1(x) \text{ and any } \hat{\mathcal{O}}_2(y): \quad [\hat{\mathcal{O}}_1(x), \hat{\mathcal{O}}_2(y)] \;=\; 0. \tag{1}$$

In these notes I shall first explain what I mean by the measurable local operators, then prove the relativistic causality for the free scalar field theory, and eventually explain how it works for the non-free fields.

Let's start with the local operators. In a QFT, such operators are constructed by taking products (or other functions) of the fields and their derivatives, all taken at the same point x. For example,

$$\hat{\Phi}(x), \quad \hat{\Phi}^2(x), \quad V(\hat{\Phi}(x)), \quad \partial_\mu\hat{\Phi}(x), \quad \partial_\mu\partial_\nu\hat{\Phi}(x), \quad f(\hat{\Phi}(x))\partial_\mu\hat{\Phi}(x)\partial_\nu\hat{\Phi}(x), \quad \ldots; \tag{2}$$

in theories with multiple fields you may also have combinations like

$$A^\mu(x)\Phi^\dagger(x)\partial_\mu\hat{\Phi}(x), \quad \ldots. \tag{3}$$

But despite the infinite variety of such local operators, it is easy to make all any such operator at x commute with any operator at y for a spacelike $x-y$: All we need is quantum fields themselves commuting with each other at spacelike separation:

$$\begin{gathered}\text{for any } x,y \text{ such that } (x-y)^2 <, 0: \\ [\hat{\Phi}(x), \hat{\Phi}_2(y)] \;=\; 0, \quad [\hat{\Phi}(x), \hat{A}^\mu(y)] \;=\; 0, \quad [\hat{A}^\mu(x), \hat{A}^\nu(y)] \;=\; 0, \quad \ldots.\end{gathered} \tag{4}$$

Indeed, if the fields themselves at x and at y commute with each other then all their derivatives also commute. For example, if $[\hat{\Phi}(x), \hat{\Phi}(y)] = 0$ — and not just for some fixed x and y but also for some neighborhoods of x and y, — then

$$[\partial_\mu\hat{\Phi}(x),\partial_\nu\partial_\lambda\hat{\Phi}(y)] \;=\; \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y^\lambda}[\hat{\Phi}(x),\hat{\Phi}(y)] \;=\; \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y^\lambda}0 \;=\; 0. \tag{5}$$

And now that we have all fields and their derivatives at x commute with all fields and their derivatives at y, the Leibniz rule tells us that all products of the fields and derivatives at x commute with all products of fields and derivatives at y, for example

$$[\partial_\mu\hat{\Phi}(x),\hat{\Phi}(y)\partial_\lambda\partial_\nu\hat{\Phi}(y)] \;=\; [\partial_\mu\hat{\Phi}(x),\hat{\Phi}(y)]\partial_\lambda\partial_\nu\hat{\Phi}(y) + \hat{\Phi}(y)[\partial_\mu\hat{\Phi}(x),\partial_\lambda\partial_\nu\hat{\Phi}(y)] \;=\; 0+0 \;=\; 0. \tag{6}$$

The bottom line is: If all the quantum fields commute at spacelike separations as in eq. (4), then all the local operators such as (2) or (3) also commute at spacelike separations, — which is precisely what we need for the relativistic causality.

A few words about measurable and un-measurable fields and operators. The bosonic quantum fields — such as the scalar field $\hat{\Phi}(x)$ or the electromagnetic fields $\hat{F}^{\mu\nu}(x)$ — can have classical-like expectation values $\left\langle\hat{\Phi}(x)\right\rangle$, $\left\langle\hat{F}^{\mu\nu}\right\rangle(x)$, *etc.*, which we can measure experimentally, so these fields are measurable. But the fermionic fields — such as the the electron fields $\hat{\Psi}_\alpha(x)$ and $\hat{\Psi}^\dagger_\alpha(x)$ we shall study later in class — are not measurable by themselves. Only the bilinear combinations of such fields — such as the current $\hat{J}^\mu(x)=\hat{\overline{\Psi}}(x)\gamma^\mu\hat{\Psi}(x)$ — can be measured experimentally. In general, local *measurable* operators are products of *even* numbers of fermionic fields and their derivatives — as well as any number of bosonic fields and their derivatives,

$$\begin{aligned}\text{measurable }\hat{O}(x) \;=\; &\prod_{i=1}^{\text{even }N}\bigl(F_i(x)\text{ or }\partial F_i(x)\text{ or }\partial\partial F_i(x)\text{ or }\cdots\bigr)\;\times\\ &\times\prod_{i=1}^{\text{any }M}\bigl(B_i(x)\text{ or }\partial B_i(x)\text{ or }\partial\partial B_i(x)\text{ or }\cdots\bigr).\end{aligned} \tag{7}$$

