

# Vacuum Energy and Effective Potentials

Quantum field theories have badly divergent vacuum energies. In perturbation theory, the leading term is the net zero-point energy

$$E_{\text{zero point}} = \sum_{\substack{\text{particle} \\ \text{species}}} \sum_{\mathbf{p},s} E_{\mathbf{p},s} \times \frac{\pm 1}{2} \quad (1)$$

where the  $\pm$  sign is  $+$  for bosons and  $-$  for fermions, while the sub-leading terms follow from the interactions between the quantum fields. However, as long as the vacuum energy remains a constant, it does not have any observable effects besides renormalizing the cosmological constant  $\Lambda$ . Consequently, most of the time we do not pay any attention to the vacuum energy or its divergences.

But sometimes the vacuum energy depends on some parameters of the theory that we may vary, and then  $\Delta E_{\text{vacuum}}$  acts as an effective potential for those parameters. For example, let's put the EM fields in a small cavity, which gives them a discrete spectrum of modes  $(\mathbf{p}, s)$  instead of the continuum. Consequently, the EM field in the cavity has a different net zero-point energy density than in infinite space, but the difference is finite,

$$\left. \frac{E_{\text{zero point}}}{\text{volume}} \right|_{\text{cavity}} = \text{divergent\_constant} + \text{finite } \Delta \mathcal{E}(\text{cavity\_size}). \quad (2)$$

This finite difference — called *the Casimir effect* — has observable consequences such as attractive force between two parallel plates at small distances from each other,

$$F_c = -\frac{\pi^2 \hbar c}{240} \times \frac{\text{Area}}{\text{distance}^4}, \quad (3)$$

see [Wikipedia article](#) for details.

In these notes I would like to focus on different effect — discovered by Sidney Coleman and Eric Weinberg — in which the fields remain in infinite volume but their masses change due to interaction with a non-zero vacuum expectation value  $\langle \varphi \rangle$  of some scalar field. The zero point energies of the fields depend on their masses, and the net vacuum energy density acts as an effective potential for the  $\varphi$  field; *it is this effective potential that the VEV  $\langle \phi \rangle$  seeks to minimize.*

For example, consider a theory of two scalar fields  $\varphi$  and  $\Phi$  with classical Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - V_0(\varphi) + \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{m_0^2}{2}\Phi^2 - \frac{g}{4}\varphi^2\Phi^2. \quad (4)$$

Suppose  $V_0(\varphi) \equiv 0$  so the  $\varphi$  field is massless and does not interact with itself, only with the other field  $\Phi$ . Classically, the vacuum expectation value  $\langle\varphi\rangle$  cannot be determined, so let us allow for a completely general (but  $x$ -independent)  $\langle\varphi\rangle$  and see how it affects the net zero-point energy of the system.

Since  $\varphi$  does not interact with itself, the shifted field  $\delta\varphi(x) = \varphi(x) - \langle\varphi\rangle$  remains massless regardless of the VEV  $\langle\varphi\rangle$ . On the other hand, the  $\langle\varphi\rangle$  does affect the mass of the other field  $\Phi$ ,

$$\mathcal{L} \supset -\frac{M^2}{2} \times \Phi^2 \quad \text{for} \quad M^2 = m_0^2 + \frac{g}{2} \times \langle\varphi\rangle^2. \quad (5)$$

Consequently, the zero-point energy of the  $\delta\varphi$  field remains constant, but the zero-point energy of  $\Phi$  depends on the  $M^2$  and hence on the  $\langle\varphi\rangle$ , so let's calculate this energy and its  $\langle\varphi\rangle$ -dependence.

