This exam has three problems, the first two problems about QED with charged scalars, and the third problem about spontaneous symmetry breaking.

Similar to the midterm exam, please do not waste time and paper by copying the homework solutions, or supplementary notes, or the textbook, or anything I have explicitly derived in class. Simply quote whichever formula you need and use it.

1. Let's extend the Quantum ElectroDynamics theory to include not only the photons γ and the electrons $e^{\mp}$ but also the scalar particles $S^{\mp}$. The Lagrangian of this extended theory is

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D}-m)\Psi + \; D_\mu\Phi^* D^\mu\Phi \;-\; M^2\Phi^*\Phi \;=\; \mathcal{L}^{\text{free}} + \mathcal{L}^{\text{int}}, \qquad (1)$$

where the quadratic part $\mathcal{L}^{\text{free}}$ describes the free fields while the cubic+quartic part $\mathcal{L}^{\text{int}}$ describes their interactions. In the Feynman rules, the propagators and the external lines follow from the $\mathcal{L}^{\text{free}}$ while the vertices follow from the $\mathcal{L}^{\text{int}}$. Altogether, the Feynman rules for QED with both electrons and charged scalars are as follows:

Photon propagator: $$A^\mu \underset{q\,\to}{\sim\!\sim\!\sim\!\sim} A^\nu \;=\; \frac{-ig^{\mu\nu}}{q^2+i0}\,, \qquad (\text{F.1})$$

Incoming photon: $$\sim\!\sim\!\sim\bullet \;=\; \mathcal{E}_\mu(k,\lambda), \qquad (\text{F.2})$$

Outgoing photon: $$\bullet\!\sim\!\sim\!\sim \;=\; \mathcal{E}^*_\mu(k,\lambda), \qquad (\text{F.3})$$

Electron propagator: $$\overline{\Psi} \underset{q\,\to}{\longrightarrow\!\!\!-\!\!\!-} \Psi \;=\; \frac{i}{\not{q}-m+i0}\,, \qquad (\text{F.4})$$

Incoming e^- or outgoing e^+: $$\longrightarrow\!\bullet \;=\; u(p,s) \text{ or } v(p,s), \qquad (\text{F.5})$$

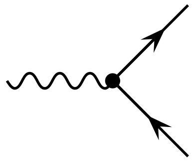
Outgoing e^- or incoming e^+: $$\bullet\!\longrightarrow \;=\; \bar{u}(p,s) \text{ or } \bar{v}(p,s), \qquad (\text{F.6})$$

Scalar propagator: $$\Phi^* \underset{q\,\to}{\cdots\cdots\blacktriangleright\cdots\cdots} \Phi \;=\; \frac{i}{q^2-M^2+i0}\,, \qquad (\text{F.7})$$

Incoming S^- or outgoing S^+: $$\cdots\blacktriangleright\cdots\bullet \;=\; 1, \qquad (\text{F.8})$$

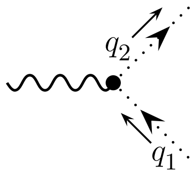
Outgoing S^- or incoming S^+: $$\bullet\cdots\blacktriangleright\cdots \;=\; 1, \qquad (\text{F.9})$$

QED vertex $ee\gamma$:



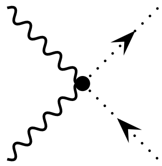
$$= +ie\gamma^\mu, \qquad (F.10)$$

Scalar QED vertex $SS\gamma$:



$$= +ie(q_1 + q_2)^\mu \qquad (F.11)$$

Seagull vertex $SS\gamma\gamma$:



$$= +2ie^2\, g^{\mu\nu}. \qquad (F.12)$$

Note: the dotted lines (F.7–9) for the charged scalars have arrows. Also note that in the $S^- S^+ \gamma$ vertex (F.11), the directions of momenta q_1 and q_2 must agree with the arrows of the scalar lines; otherwise, the vertex becomes $+ie(q_1 - q_2)^\mu$ or $+ie(-q_1 + q_2)^\mu$ or $+ie(-q_1 - q_2)^\mu$.

(a) The QED Feynman rules (F.1–6) and (F.10) were explained in class. Explain the remaining rules (F.7–9) and (F.11–12) in terms of the Lagrangian (1).
Note: don't re-derive the Feynman rules as such, just explain why the scalar propagators and external lines have arrows, why do those arrows point as in eqs. (F.7–9), and why the vertices are as in eqs. (F.11–12).

(b) Given the Feynman rules, draw the tree diagram(s) for the scalar pair production $e^- e^+ \to S^- S^+$ and calculate the tree-level amplitude $\left\langle S^-, S^+ \middle| \mathcal{M} \middle| e^-, e^+ \right\rangle$.
Hint: Mind the arrow directions on the dotted lines of scalars.

