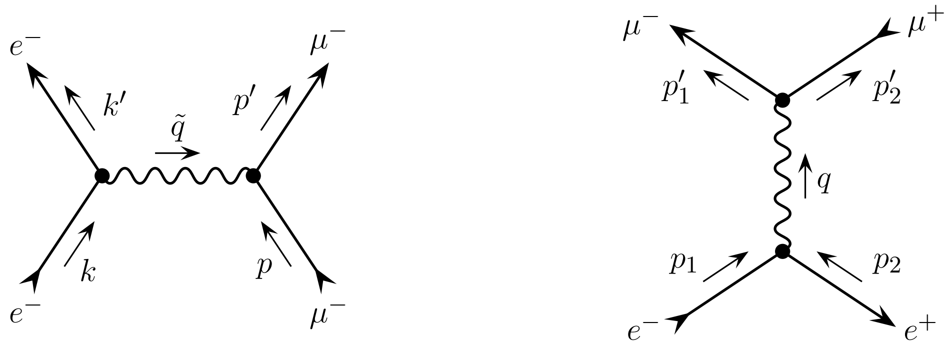
# Crossing Symmetry

Consider 2 processes in QED: (a) elastic electron-muon scattering $e^-\mu^- \to e^-\mu^-$, and (b) muon pair production $e^-e^+ \to \mu^-\mu^+$. At the tree level there is one Feynman diagram for each process,




(1)

and the two diagrams look like the same diagram from two different points of view. Indeed, both diagrams have identical arrangements of lines and vertices, and the only difference between them is the interpretation of the external lines as incoming or outgoing and the signs of the corresponding external momenta. Here is the correspondence table:

| **scattering** | **momenta** | **pair production** |
|---|---|---|
| incoming $e^-$ | $k \leftrightarrow +p_1$ | incoming $e^-$ |
| outgoing $e^-$ | $k' \leftrightarrow -p_2$ | incoming $e^+$ |
| outgoing $\mu^-$ | $p' \leftrightarrow +p'_1$ | outgoing $\mu^-$ |
| incoming $\mu^-$ | $p \leftrightarrow -p'_2$ | outgoing $\mu^+$ |

(2)

Now let's compare the tree-level amplitudes and $\overline{|\mathcal{M}|^2}$ of the two processes. For the pair production,

$$\mathcal{M}^{\text{pair}} \;=\; \frac{e^2}{s} \times \bar{v}(e^+)\gamma^\nu u(e^-) \times \bar{u}(\mu^-)\gamma_\nu v(\mu^+) \tag{3}$$

where $s = q^2 = (p_1 + p_2)^2$, and after summing over all the spins we get

$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{pair}}\right|^2 \;=\; \frac{e^4}{s^2} \times \operatorname{tr}\Big((\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\lambda\Big) \times \operatorname{tr}\Big((\not{p}'_1 + M_\mu)\gamma_\nu(\not{p}'_2 - M_\mu)\gamma_\lambda\Big). \tag{4}$$

For the scattering,

$$\mathcal{M}^{\rm scatt} \;=\; \frac{e^2}{t}\times\bar u(e^{-\prime})\gamma^\nu u(e^-)\times\bar u(\mu^{-\prime})\gamma_\nu u(\mu^-) \tag{5}$$

where $t=\tilde q^2=(k-k')^2$, and after summing over all the spins we get

$$\sum_{\rm spins}\left|\mathcal{M}^{\rm scatt}\right|^2 \;=\; \frac{e^4}{t^2}\times{\rm tr}\Big((\not k'+m_e)\gamma^\nu(\not k+m_e)\gamma^\lambda\Big)\times{\rm tr}\Big((\not p'+M_\mu)\gamma_\nu(\not p+M_\mu)\gamma_\lambda\Big). \tag{6}$$

The right hand sides of eqs. (4) and (6) are exactly the same analytic functions of the momenta, provided we identify the momenta in the two processes according to the table (2),

$$k\leftrightarrow +p_1\,,\quad k'\leftrightarrow -p_2\,,\quad p\leftrightarrow -p_2'\,,\quad p'\leftrightarrow +p_1'\,. \tag{7}$$