In a box of finite volume  $L^3$  particle momenta are discrete and

$$E_{\text{zero point}} = +\frac{1}{2} \sum_{\mathbf{p}} \sqrt{M^2 + \mathbf{p}^2}. \quad (6)$$

For large  $L$ , the momenta  $\mathbf{p}$  have a uniform density  $(L/2\pi)^3$  in the momentum space, thus

$$\sum_{\mathbf{p}} \xrightarrow{L \rightarrow \infty} L^3 \times \int \frac{d^3\mathbf{p}}{(2\pi)^3}, \quad (7)$$

so the *density* of the zero-point energy obtains from the momentum integral

$$\mathcal{E} \stackrel{\text{def}}{=} \frac{E_{\text{zero point}}}{\text{volume}} = \frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sqrt{M^2 + \mathbf{p}^2}. \quad (8)$$

We are interested in the dependence of this energy density on the VEV  $\langle\varphi\rangle$ , so let's focus on the difference

$$\Delta\mathcal{E}(\langle\varphi\rangle) \stackrel{\text{def}}{=} \mathcal{E}(\langle\varphi\rangle) - \mathcal{E}(0) = \frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left( \sqrt{M^2(\langle\varphi\rangle) + \mathbf{p}^2} - \sqrt{m_0^2 + \mathbf{p}^2} \right). \quad (9)$$

It is this difference that acts as the effective potential for the  $\varphi$  field.

Before we try to evaluate the integral (9), let us re-write it as a 4D momentum integral. The difference of any two energies  $E_1$  and  $E_2$  can be written as

$$\begin{aligned} E_2 - E_1 &= \int_{E_1^2}^{E_2^2} \frac{dE^2}{2E} = \int_{E_1^2}^{E_2^2} dE^2 \times \int_{-\infty}^{+\infty} \frac{dp_4}{2\pi} \frac{1}{E^2 + p_4^2} = \int_{-\infty}^{+\infty} \frac{dp_4}{2\pi} \int_{E_2^2}^{E_1^2} dE^2 \frac{1}{E^2 + p_4^2} \\ &= \int_{-\infty}^{+\infty} \frac{dp_4}{2\pi} \log \frac{E_2^2 + p_4^2}{E_1^2 + p_4^2} \end{aligned}$$

Plugging this formula into eq. (9) gives us

$$\Delta\mathcal{E} = \frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{dp_4}{2\pi} \log \frac{M^2 + \mathbf{p}^2 + p_4^2}{m_0^2 + \mathbf{p}^2 + p_4^2} = \frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \log \frac{M^2 + p_E^2}{m_0^2 + p_E^2} \quad (10)$$

where  $p_E^\mu = (\mathbf{p}, p_4)$  acts as a 4D Euclidean momentum. To clarify the physical meaning of this formula, let's expand the logarithm in the integrand into powers of

$$M^2 - m_0^2 = \frac{g \langle \varphi \rangle^2}{2} \quad (11)$$

and hence into powers of  $\langle \varphi \rangle$ ,

$$\log \frac{M^2 + p_E^2}{m_0^2 + p_E^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \times \frac{\left(\frac{g}{2} \langle \varphi \rangle^2\right)^n}{(p_E^2 + m_0^2)^n}. \quad (12)$$

Consequently,

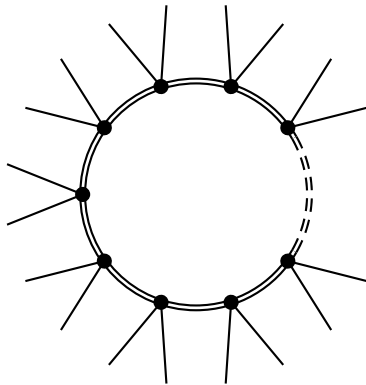
$$\begin{aligned} \Delta\mathcal{E} &= \frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \times \frac{\left(\frac{g}{2} \langle \varphi \rangle^2\right)^n}{(p_E^2 + m_0^2)^n} \\ &= \sum_{n=1}^{\infty} \frac{\langle \varphi \rangle^{2n}}{2n \times 2^n} \times \int \frac{d^4 p_E}{(2\pi)^4} \frac{(-1)^{n-1} g^n}{(p_E^2 + m_0^2)^n} \end{aligned} \quad (13)$$

or in terms of Minkowski 4-momentum  $p^\mu$ ,

$$\Delta\mathcal{E} = \sum_{n=1}^{\infty} \frac{-i \langle \varphi \rangle^{2n}}{2n \times 2^n} \times \int \frac{d^4 p}{(2\pi)^4} (-ig)^n \times \left( \frac{i}{p^2 - m_0^2 + i0} \right)^n. \quad (14)$$

For each  $n$ , the momentum integral here evaluates the 1-loop Feynman diagram with  $2n$

external  $\varphi$  legs carrying zero momenta,



$$= \int \frac{d^4 p}{(2\pi)^4} (-ig)^n \times \left( \frac{i}{p^2 - m_0^2 + i0} \right)^n \quad (15)$$

where the internal propagators belong to the massive  $\Phi$  field. This gives us a formula for the  $\Delta\mathcal{E}$  in terms of Feynman diagrams.