Consequently, to assure that all such *measurable* operators commute at spacelike-separated points, the fermionic fields should either commute or anticommute with each other. Altogether, we need

For any bosonic fields $\hat{B}_1$ and $\hat{B}_2$, and any fermionic fields $\hat{F}_1$ and $\hat{F}_2$,

$$
\text{when } (x-y)^2 < 0,
$$

$$
\hat{B}_1(x) \times \hat{B}_2(y) \;=\; +\hat{B}_2(y) \times \hat{B}_1(x), \tag{8}
$$

$$
\hat{F}_i(x) \times \hat{B}_j(y) \;=\; +\hat{B}_j(y) \times \hat{F}_i(x), \tag{9}
$$

$$
\hat{F}_1(x) \times \hat{F}_2(y) \;=\; -\hat{F}_2(y) \times \hat{F}_1(x). \tag{10}
$$

By itself, the relativistic causality is consistent with either '+' or '−' sign on the last line (10), but other considerations fix that sign to be negative. Thus, at spacelike separations, the fermionic fields anticommute rather than commute with each other.

In October, we shall study the fermionic fields and their anticommutation in great detail. But for now, let's focus on the bosonic fields and see how and why they commute at spacelike distances.

Relativistic Causality for the Free Scalar Field

Let's prove the relativistic causality for the free scalar field. That is, let's prove that

$$
[\hat{\Phi}(x), \hat{\Phi}(y)] \;=\; 0 \quad \text{whenever} \quad (x-y)^2 \;<\; 0. \tag{11}
$$

As we saw last lecture, a free scalar field expands into creation and annihilation operators according to

$$
\hat{\Phi}(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \left(e^{-ikx}\, \hat{a}_{\mathbf{k}} \;+\; e^{+ikx}\, \hat{a}^\dagger_{\mathbf{k}} \right)^{k^0=+\omega_{\mathbf{k}}}. \tag{12}
$$

Expanding both $\hat{\Phi}(x)$ and $\hat{\Phi}(y)$ in eq. (11) in this manner, we get

$$
[\hat{\Phi}(x), \hat{\Phi}(y)] \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \int \frac{d^3\mathbf{k}'}{(2\pi)^3 2\omega_{\mathbf{k}'}} \begin{pmatrix} e^{-ikx-ik'y} \times [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] \\ +\, e^{-ikx+ik'y} \times [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \\ +\, e^{+ikx-ik'y} \times [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] \\ +\, e^{+ikx+ik'y} \times [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \end{pmatrix}. \tag{13}
$$

In the expansion (12), the operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ are all evaluated at $t = 0$ — or equivalently in the Schrödinger picture, — so they obey the equal-time commutation relations

$$\begin{aligned}
[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] &= [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0, \\
[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] &= +2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \\
[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] &= -2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}').
\end{aligned} \tag{14}$$

Plugging these commutators into eq. (13), we get

$$\begin{aligned}
(\cdots) &= 0 + 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \times e^{-ikx+ik'y} - 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \times e^{+ikx-ik'y} + 0 \\
&= 2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \times \left(e^{-ikx+ik'y} - e^{+ikx-ik'y}\right),
\end{aligned} \tag{15}$$

and therefore

$$\begin{aligned}
[\hat{\Phi}(x), \hat{\Phi}(y)] &= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\int\frac{d^3\mathbf{k}'}{(2\pi)^3 2\omega_{\mathbf{k}'}}\,2\omega_{\mathbf{k}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \times \left(e^{-ikx+ik'y} - e^{+ikx-ik'y}\right) \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left(e^{-ikx+iky} - e^{+ikx-iky}\right)^{k_0=+\omega_{\mathbf{k}}}.
\end{aligned} \tag{16}$$