Indeed, under this mapping,

$$\begin{aligned} t^{\rm scatt} \;=\; (k-k')^2 \;&\leftrightarrow\; s^{\rm pair} \;=\; (p_1+p_2)^2,\\ {\rm tr}\Big((\not k'+m_e)\gamma^\nu(\not k+m_e)\gamma^\lambda\Big)^{\rm scatt} \;&\leftrightarrow\; -\,{\rm tr}\Big((\not p_2-m_e)\gamma^\nu(\not p_1+m_e)\gamma^\lambda\Big)^{\rm pair},\\ {\rm tr}\Big((\not p'+M_\mu)\gamma_\nu(\not p+M_\mu)\gamma_\lambda\Big)^{\rm scatt} \;&\leftrightarrow\; -\,{\rm tr}\Big((\not p_1'+M_\mu)\gamma_\nu(\not p_2'-M_\mu)\gamma_\lambda\Big)^{\rm pair}, \end{aligned} \tag{8}$$

and hence

$$\sum_{\rm spins}\left|\mathcal{M}^{\rm scatt}\right|^2\;\leftrightarrow\;\sum_{\rm spins}\left|\mathcal{M}^{\rm pair}\right|^2. \tag{9}$$

To be precise, the correspondence in eq. (9) involves analytic continuation rather than outright equality because positive particle energies in scattering map onto negative energies in pair production and vice verse. Thus,

$$\sum_{\rm spins}\left|\mathcal{M}^{\rm pair}\right|^2 \;=\; F(p_1,p_2,p_1',p_2') \quad\text{and}\quad \sum_{\rm spins}\left|\mathcal{M}^{\rm scatt}\right|^2 \;=\; F(k,-k',p',-p) \tag{10}$$

*for the same analytic function $F$ of the momenta,* but for the pair production this function is evaluated for $p_2^0>0$ and $p_2'^0>0$, while for the scattering we use it for $p_2^0=-k'^0<0$ and $p_2'^0=-p^0<0$.

Relations such as (9) between processes described by similar Feynman diagrams — but with different identifications of the external legs as incoming or outgoing — are called the *crossing symmetries*. These symmetries apply not not only to the spin-summed $|\mathcal{M}|^2$ but also to the amplitudes $\mathcal{M}$ themselves, provided one properly maps the spin states of the incoming and the outgoing particles onto each other. For example, in the ultra-relativistic limit of the muon pair production, the polarized amplitudes (for particles of definite helicities) are

$$\begin{aligned} \left\langle \mu_L^-,\mu_R^+\right|\mathcal{M}\left|e_L^-,e_R^+\right\rangle &= \left\langle \mu_R^-,\mu_L^+\right|\mathcal{M}\left|e_R^-,e_L^+\right\rangle = 2e^2\times\frac{u^{\rm pair}}{s^{\rm pair}}\times e^{i\,\rm phase},\\ \left\langle \mu_R^-,\mu_L^+\right|\mathcal{M}\left|e_L^-,e_R^+\right\rangle &= \left\langle \mu_L^-,\mu_R^+\right|\mathcal{M}\left|e_R^-,e_L^+\right\rangle = 2e^2\times\frac{t^{\rm pair}}{s^{\rm pair}}\times e^{i\,\rm phase},\\ \text{for all other sets of helicities,}\quad &\left\langle \mu^-,\mu^+\right|\mathcal{M}\left|e^-,e^+\right\rangle = 0. \end{aligned} \tag{11}$$

In the similar ultra-relativistic limit of the electron-muon scattering, the polarized amplitudes are

$$\begin{aligned} \left\langle e_L^-,\mu_L^-\right|\mathcal{M}\left|e_L^-,\mu_L^-\right\rangle &= \left\langle e_R^-,\mu_R^-\right|\mathcal{M}\left|e_R^-,\mu_R^-\right\rangle = 2e^2\times\frac{s^{\rm scatt}}{t^{\rm scatt}}\times e^{i\,\rm phase},\\ \left\langle e_R^-,\mu_L^-\right|\mathcal{M}\left|e_R^-,\mu_L^-\right\rangle &= \left\langle e_L^-,\mu_R^-\right|\mathcal{M}\left|e_L^-,\mu_R^-\right\rangle = 2e^2\times\frac{u^{\rm scatt}}{t^{\rm scatt}}\times e^{i\,\rm phase},\\ \text{for all other sets of helicities,}\quad &\left\langle e^-,\mu^-\right|\mathcal{M}\left|e^-,\mu^-\right\rangle = 0. \end{aligned} \tag{12}$$