Note that the combinatorical factors in eq. (14) are different from the Feynman rules for scattering amplitudes. In scattering amplitudes, all the external legs are treated as distinct, and we should add up diagrams related by non-trivial permutations of the external legs, for example

$$\frac{(2n)!}{2n \times 2^n}$$

diagrams that look like (15). But in eq. (14) the net combinatorial factor is

$$\frac{1}{2n \times 2^n},$$

which corresponds to a single diagram with  $(2n) \times 2^n$  symmetries; in terms of the diagram (15), this means treating the external legs as identical — especially since they all carry the same momentum  $p = 0$ .

#### FEYNMAN RULES FOR THE EFFECTIVE POTENTIAL

In light of the above, let me formulate the Feynman rules for the effective potentials  $V_{\text{eff}}(\langle\varphi\rangle)$ . Besides the usual vertices, propagators, and loops, there are also *vacuum legs*

$$(\text{vertex}) \text{---}\text{X} = \langle\varphi\rangle$$

Physically, the vacuum legs correspond to insertions of the scalar VEVs into the Lagrangian and hence into the amplitudes, so the vacuum legs (also called the vacuum-insertion lines)

belong only to the scalar species with  $\langle\varphi\rangle \neq 0$ . Although graphically the vacuum legs may look like the external lines, they do not correspond to any incoming or outgoing particles, so the momentum flowing through a vacuum leg is always zero. Combinatorically, symmetries of a diagram may permute the vacuum legs belonging to VEVs of the same scalar field  $\varphi$ .

Also, in a vacuum-energy diagram we drop the overall momentum conservation factor  $(2\pi)^4\delta^{(4)}(p_{\text{net}})$  since there are no non-zero external momenta.

With these Feynman rules, eq. (14) becomes simply

$$\Delta\mathcal{E}_{\text{zero point}} = i \sum_{n=1}^{\infty} \text{[Diagram: A circular vacuum diagram with two concentric lines and 12 external vacuum legs, each ending in a cross symbol.]} \quad (16)$$

For the general quantum field theories — which may have multiple scalar, vector, and fermionic fields, and several scalars may have non-zero VEVs — the effective potential also follows from Feynman diagrams with vacuum legs (but no other kinds of external legs),

$$V_{\text{eff}}(\langle\varphi\rangle) = i \sum \text{all connected vacuum diagrams.} \quad (17)$$

Note: this sum includes diagrams with any number of loops,  $L = 0, 1, 2, \dots$

★ In theories with  $V_0(\varphi) \neq 0$ , this expansion begins with tree diagrams. At the tree level,

$$V_{\text{eff}}^{\text{tree}}(\langle\varphi\rangle) = \text{classical } V_0(\varphi). \quad (18)$$

Indeed, suppose the  $\varphi$  field has a classical potential

$$V_0(\varphi) = \frac{m_\varphi^2}{2} \times \varphi^2 + \frac{\lambda_\varphi}{24} \times \varphi^4, \quad (19)$$

then at the tree level

$$\begin{aligned}
-iV_{\text{eff}}^{\text{tree}} &= \text{X} \text{---} \bullet \text{---} \text{X} \quad + \quad \begin{array}{c} \text{X} \quad \text{X} \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \text{X} \quad \text{X} \end{array} \\
&= \frac{-im_\varphi^2}{2} \times \langle \varphi \rangle^2 + \frac{-i\lambda_\varphi}{24} \times \langle \varphi \rangle^4 \equiv -iV_0(\langle \varphi \rangle).
\end{aligned} \tag{20}$$