For future reference let us define

$$D(z^\mu) \stackrel{\text{def}}{=} \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\left[e^{-ik^\mu z_\mu}\right]^{k^0=+\omega_{\mathbf{k}}}, \tag{17}$$

then in terms of this $D(z)$, eq. (16) amounts to

$$[\hat{\Phi}(x), \hat{\Phi}(y)] = D(x-y) - D(y-x) \tag{18}$$

The RHS of this formula happens to vanish for spacelike $x-y$, and that's what establishes the relativistic causality for the free scalar field $\hat{\Phi}(x)$. To see how this works, we need a couple of lemmas:

Lemma 1: $D(z)$ is invariant under *orthochronous* Lorentz transforms,

$$\forall\, L^\mu_{\ \nu} \in O^+(3,1), \quad D(L^\mu_{\ \nu} z^\nu) = D(z^\mu). \tag{19}$$

Proof: Let $z'^\mu = L^\mu_{\ \nu} z^\nu$ for some orthochronous Lorentz transform L, and let $k'^\mu = L^\mu_{\ \nu} k^\nu$ for the same L. Then $k'^\mu z'_\mu = k^\nu z_\nu$ and hence $\exp(-ik'z') = \exp(-ikz)$. At the same time, the integration measure in eq. (17) is invariant under the $k \to k'$ change of variables,

$$\frac{d^3\mathbf{k}'}{(2\pi)^3 2\omega_{\mathbf{k}'}} \;=\; \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\,, \tag{20}$$

and the integration range — the entire mass shell $k^0 = +\omega_{\mathbf{k}}$ — is also invariant. Consequently,

$$D(z') \;=\; \int \frac{d^3\mathbf{k}'}{(2\pi)^3 2\omega_{\mathbf{k}'}}\, e^{-ik'z'} \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\, e^{-ikz} \;=\; D(z), \tag{21}$$

quod erat demonstrandum.

Note that the integration range in eq. (17) — the mass shell $k^0 = +\omega_{\mathbf{k}}$ — is invariant under the orthochronous Lorentz symmetries, but not under the time reversal. In particular, it's not invariant under $L = PT : z^\mu \mapsto -z^\mu$, so *in general* $D(-z^\mu) \neq D(+z^\mu)$. Instead, eq. (17) gives us $D(-z^\mu) = D^*(+z^\mu)$, which is equal to $D(+z^\mu)$ only if it happens to be real.

Lemma 2: If a 4-vector z^μ is spacelike, $z^\mu z_\mu < 0$, then there exists an orthochronous Lorentz transform L such that $L^\mu_{\ \nu} z^\nu = -z^\mu$.
Proof: for a spacelike vector there exists a Lorentz boost B which eliminates its time component, $z'^\mu = B^\mu_{\ \nu} z^\nu = (0, \mathbf{z}')$. Indeed, let

$$\mathbf{v} \;=\; -\frac{z^0 \mathbf{z}}{\mathbf{z}^2}\,; \tag{22}$$

for a spacelike z^μ with $|z^0| < |\mathbf{z}|$ this formula yields a slower-than-light velocity ($|\mathbf{v}| < \mathbf{1}$ in $c = 1$ units), so let B be the active Lorentz boost by this velocity $\mathbf{v}$. Consequently,

$$(z')^0 \;=\; \gamma_{\mathbf{v}}\big(z^0 + \mathbf{v}\cdot\mathbf{z}\big) \;=\; \gamma_{\mathbf{v}}\left(z^0 \;-\; \frac{z^0\mathbf{z}}{\mathbf{z}^2}\cdot\mathbf{z}\right) \;=\; 0. \tag{23}$$

Next, let R be a 180° space rotation around some axis $\perp$ to the $\mathbf{z}'$, then $z'' = Rz' = (0, -\mathbf{z}') = -z'$. Finally, let's boost back by velocity $-\mathbf{v}$, so that $z''' = B^{-1}z''$.

Altogether, $z''' = B^{-1}z'' = B^{-1}Rz' = B^{-1}RBz$, so we may identify $L = B^{-1}RB$; by construction, L is an orthochronous Lorentz symmetry, in fact it's a continuous Lorentz symmetry, $L \in SO^+(3,1)$. At the same time,

$$Bz''' \;=\; z'' \;=\; -z' \;=\; -Bz \quad\Longrightarrow\quad z''' \;=\; -z, \tag{24}$$

quod erat demonstrandum.