(c) Average $|\mathcal{M}|^2$ over the incoming particles' spins and calculate the partial cross-section for the scalar pair production.
For simplicity, neglect the electron's mass m. But don't neglect the scalar's mass M.

(d) Compare the angular dependence of the scalar pair production's partial cross-section with that of the muon pair production's we have studied in class. Also, calculate the total cross-section $\sigma_{\rm tot}(e^-e^+ \to S^-S^+)$ and compare its energy dependence to that of the $\sigma_{\rm tot}(e^-e^+ \to \mu^-\mu^+)$.

2. Continuing the first problem about QED with charged scalars, consider the annihilation of a scalar and an anti-scalar into two photons, $S^+S^- \to \gamma\gamma$.

   (a) Draw and evaluate **all** tree diagrams contributing to the $\langle\gamma\gamma|\,\mathcal{M}\,|S^+S^-\rangle$ amplitude. Make sure the amplitude respects the Bose symmetry between the two photons.

   (b) Write the tree amplitude as $\mathcal{M} = \mathcal{M}^{\mu\nu} \times \mathcal{E}^*_\mu(k_1,\lambda_1)\mathcal{E}^*_\nu(k_2,\lambda_2)$ and verify the Ward identities

   $$k_1^\mu \times \mathcal{M}_{\mu\nu} \;=\; 0, \qquad k_2^\nu \times \mathcal{M}_{\mu\nu} \;=\; 0. \tag{2}$$

   Hint: If these identities seem to be broken, go back to part (a) and make sure you have not missed a diagram. If this does not help, check your signs.

   (c) Sum $|\mathcal{M}|^2$ over the outgoing photon polarizations and calculate the partial cross-section of the $S^+S^- \to \gamma\gamma$ annihilation.

   - For simplicity, assume $E \gg M$ and neglect the scalar mass M in your calculations.

   ⋆ Extra credit if you do take M into account, and do it right. But beware: the kinematics is much messier for $M \neq 0$, and you might need several hours just to work through the algebra. If you use Mathematica, make sure to liberally comment your code. Also, if you sum over photon polarizations using a different method from what I have explained in class, you must prove that your method works.

3. Finally, consider an $SU(2)$ gauge theory with the following matter fields: a doublet of left-handed Weyl spinors $\psi^i_L(x)$, a doublet of right-handed Weyl spinors $\psi^i_R(x)$, and a triplet of *complex* scalar field $\Phi^a(x)$. The Lagrangian of the theory is

$$\begin{aligned}\mathcal{L} \;=\; & -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; D_\mu\Phi^{*a}D^\mu\Phi^a \;+\; i\psi^\dagger_{Li}\bar\sigma^\mu D_\mu\psi^i_L \;+\; i\psi^\dagger_{Ri}\sigma^\mu D_\mu\psi^i_R \\ & -\; y\Phi^a \times \psi^\dagger_{Ri}\left(\tau^a\right)^i{}_j\psi^j_L \;-\; y\Phi^{*a} \times \psi^\dagger_{Lj}\left(\tau^a\right)^j{}_i\psi^i_R \;-\; V(\Phi,\Phi^*),\end{aligned} \tag{3}$$

where τ^a are the isospin Pauli matrices, y is the Yukawa coupling, the $SU(2)$ gauge fields

$A^a_\mu(x)$ and the gauge coupling g are hiding inside

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - g\epsilon^{abc}A^b_\mu(x)A^c_\nu(x), \tag{4}$$

$$D_\mu\psi^i_L(x) = \partial_\mu\psi^i_L(x) + \frac{ig}{2}A^a_\mu(x)\big(\tau^a\big)^i{}_j\psi^j_L(x), \tag{5}$$

$$D_\mu\psi^i_R(x) = \partial_\mu\psi^i_R(x) + \frac{ig}{2}A^a_\mu(x)\big(\tau^a\big)^i{}_j\psi^j_R(x), \tag{6}$$

$$D_\mu\Phi^a(x) = \partial_\mu\Phi^a(x) - g\epsilon^{abc}A^b_\mu(x)\Phi^c(x), \tag{7}$$

$$D_\mu\Phi^{*a}(x) = \partial_\mu\Phi^{*a}(x) - g\epsilon^{abc}A^b_\mu(x)\Phi^{*c}(x), \tag{8}$$

and V is the scalar potential

$$V = \lambda\times\big(\Phi^{*a}\Phi^a\big)^2 + \lambda'\times\sum_a\big|\epsilon^{abc}\Phi^{*b}\Phi^c\big|^2 + m^2\times\Phi^{*a}\Phi^a \tag{9.a}$$

$$= (\lambda+\lambda')\times\big(\Phi^{*a}\Phi^a\big)^2 - \lambda'\times\big|\Phi^a\Phi^a\big|^2 + m^2\times\Phi^{*a}\Phi^a. \tag{9.b}$$

(a) Show that besides the local $SU(2)$ symmetry, the Lagrangian (3) has a $U(1)\times U(1)$ global symmetry. For each $U(1)$ factor, spell out the charges of the scalar and the fermionic fields.