Up to overall phases, the scattering amplitudes look exactly like the pair production amplitudes, provided we cross the Mandelstam variables $(s,t,u)$ according to eqs. (7),

$$\begin{aligned} s^{\rm pair} &= (p_1+p_2)^2 = (p_1'+p_2')^2 \leftrightarrow t^{\rm scatt} = (k-k')^2 = (p'-p)^2,\\ t^{\rm pair} &= (p_1-p_1')^2 = (p_2'-p_2)^2 \leftrightarrow u^{\rm scatt} = (p'-k')^2 = (k-p)^2,\\ u^{\rm pair} &= (p_1-p_2')^2 = (p_1'-p_2)^2 \leftrightarrow s^{\rm scatt} = (p'+k')^2 = (p+k)^2, \end{aligned} \tag{13}$$

and also cross the helicities as

$$\lambda(e^{-\prime}) = -\lambda(e^+),\quad \lambda(\mu^-) = -\lambda(\mu^{+\prime}). \tag{14}$$

In other words,

$$\begin{aligned} \mathcal{M}^{\rm pair} &= f(s=s^{\rm pair},t=t^{\rm pair},u=u^{\rm pair};\text{helicities}),\\ \mathcal{M}^{\rm scatt} &= f(s=t^{\rm scatt},t=u^{\rm scatt},u=s^{\rm scatt};\text{crossed helicities})\times e^{i\,\rm phase}, \end{aligned} \tag{15}$$

*for exactly the same analytic finction $f(s,t,u)$.* However, the scattering and the pair production use this function for different domains of $s,t,u$: for the pair production $s>0$ while $t,u<0$, but for the scattering $u>0$ while $s,t<0$.

There are similar crossing relations for amplitudes involving more than 4 incoming+outgoing particles, although they use more variables than just $s$, $t$, and $u$. Most generally, take any $m$-particle $\to$ $n$-particle amplitude

$$\big\langle (s_1',p_1),\ldots,(s_n',p_n')\big|\,\mathcal{M}\,\big|(s_1,p_1),\ldots,(s_m,p_m)\big\rangle \tag{16}$$

where the $s$'s label particle species and other discrete quantum numbers (like helicities) and $p$'s are their momenta. Let's focus on the momentum dependence, so we may write

$$\big\langle (s_1',p_1),\ldots,(s_n',p_n')\big|\,\mathcal{M}\,\big|(s_1,p_1),\ldots,(s_m,p_m)\big\rangle \;=\; F(p_1,\ldots,p_m;p_1',\ldots,p_n') \tag{17}$$

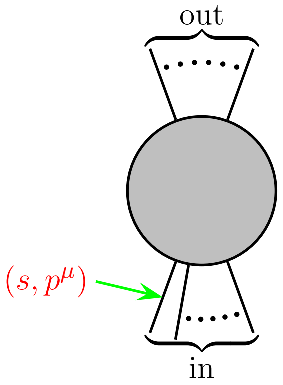
for some *analytic function* $F$ that may be *analytically continued to complex momenta.* Or rather, to complex momenta $p_i^\mu$ and $p_j'^\mu$ that lie on the *complex continuation of the mass shell* and satisfy net momentum conservation, thus

$$\text{complex } p_i^\mu,p_j'^\mu \text{ but: each } p_i^\mu p_{i\mu} \;=\; M_i^2,\quad \text{each } p_j'^\mu p_{j\mu}' \;=\; M_j'^2,\quad \sum_i p_i^\mu \;=\; \sum_j p_j'^\mu. \tag{18}$$