BTW, orthochronous Lorentz symmetries sending z^μ into $-z^\mu$ exist only for spacelike z^μ. For a timelike z^μ, a Lorentz symmetry turning it into $-z^\mu$ would involve a time reversal rather than be orthochronous. Specifically, for a timelike z, $L = \tilde{B}^{-1} T \tilde{B}$ where T is time reversal $T(z^0, \mathbf{z}) = (-z^0, +\mathbf{z})$ while $\tilde{B}$ is a Lorentz boost to the rest frame of the timelike vector z^μ, $z' = \tilde{B} z = (z'^0, \mathbf{0})$.

Now let's go back to eq. (18) and use the Lemmas 1–2. Identifying $x - y$ as z, we see that for a spacelike $x - y$ there is an orthochronous Lorentz transform turning $x - y$ into $y - x$. But D is invariant under all such transforms, hence $D(x - y) = D(y - x)$. Consequently, the RHS of eq. (18) vanishes, so the commutator on the LHS must also vanish,

$$[\hat{\Phi}(x), \hat{\Phi}(y)] \;=\; 0 \quad \text{for any spacelike interval } x - y. \tag{25}$$

And this completes the proof of relativistic causality for the free scalar field.

Let me conclude this section with a few notes of the D function. For $z^0 = 0$ we have

$$D(z) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\, e^{i\mathbf{k}\cdot\mathbf{z}}, \tag{26}$$

and as a Fourier transform of a real symmetric function $1/2\omega_{\mathbf{k}}$ this integral is real for any $\mathbf{z}$. Consequently, by invariance under orthochronous Lorentz transforms, *for any spacelike z, $D(z)$ is real.*

On the other hand, for $z = (t, \mathbf{0})$ we have

$$D(t, \mathbf{0}) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\, e^{-it\omega_{\mathbf{k}}}, \tag{27}$$

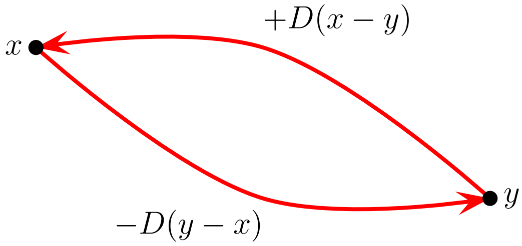
which is complex rather than real. Also $D(-t, \mathbf{0})$ is a complex conjugate of $D(+t, \mathbf{0})$. Consequently, for any timelike z, $D(z)$ is complex and $D(-z) = D^*(+z)$ instead of $D(+z)$. In terms of eq. (18), this means that

$$\text{for a timelike } x - y, \quad [\hat{\Phi}(x), \hat{\Phi}(y)] \;=\; 2i\,\mathrm{Im}(D(x - y)) \;\neq\; 0. \tag{28}$$

Thus, for a timelike $x - y$ we may send a signal from x to y or from y to x, depending on the sign of $y^0 - x^0$. But for a spacelike $x - y$ no signal would propagate between x and y.

Interpretation

In eq. (18), the commutator $[\hat{\Phi}(x), \hat{\Phi}(y)]$ vanishes for spacelike $x-y$ due to cancellation between the $D(x-y)$ and the $D(y-x)$ terms on the RHS. In our derivation — eqs. (13) through (16), — the $D(x-y)$ term came from annihilation operators $\hat{a}_{\mathbf{k}}$ in the expansion of the $\hat{\Phi}(x)$ and the matching creation operators $\hat{a}^\dagger_{\mathbf{k}}$ in the expansion of $\hat{\Phi}(y)$. Thus, we may say that the $+D(x-y)$ term stems from particles being created at y, traveling from y to x, and eventually being annihilated at x. On the other hand, the $-D(y-x)$ term comes from the creation operators $\hat{a}^\dagger_{\mathbf{k}}$ in $\hat{\Phi}(x)$ and the matching annihilation operators $\hat{a}_{\mathbf{k}}$ in $\hat{\Phi}(y)$, so we can say that it stems from the particles being created at x, traveling from x to y, and being annihilated at y.