To keep the scalar potential (9) bounded from below, we must have $\lambda>0$ and $\lambda'+\lambda>0$; however, the λ' and the m^2 parameters may have either sign. Accordingly, the theory has 3 distinct regimes with different scalar VEVs and different patterns of spontaneous symmetry breaking:

(**I**) For $m^2>0$ and either sign of λ', the scalar potential has a unique minimum at $\langle\Phi\rangle=0$.

(**II**) For $m^2<0$ and $\lambda'>0$, the potential has a degenerate family of minima, which are related by the continuous symmetries (local or global) to

$$\langle\Phi\rangle = \frac{v}{\sqrt{2}}\times\begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad v = \sqrt{\frac{-m^2}{\lambda}}. \tag{10}$$

(**III**) For $m^2<0$ and $\lambda'<0$ (but $\lambda'+\lambda>0$), the potential has a different degenerate family of minima, which are related by the continuous symmetries to

$$\langle\Phi\rangle = \frac{v}{2} \times \begin{pmatrix} +1 \\ -i \\ 0 \end{pmatrix}, \qquad v = \sqrt{\frac{-m^2}{\lambda+\lambda'}}. \tag{11}$$

(b) Verify these minima for each of the regimes of the theory.
Hint#1: a complex isovector Φ^a of the $SU(2) \cong SO(3)$ symmetry is equivalent to two independent real isovectors $\operatorname{Re}\Phi^a$ and $\operatorname{Im}\Phi^a$, but both of them are subject to the same $SO(3)$ gauge symmetries.
Hint#2: for $m^2 < 0$, first minimize the potential for a given $\Phi^{*a}\Phi^a = (v^2/2)$, and then minimize with respect to v.

In the regime **(I)** there is no spontaneous symmetry breaking, no Higgs mechanism, and the particle spectrum of the theory can be read directly from the Lagrangian (3). Your main task in this problem is to analyze the other two regimes **(II)** and **(III)**.

(c) For each of the regimes **(II)** and **(III)**, find which continuous symmetries — global or local — are spontaneously broken by the respective scalar VEVs (10) or (11), and which symmetries — or combinations of symmetries — remain unbroken.
Hint: in both regimes the unbroken continuous symmetry group is $U(1)_V \times U(1)_X$, but the $U(1)_X$ factors in the two regimes are quite different from each other.

(d) For each regime **(II)** and **(III)**, reorganize the vector and the scalar fields of the theory into linear combinations having definite charges with respect to the unbroken symmetries (global, local, or mixed). Then use the broken symmetries to predict which vector fields should become massive and which should remain massless, which scalar fields should be eaten by the Higgs mechanism, and which of the remaining scalar fields should be massless Goldstone bosons.

(e) Calculate the masses of the vector fields and check them against the predictions from part (d).

(f) Argue that the unitary gauge conditions for the problem at hand are:

$$\text{for regime } \mathbf{II}: \quad \operatorname{Re}\Phi^1 \;=\; \operatorname{Re}\Phi^2 \;=\; 0, \tag{12}$$

$$\text{for regime } \mathbf{III}: \quad \operatorname{Re}\Phi^3 \;=\; \operatorname{Im}\Phi^3 \;=\; 0, \quad \operatorname{Im}(\Phi^1 + i\Phi^2) \;=\; 0. \tag{13}$$

Then calculate the masses of the un-fixed scalar fields and check them against the predictions from part (d).

(g) Finally, consider the fermions. For each regime — **(II)** or **(III)** — diagonalize the fermionic mass matrix due to Yukawa couplings × scalar VEVs, then combine the left-handed and the right-handed Weyl fermions into Dirac fermions of definite masses and charges (with respect to the unbroken $U(1)_V \times U(1)_X$ symmetries).

Note: the ψ_L and the ψ_R components of the same *massive* Dirac fermion must have similar charges. But for a massless Dirac fermion, a symmetry may act in the axial or chiral fashion, hence different charges for the ψ_L and the ψ_R components.