- ★ The one-loop vacuum diagrams such as (16) comprise the zero-point energy density  $\mathcal{E}$  of the fields whose propagators run around the loop, or rather the  $\Delta\mathcal{E}$  due to masses of those fields being affected by their couplings to the VEVs. However, at this level of approximation we ignore any interactions between the quantum fields except for their couplings to the  $\langle \varphi \rangle$ .
- ★ Interactions between the quantum fields lead to Hamiltonian terms of the form

$$\begin{aligned}
\hat{H} \supset \hat{a}\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger, \\
\hat{a}\hat{a}\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}, \quad \hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger.
\end{aligned} \tag{21}$$

These terms affect the ground state energy of the theory at second and higher orders of the perturbation theory. In the Feynman diagram language, these higher-order corrections correspond to the two-loop and multi-loop diagrams in the sum (17). I shall return to this issue later in these notes.

## ONE LOOP CALCULATION

Now that we have the general rules, let's actually calculate the one-loop effective potential for our example of a scalar field  $\Phi$  whose  $M^2$  depends on the VEV of another scalar  $\varphi$  according to eq. (5).

Instead of separately evaluating the one-loop diagrams with different numbers of vacuum legs, it's more convenient to add them up before integrating over the loop momentum. In other words, let's go back from the one-loop formula (16) to eq. (10) for the zero-point energy

of the  $\Phi$  field, thus

$$\text{net } V_{\text{eff}}^{1\text{loop}}(\langle\varphi\rangle) = \Delta\mathcal{E} = \frac{1}{2} \int \frac{d^4 p_E}{(2\pi)^4} \log \frac{M^2 + p_E^2}{m_0^2 + p_E^2}. \quad (22)$$

The momentum integral here diverges quadratically, but we may reduce the divergence by taking derivatives WRT  $M^2$ . Indeed,

$$\begin{aligned} F(M^2) &= \int \frac{d^4 p_E}{(2\pi)^4} \log \frac{M^2 + p_E^2}{m_0^2 + p_E^2} && \text{diverges as } \Lambda^2, \\ \frac{dF}{dM^2} &= \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{M^2 + p_E^2} && \text{diverges as } \Lambda^2, \\ \frac{d^2 F}{(dM^2)^2} &= \int \frac{d^4 p_E}{(2\pi)^4} \frac{-1}{(M^2 + p_E^2)^2} && \text{diverges as } \log \Lambda, \\ \frac{d^3 F}{(dM^2)^3} &= \int \frac{d^4 p_E}{(2\pi)^4} \frac{2}{(M^2 + p_E^2)^3} && \text{converges.} \end{aligned} \quad (23)$$

Specifically,

$$\frac{d^3 F}{(dM^2)^3} = \int \frac{d^4 p_E}{(2\pi)^4} \frac{2}{(M^2 + p_E^2)^3} = \frac{1}{16\pi^2} \int_0^\infty \frac{2p_E^2 dp_E^2}{(M^2 + p_E^2)^3} = \frac{1}{16\pi^2} \times \frac{1}{M^2} \quad (24)$$

and hence

$$F(M^2) = \frac{M^4}{32\pi^2} \times \log \frac{M^2}{m_0^2} + A \times (M^4 - m_0^2) + B \times (M^2 - m_0^2) \quad (25)$$

for some divergent constants  $A$  and  $B$ . Consequently,

$$V_{\text{eff}}^{1\text{loop}}(\langle\varphi\rangle) = \frac{1}{64\pi^2} \times \left( m_0^2 + \frac{g\langle\varphi\rangle^2}{2} \right)^2 \times \log \left( 1 + \frac{g\langle\varphi\rangle^2}{2m_0^2} \right) + \frac{a}{24} \times \langle\varphi\rangle^4 + \frac{b}{2} \times \langle\varphi\rangle^2. \quad (26)$$

for some related constants  $a$  and  $b$ .