(29)

At spacelike $x-y$ both processes are superluminal: In one process, the particles move FTL but forward in time, in the other — FTL and backward in time. Both processes are unphysical but seems to contribute a non-zero effect $\pm D(x-y)$. However, *the two processes precisely cancel each other,* and that's what provides for the relativistic causality: no net signal can travel either form x to y or from y to x.

Note that we need the quantum field theory to achieve the cancellation between particles moving from x to y and from y to x. In a relativistic quantum mechanics of a single particle, there is only one direction of travel and hence no cancellation. Instead, there is a small but non-zero amplitude for a faster-then light travel, as you will see in your next homework, set 3 problem 1. In fact, the amplitude you should get that problem is related to the D function as

$$U(x \leftarrow y) \stackrel{\text{def}}{=} \langle \mathbf{x} | \exp\bigl(-it\hat{H}\bigr) | \mathbf{y} \rangle = 2i\frac{\partial}{\partial t} D(x-y) \quad \text{where } t = x^0 - y^0. \tag{30}$$

The picture of relativistic causality stemming from cancellation between particles traveling in both directions of time becomes more mysterious when we consider the charged scalar fields

$\hat{\Phi}(x)$ and $\hat{\Phi}^\dagger(x)$. As we saw last lecture, the free fields expand into creation and annihilation operators as

$$\begin{aligned}\hat{\Phi}(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left(e^{-ikx}\,\hat{a}_\mathbf{k} \;+\; e^{+ikx}\,\hat{b}^\dagger_\mathbf{k}\right)^{k^0=+\omega_\mathbf{k}}, \\ \hat{\Phi}^\dagger(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_\mathbf{k}} \left(e^{-ikx}\,\hat{b}_\mathbf{k} \;+\; e^{+ikx}\,\hat{a}^\dagger_\mathbf{k}\right)^{k^0=+\omega_\mathbf{k}}.\end{aligned} \tag{31}$$

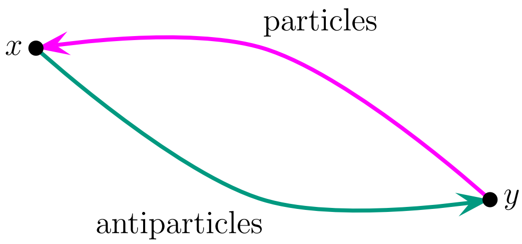
Given the equal-time commutation relations between the creation and the annihilation operators, we immediately see that

$$[\hat{\Phi}(x), \hat{\Phi}(y)] \;=\; 0 \quad \text{and} \quad [\hat{\Phi}^\dagger(x), \hat{\Phi}^\dagger(y)] \;=\; 0 \quad \text{for any } x \text{ and } y, \tag{32}$$

but the other commutators $[\hat{\Phi}^\dagger(x), \hat{\Phi}(y)]$ and $[\hat{\Phi}(x), \hat{\Phi}^\dagger(y)]$ are non-trivial. Proceeding similarly to what we did to the real quantum field, we arrive at

$$[\hat{\Phi}^\dagger(x), \hat{\Phi}(y)] \;=\; [\hat{\Phi}(x), \hat{\Phi}^\dagger(y)] \;=\; D(x-y) \;-\; D(y-x), \tag{33}$$

which vanishes for the spacelike $x-y$ but not for timelike $x-y$. Focusing on the $[\hat{\Phi}(x), \hat{\Phi}^\dagger(y)]$ commutator, we have the $+D(x-y)$ term stemming from the $\hat{a}_\mathbf{k}$ operators in $\hat{\Phi}(x)$ and the matching $\hat{a}^\dagger_\mathbf{k}$ operators in $\hat{\Phi}^\dagger(y)$, while the $-D(y-x)$ term stems from the $\hat{b}^\dagger_\mathbf{k}$ operators in $\hat{\Phi}(x)$ and the matching $\hat{b}_\mathbf{k}$ in $\hat{\Phi}^\dagger(y)$. Thus, pictorially the particles travel from y to x while the antiparticles travel from x to y,