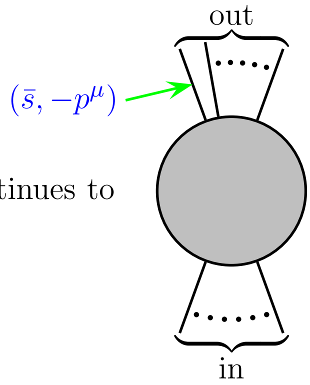
After going through such complex momenta, we may analytically continue $F$ back to the real momenta with $p^\mu p_\mu = M^2$, but with negative energies for some of the incoming or outgoing particles. The result of this analytic continuation is the amplitude for a different physical process — in which incoming particles with negative energies become outgoing antiparticles with $p'^\mu = -p^\mu$ and vice verse. For example, if after analytic continuation the only negative energy is $p_1^0$, then

$$F^{\text{cont}}(p_1,p_2\ldots,p_m;p_1',\ldots,p_n') \;=\; \big\langle (\bar s_1,-p_1)(s_1',p_1),\ldots,(s_n',p_n')\big|\,\mathcal{M}\,\big|(s_2,p_2),\ldots,(s_m,p_m)\big\rangle\,, \tag{19}$$

or diagrammatically




analytically continues to




(20)

In general, we may cross any number of external lines from incoming to outgoing or vice verse (as long as we can satisfy the net momentum conservation rule for the on-shell momenta). Thus, an $m$-particle $\to$ $n$-particle amplitude (16) analytically continues to all of the amplitudes

$$\Big\langle \text{any combo of } (s,p^\mu)'_j \text{ and } (\bar{s},-p^\mu)_i \Big|\, \mathcal{M} \,\Big| \text{the remaining } (s,p^\mu)_i \text{ and } (\bar{s},-p^\mu)'_j \Big\rangle \tag{21}$$

with the same net number $m+n$ of external particles but different arrangements of those external particles to incoming and outgoing.

Caution: The crossing relations may change the overall phases of the amplitudes involved, and even the overall signs of the spin-sums $\sum|\mathcal{M}|^2$! Indeed, consider some generic process $X \to Y$, its reversal $Y \to X$, and the amplitudes

$$F(s,t,u) \;=\; \langle Y|\,\mathcal{M}\,|X\rangle \quad \text{and} \quad \overline{F}(s,t,u) \;=\; \langle X|\,\mathcal{M}\,|Y\rangle\,. \tag{22}$$

For the *real, on-shell momenta* the two amplitudes are complex conjugates of each other, $\overline{F}(s,t,u) = F^*(s,t,u)$. However, the analytic continuation to complex momenta generally does *not* preserve such relations, thus

$$\overline{F}_{\rm cont}(s,t,u) \;=\; \big(F_{\rm cont}(s^*,t^*,u^*)\big)^* \;\neq\; F^*_{\rm cont}(s,t,u) \quad \text{for complex } s,t,u. \tag{23}$$

When we analytically continue to the crossed momenta — which are real and on-shell but have negative energies for some particles — the $F$ and $\overline{F}$ amplitudes go back to complex conjugates of each other *up to an overall sign.* Specifically,

$$\overline{F}(\text{crossed } s,t,u) \;=\; F^*(\text{crossed } s,t,u) \times (-1)^{\#\text{FC}} \tag{24}$$

where #FC is the number of crossed fermionic lines, *i.e.*, the number of formerly incoming fermions that became outgoing or vice verse.

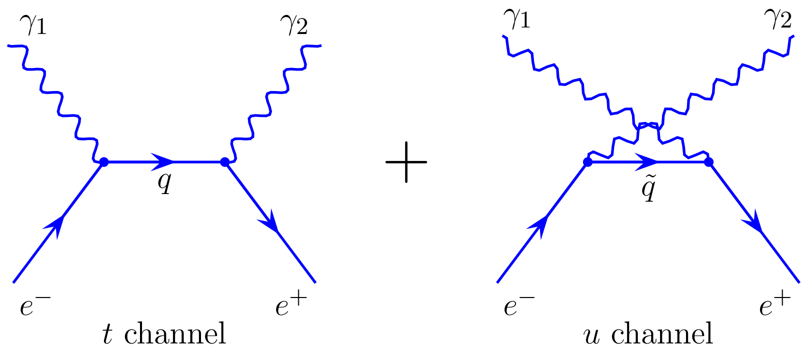
Consequently, the spin sums for two processes related by crossing are related by analytic continuation of momenta — which amount to exchanges between $s$, $t$, and $u$ — and a sign factor $(-1)^{\#\text{FC}}$. Thus, given

$$\sum_{\text{spins}} |\mathcal{M}|^2 \;=\; \mathcal{F}(s,t,u) \;\stackrel{\text{def}}{=}\; \sum_{\text{spin}} F(s,t,u) \times \overline{F}(s,t,u) \tag{25}$$