To calculate these constants we go back to the Feynman diagram expansion (16) and note that only the one-propagator and two-propagator diagrams suffer from the UV divergence. Thus, we identify

$$\frac{b}{2} \times \langle \varphi \rangle^2 = \text{Diagram 1} \quad \text{and} \quad \frac{a}{24} \times \langle \varphi \rangle^4 = \text{Diagram 2} \quad (27)$$

and hence

$$b = \frac{g}{32\pi^2} \left( \Lambda^2 - m_0^2 \log \frac{\Lambda^2}{m_0^2} + \text{finite} \right), \quad a = -\frac{3g^2}{32\pi^2} \left( \log \frac{\Lambda^2}{m_0^2} + \text{finite} \right). \quad (28)$$

Note that the diagrams (27) look exactly like the diagrams that renormalize the mass<sup>2</sup> and the self-coupling  $\lambda_\varphi$  of the field  $\varphi$ , so their divergences must be canceled by the counterterms  $\delta_{(\varphi)}^m$  and  $\delta_{(\varphi)}^\lambda$ . The same counterterms also appear in the effective potential as

$$V_{\text{eff}} \supset = \text{Diagram 3} + \text{Diagram 4} = \delta_{(\varphi)}^m \times \frac{\langle \varphi \rangle^2}{2} + \delta_{(\varphi)}^\lambda \times \frac{\langle \varphi \rangle^4}{24}, \quad (29)$$

so they cancel the divergences of the  $a$  and  $b$ . Working through the finite parts of the counterterms (never mind the details), we finally arrive at the *Coleman–Weinberg effective potential* for the two-scalar model.

$$V_{\text{eff}}(\langle \varphi \rangle) = \frac{1}{64\pi^2} \times \left( m_0^2 + \frac{g \langle \varphi \rangle^2}{2} \right)^2 \times \log \left( 1 + \frac{g \langle \varphi \rangle^2}{2m_0^2} \right) + \left( \frac{\lambda_\varphi}{24} - \frac{3g^2}{512\pi^2} \right) \times \langle \varphi \rangle^4 + \left( \frac{m_\varphi^2}{2} - \frac{gm_0^2}{128\pi^2} \right) \times \langle \varphi \rangle^2. \quad (30)$$

In a general quantum field theory, the scalar VEV  $\langle \varphi \rangle$  affects the masses of many particles



of different spins, including scalars, vectors, spinors, *etc.* The general form of the Coleman–Weinberg effective potentials for such theories is

$$V = \sum_{\substack{\text{massive} \\ \text{particles}}} (2S + 1)(-1)^{2S} \times \frac{M^4(\langle\varphi\rangle)}{64\pi^2} \times \log \frac{M^2(\langle\varphi\rangle)}{\mu^2} + \frac{\tilde{\lambda}_\varphi \langle\varphi\rangle^4}{24} + \frac{\tilde{m}_\varphi^2 \langle\varphi\rangle^2}{2} \quad (31)$$

where  $S$  is the particle’s spin and  $M(\langle\varphi\rangle)$  is its VEV-dependent mass. Also,  $\tilde{\lambda}_\varphi$  is the classical self-coupling of the scalar field  $\varphi$  plus a finite  $O(g^2)$  correction, and likewise for the scalar mass  $\tilde{m}_\varphi^2$ . Finally,  $\mu$  is an arbitrary mass scale for measuring particles’ masses; a redefinition of  $\mu$  can be canceled by a suitable redefinition of the  $\tilde{\lambda}_\varphi$  and  $\tilde{m}_\varphi^2$ .

### COLEMAN–WEINBERG EFFECT

In the ground state of a quantum field theory, the vacuum expectation values  $\langle\varphi\rangle$  of scalar fields minimize the net energy density of the theory. Presumably there are no field gradients in the vacuum states, so the net energy density is the *effective potential*  $V_{\text{eff}}(\langle\varphi\rangle)$ , including the classical potential as well as the quantum corrections from the loop diagrams. And sometimes, the minima of this effective potential has different symmetries than what we would expect from the classical potential.

As an example, consider a massless charged scalar field  $\phi$  coupled to the EM field  $A^\mu$ ,

$$\mathcal{L}_{\text{phys}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^*D^\mu\phi - \frac{\lambda}{4}\phi^*\phi^2. \quad (32)$$

The classical scalar potential  $V_{\text{cl}} = \frac{\lambda}{4}(\phi^*\phi)^2$  has a unique minimum at  $\phi = 0$ , so we expect the  $U(1)$  gauge symmetry of the theory to remain unbroken. However, we shall see in a moment that the Coleman–Weinberg effective potential can lead to  $\langle\phi\rangle \neq 0$  and hence to Higgsing of the gauge symmetry.