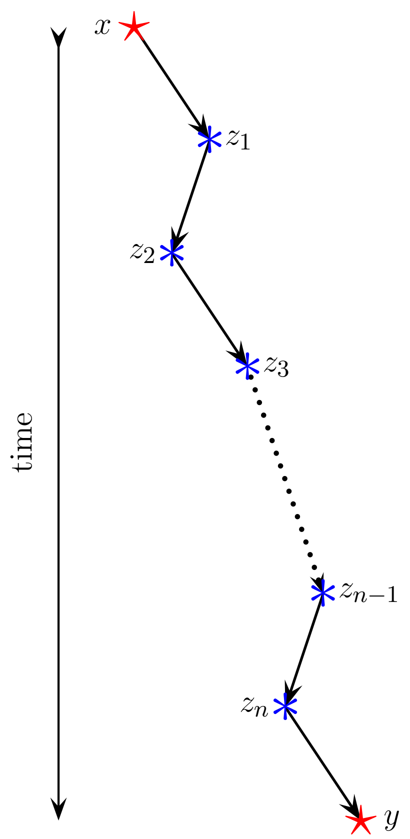
(34)

In particular, for $x^0 > y^0$ *the particles travel forward in time while the antiparticles travel backward in time.* However, for spacelike $x-y$ both particles and antiparticles move FTL, and who is going forward in time and who backward is relative to a frame of reference, so the two effects precisely cancel each other. And that's how the relativistic causality works for the charged scalar fields.

Interacting Field Theories

Thus far, we have seen how relativistic causality works for the free scalar fields. Later in class (October), I shall discuss the spinor fields — like the electron field — in some detail; in particular, here are my notes on the relativistic causality for the spinor fields. As for the higher-spin bosonic or fermionic fields, they work similarly to the scalar or spinor fields — except for a messier algebra — so I am not going to dwell on them.

Instead, let us consider how the relativistic causality works for the interacting quantum field theories. For simplicity, let's assume the interactions are week enough to be treated as perturbations. In relativistic QFTs all interactions are local — they happen at a particular point z, although z could be anywhere in spacetime. Consequently, a signal can travel from some point x to another point y either directly or via a chain of intermediate points $z_1, \ldots, z_n$ where some interaction modifies the signal. That is:



$\star$ A signal is emitted at point x.

• Free propagation of the signal from x to z_1.
By free propagation I mean propagates as in the free field theory without interactions.

$*$ Local interaction at z_1

• Free propagation from z_1 to z_2.

$*$ Local interaction at z_2.

• Free propagation from z_2 to z_3.

— ....

• Free propagation from z_{n-1} to z_n.

$*$ Local interaction at z_n.

• Free propagation from z_n to y.

$\star$ The signal is detected at y.

By relativistic causality of the free field theory, every free propagation step must lie in the future light cone, thus

$$\begin{aligned}
z_1^0 - x^0 &> |\mathbf{z}_1 - \mathbf{x}|\,, \\
z_2^0 - z_1^0 &> |\mathbf{z}_2 - \mathbf{z}_1|\,, \\
z_3^0 - z_2^0 &> |\mathbf{z}_3 - \mathbf{z}_2|\,, \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\
z_n^0 - z_{n-1}^0 &> |\mathbf{z}_n - \mathbf{z}_{n-1}|\,, \\
y^0 - z_n^0 &> |\mathbf{y} - \mathbf{z}_n|\,.
\end{aligned} \tag{35}$$

Totaling all these inequalities, we immediately get

$$y^0 - x^0 \;>\; |\mathbf{z}_1 - \mathbf{x}| \;+\; |\mathbf{z}_2 - \mathbf{z}_1| \;+\; \cdots\; |\mathbf{z}_n - \mathbf{z}_{n-1}| \;+\; |\mathbf{y} - \mathbf{z}_n| \;>\; |\mathbf{y} - \mathbf{x}|\,, \tag{36}$$

which means that the entire route from x to y lies within the future light cone. In other words, *a signal can propagate only to points y within the future light cone from the origin point x, and this is precisely what the relativistic causality requires.*

The bottom line is, if the free quantum field theory respects the relativistic causality and if all the interactions are local, then the perturbation theory — to all orders — also respects the relativistic causality.

Beyond the perturbation theory, proving relativistic causality of the non-perturbative effects is much harder, and I do not think this has ever been done for the general case. However, all the *known* non-perturbative effects in quantum field theories happen to respect the relativistic causality.