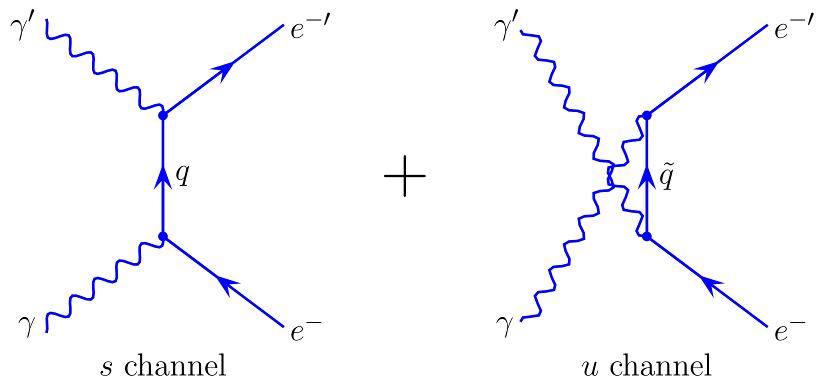
for some process, the crossed process would have

$$\sum_{\text{spins}} |\mathcal{M}|^2_{\text{crossed}} \;=\; \mathcal{F}(\text{crossed } s,t,u) \times (-1)^{\#\text{FC}}. \tag{26}$$

For example, consider the electron-positron annihilation $e^-e^+ \to \gamma\gamma$ and the Compton scattering of a photon and an electron $\gamma e^- \to \gamma e^-$. At the tree level, there are two Feynman diagrams for each process:

Annihilation:  (27)

Compton:




(28)

Clearly, the two processes are related by crossing of one fermion (an incoming positron becomes an outgoing electron) and one boson (one of the outgoing photons becomes incoming). In terms of the $(s,t,u)$ variables for the two processes, the crossing exchanges $s \leftrightarrow t$, *i.e.*,

$$t^a \leftrightarrow s^c, \qquad s^a \leftrightarrow t^c, \qquad u^a \leftrightarrow u^c, \tag{29}$$

hence

$$\mathcal{M}^{\text{annihilation}} = F(s,t,u), \qquad \mathcal{M}^{\text{Compton}} = F(t,s,u) \tag{30}$$

for the same analytic function $F$. Note: this relation works to all orders of QED perturbation theory, as long as we compute both amplitudes to the same order in $\alpha = e^2/4\pi$.

However, after summing over fermions spins and photon polarizations, we obtain

$$\begin{aligned} \frac{1}{4}\sum_{\lambda_1,\lambda_2}\sum_{s_-,s_+}\left|\mathcal{M}^{\text{annihilation}}\right|^2 &= +\mathcal{F}(s,t,u) \\ \text{but}\quad \frac{1}{4}\sum_{\lambda,\lambda'}\sum_{s,s'}\left|\mathcal{M}^{\text{Compton}}\right|^2 &= -\mathcal{F}(t,s,u), \end{aligned} \tag{31}$$

for the same analytic function $\mathcal{F} = \frac{1}{4}\sum F \times \overline{F}$. The sign difference between the two spin sums (31) is due to crossing of an odd number (*i.e.*, 1) of fermions between the two processes, and it is necessary to assure that both the annihilation and the Compton scattering have positive un-polarized cross sections.

Indeed, next lecture we shall calculate at the tree level

$$\mathcal{F}(s,t,u) \;=\; 2e^2\left[\frac{u-m^2}{t-m^2}+\frac{t-m^2}{u-m^2}+1-\left(1+\frac{2m^2}{t-m^2}+\frac{2m^2}{u-m^2}\right)^2\right]. \qquad (32)$$

This expression is positive for the annihilation process with $t, u < 0$, but crossing to the Compton scattering where $t \to s > m^2$ while $u \to u < m^2$ produces negative $\mathcal{F}(t,s,u)$ for the Compton scattering. Thus, to get a positive partial cross-section, we really need that minus sign on the second line of eq. (31).