To see how this works, suppose that  $\phi$  somehow acquires a non-zero VEV  $\langle\phi\rangle = h/\sqrt{2}$ . Consequently, the photon gets a mass  $M_\gamma = h \times e$ , the imaginary component of the complex scalar field becomes the longitudinal polarization of the photon, while the real part of the

scalar field gives rise to the physical Higgs particle of mass  $M_H = h \times \sqrt{\lambda/2}$ . Plugging these masses into the Coleman–Weinberg effective potential, we get

$$V(h) = \frac{\lambda h^4}{16} + 3 \times \frac{e^4 h^4}{64\pi^2} \times \log \frac{(eh)^2}{\mu^2} + \frac{(\lambda^2/4)h^4}{64\pi^2} \times \log \frac{(\lambda/2)h^2}{\mu^2}. \quad (33)$$

Now suppose  $\lambda \sim e^4 \ll e^2$ . In this case,  $M_\gamma \gg M_H$  and the dominant contribution to the Coleman–Weinberg potential comes from the photon’s mass. Consequently,

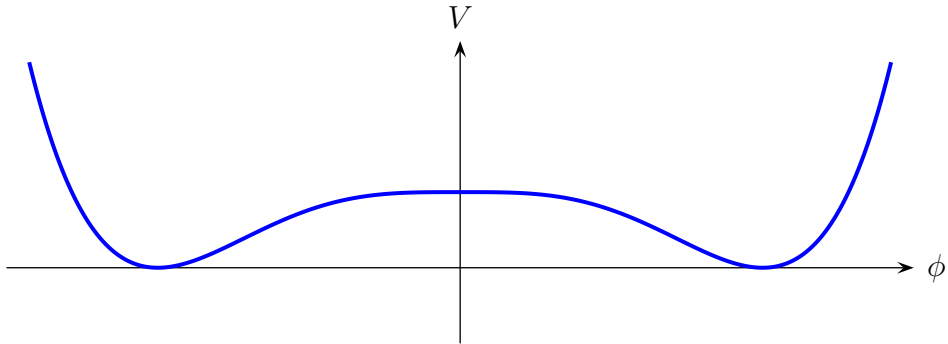
$$V(h) \approx \frac{\lambda h^4}{16} + 3 \times \frac{e^4 h^4}{64\pi^2} \times \log \frac{(eh)^2}{\mu^2} = \frac{3\alpha^2}{4} \times h^4 \times \left( \log \frac{e^2 h^2}{\mu^2} + \frac{\lambda}{12\alpha^2} \right), \quad (34)$$

which we may rewrite as

$$V(h) \approx \frac{3\alpha^2}{4} \times h^4 \times \left( \log \frac{h^2}{v^2} - \frac{1}{2} \right) \quad (35)$$

$$\text{for } v = \frac{\mu}{e} \times \exp \left( -\frac{1}{4} - \frac{\lambda}{24\alpha^2} \right). \quad (36)$$

The pictures below show this potential as a function of complex  $\langle \phi \rangle$ . Here is the profile along the real- $\phi$  axis:

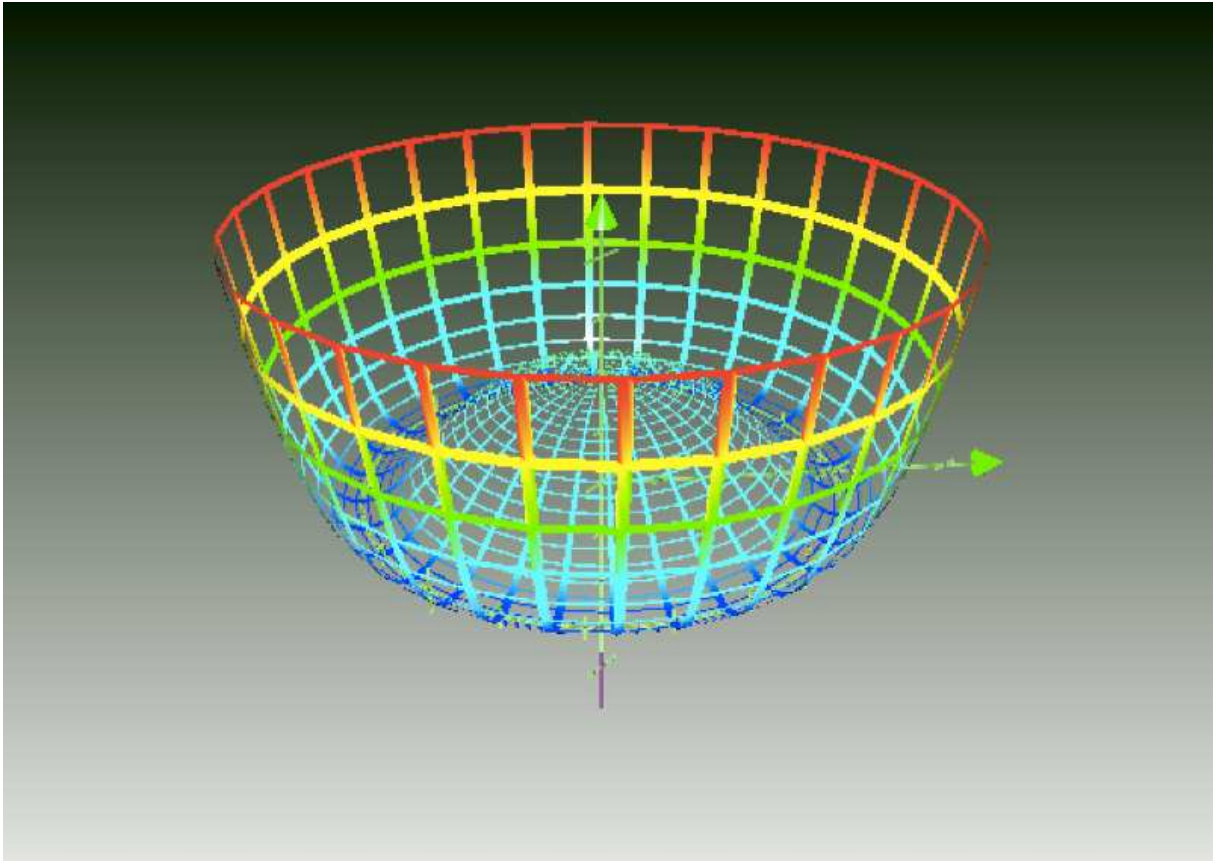


Note *local maximum* instead of minimum at  $\phi = 0$ , while the minima lie at  $\phi = \pm v/\sqrt{2}$ .

In the complex  $\phi$  plane, there is a whole ring of minima at

$$|\langle\phi\rangle|^2 = \frac{v^2}{2} > 0. \quad (37)$$

Indeed, here is the 3D picture of  $V(\phi)$ :



We see that the Coleman–Weinberg mechanism can lead to spontaneous symmetry breaking in a quantum theory even when the classical potential has a symmetric minimum. Moreover, this phenomenon is not limited to the classically massless Higgs scalars but persists for positive classical Higgs mass<sup>2</sup> as long as this positive mass<sup>2</sup> is small enough,

$$\text{mass}^2 < (3 \exp(-\frac{1}{2}))\alpha^2 v^2. \quad (38)$$

*Optional exercise to the students:* Show that in this case, the net potential (classical plus CW) has a local minimum at  $\langle\phi\rangle = 0$ , but the global minima form a ring of positive radius  $|\langle\phi\rangle| = O(v) > 0$ .

## FORMAL ANALYSIS

To complete these notes, let me give you a formal explanation of the relation between the effective potential and the vacuum diagrams, especially the multi-loop vacuum diagram. The key here are the field correlation functions

$$\mathcal{F}_n(x_1, \dots, x_n) = \langle \Omega | \mathbf{T} \hat{\Phi}_H(x_1) \cdot \hat{\Phi}_H(x_n) | \Omega \rangle. \quad (39)$$

As we have learned last week,

$$\mathcal{F}_n = \lim_T C(T) \times \sum \left[ \begin{array}{l} \text{all Feynman diagrams} \\ \text{with } n \text{ external vertices} \end{array} \right], \quad (40)$$

where the limit is  $T \rightarrow (+1 - i\epsilon) \times \infty$ ,

$$C(T) = \frac{\exp(2iT E_{\text{vac}})}{|\langle 0 | \Omega \rangle|^2}, \quad (41)$$

and in the coordinate-space Feynman rules for the diagrams we integrate each vertex location over

$$\int d^4z = \int_{-T}^{+T} dz^0 \int_{\text{whole space}} d^3\mathbf{z}. \quad (42)$$

In particular, for  $n = 0$

$$\begin{aligned} \mathcal{F}_0 &= \langle \Omega | \Omega \rangle = 1 \\ \text{but also} &= \lim_T C(T) \times \sum \left[ \begin{array}{l} \text{all Feynman diagrams} \\ \text{without external vertices} \end{array} \right], \end{aligned} \quad (43)$$

hence

$$\frac{1}{C(T)} = |\langle 0 | \Omega \rangle|^2 \times \exp(-2iT E_{\text{vac}}) \xrightarrow{T \rightarrow (+1 - i\epsilon)\infty} \sum \left[ \begin{array}{l} \text{all Feynman diagrams} \\ \text{without external vertices} \end{array} \right]. \quad (44)$$

Note: at this point, the sum on the RHS includes all the vacuum diagrams, connected and

disconnected. However, back in October we have learned that

$$\sum \left[ \text{all vacuum diagrams} \right] = \exp \left( \sum \left[ \begin{array}{c} \text{connected vacuum} \\ \text{diagrams only} \end{array} \right] \right), \quad (45)$$

*cf. my notes on Feynman rules* (pages 14–15). Plugging this formula into eq. (44) and taking the logarithm of both sides, we arrive at

$$\log \left( |\langle 0|\Omega\rangle|^2 \right) - 2iT E_{\text{vac}} \xrightarrow{T \rightarrow (+1-i\epsilon)\infty} \sum \left[ \begin{array}{c} \text{connected vacuum} \\ \text{diagrams only} \end{array} \right] \quad (46)$$

and hence

$$2 \operatorname{Re}(T) \times E_{\text{vac}} \xrightarrow{T \rightarrow (+1-i\epsilon)\infty} - \operatorname{Im} \left( \sum \left[ \begin{array}{c} \text{connected vacuum} \\ \text{diagrams only} \end{array} \right] \right). \quad (47)$$

Next, the coordinate-space Feynman rules include integration  $\int d^4z$  over all the internal vertex locations  $z_1, \dots, z_N$ , but for a vacuum diagram the integrand depends only on the differences  $z_i - z_j$ , so integral over one of the vertices yields simply the overall factor of

$$4\text{-volume}(\text{spacetime}) = 2T \times \text{volume}(\text{space}). \quad (48)$$

where the time duration is  $2T$  rather than  $\infty$  because of eq. (42). Consequently, modifying the Feynman rules for the vacuum diagrams to skip the  $\int d^4z$  integral over one of the vertices, we get

$$\frac{\text{vacuum energy}}{\text{space volume}} = - \lim_T \operatorname{Im} \left( \sum \left[ \text{connected vacuum diagrams} \right]_{\text{mod}} \right). \quad (49)$$

Note that the LHS of this formula is the vacuum energy *density*, which is precisely the effective potential we are seeking to calculate.

Finally, once we have dropped *one*  $\int d^4z$  integral from our Feynman rules, we can take the  $T \rightarrow (+1 - i\epsilon)\infty$  limit for each connected diagram, which means integrating the remaining vertex locations over the entire spacetime. This allows us to go from the coordinate-space

Feynman rules to the momentum-space Feynman rules in the usual way — with the Kirchhoff Law for the momenta flowing into every vertex. In the momentum space, the one  $\int d^4z$  vertex-location integral we have dropped translates to dropping the overall momentum-conservation factor

$$(2\pi)^4 \delta^{(4)}(\sum p_{\text{ext}}), \quad (50)$$

— which would be inappropriate for a vacuum diagram without any external momenta. Thus, the effective potential obtains as

$$V_{\text{eff}} = -\text{Im} \sum \left[ \begin{array}{c} \text{connected vacuum diagrams} \\ \text{without the (50) factor} \end{array} \right]. \quad (51)$$

*Quod erat demonstrandum.*