Please do not waste time and paper by copying the posted homework solutions or supplementary notes. If you need to use any homework result, simply reference the appropriate question or equation and go ahead. Likewise, don't re-derive anything I derived in class.

1. The first problem is about tree-level gluon scattering, $gg \to gg$.
   (a) Draw all tree diagrams for this process. Use crossing symmetry to write the net amplitude as

$$\mathcal{M}(g_1^a, g_2^b, g_3^c, g_4^d) \;=\; G^{abcd} \times \mathcal{M}_s \;+\; G^{acdb} \times \mathcal{M}_t \;+\; G^{adbc} \times \mathcal{M}_u \tag{1}$$

   where the G^{abcd}, *etc.*, are group factors depending on the colors of the four gluons, while the $\mathcal{M}_s$, $\mathcal{M}_t$, and $\mathcal{M}_u$ amplitudes depend on the gluons' momenta and polarizations. Thanks to the crossing symmetry,

$$\mathcal{M}_s \;\equiv\; \mathcal{M}(1,2,3,4), \qquad \mathcal{M}_t \;\equiv\; \mathcal{M}(1,3,4,2), \qquad \mathcal{M}_u \;\equiv\; \mathcal{M}(1,4,2,3), \tag{2}$$

   for the same analytic function $\mathcal{M}$ applied to 3 different ordering of the four gluons. (For simplicity, treat all 4 gluons as incoming, $k_1 + k_2 + k_3 + k_4 = 0$.)

   (b) Show that group factor G^{abcd} has the same index symmetry as the Riemann tensor in gravity,

$$G^{abcd} \;=\; -G^{bacd} \;=\; -G^{abdc} \;=\; +G^{cdab}, \tag{3}$$

$$G^{abcd} \;+\; G^{acdb} \;+\; G^{adbc} \;=\; 0. \tag{4}$$

   Eqs. (3) should be obvious (if they are not, you have a wrong G^{abcd}), but eq. (4) takes some work. To prove it, use the identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

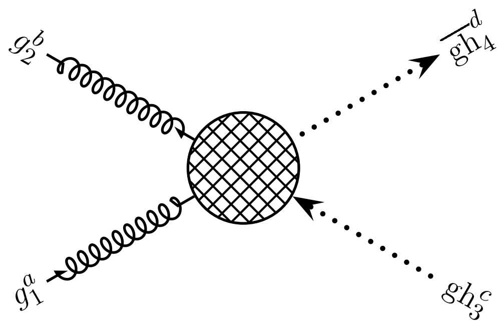
   (c) Sum / average the 4–gluon $|\text{amplitude}|^2$ over all the colors and show that

$$\overline{|\mathcal{M}|^2} \;=\; \frac{C^2(G)}{2\dim(G)} \times \left(3|\mathcal{M}_s|^2 \,+\, 3|\mathcal{M}_t|^2 \,+\, 3|\mathcal{M}_u|^2 \,-\, |\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u|^2\right). \tag{5}$$

(d) Prove the weak Ward identity for the 4–gluon amplitude (1): If one gluon — say, gluon#3 — has $e_3^\mu = k_3^\mu$ while the other three gluons are transverse, $(e_i k_i) = 0$ for $i = 1, 2, 4$, then $\mathcal{M} = 0$.

Hint: Show that in this case $\mathcal{M}_s = \mathcal{M}_t = \mathcal{M}_u$, then use eq. (4).

Let's change gears for a moment and consider the tree amplitude involving two gluons, a ghost, and an antighost,



(e) Draw all tree diagrams for this process, calculate the net amplitude, and bring it to the form

$$\widetilde{\mathcal{M}}_{\text{net}} \;=\; \widetilde{\mathcal{M}}_s \times G^{abcd} \;+\; \widetilde{\mathcal{M}}_t \times G^{acdb} \;+\; \widetilde{\mathcal{M}}_u \times G^{adbc} \tag{6}$$

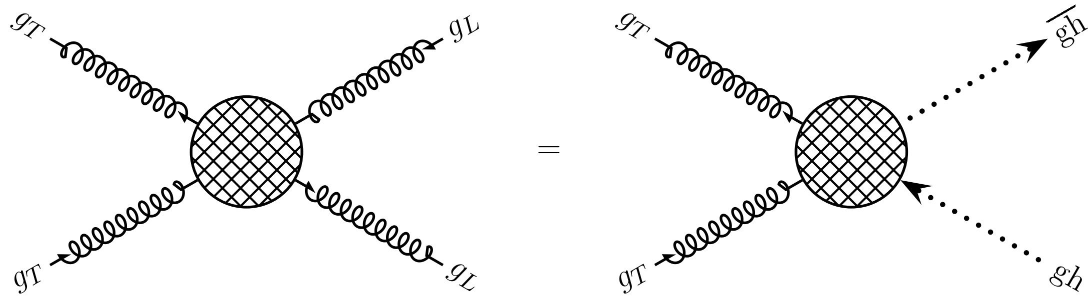
where G^{abcd} is the same function of the adjoint color indices as in parts (a–d), while $\widetilde{\mathcal{M}}_s$, $\widetilde{\mathcal{M}}_t$, and $\widetilde{\mathcal{M}}_u$ are some functions of the four particle's momenta and of the two gluon's polarizations. But unlike the 4-gluon amplitude, there are no crossing symmetries like eq. (2) for the $\widetilde{\mathcal{M}}_s$, $\widetilde{\mathcal{M}}_t$, and $\widetilde{\mathcal{M}}_u$.

Assume both gluons to be transversely polarized and treat all 4 momenta as incoming.

Now go back to the 4 gluon amplitude and suppose that only two gluons are transverse while the other two have unphysical polarizations (longitudinal or temporal). Or rather, let the two unphysical gluons have null polarization vectors, specifically

$$e_3^\mu \;=\; k_3^\mu \;=\; (\omega_3, \mathbf{k}_3), \quad e_4^\mu \;=\; \frac{(\omega_4, -\mathbf{k}_4)}{2\omega_4^2}, \quad e_3^2 \;=\; e_4^2 \;=\; 0, \quad e_3 k_3 \;=\; 0, \quad e_4 k_4 \;=\; +1. \tag{7}$$

(f) Show that in this case, the 4–gluon amplitude is exactly equal to the amplitude where the longitudinal gluons are replaced with a ghost and an antighost,



Hint: show that for two physical and two unphysical gluon polarizations as in eq. (7)

$$\mathcal{M}_s \;=\; \mathcal{M}_c \,+\, \widetilde{\mathcal{M}}_s\,, \quad \mathcal{M}_t \;=\; \mathcal{M}_c \,+\, \widetilde{\mathcal{M}}_t\,, \quad \mathcal{M}_u \;=\; \mathcal{M}_c \,+\, \widetilde{\mathcal{M}}_u\,, \tag{8}$$

for the same $\mathcal{M}_c$ in all 3 cases, while the $\widetilde{\mathcal{M}}_s$, $\widetilde{\mathcal{M}}_t$, and $\widetilde{\mathcal{M}}_u$ are exactly as in part (e). Then use eq. (4) similarly to part (d).

Finally, let's calculate the amplitudes (2) and the partial cross-section for the four transverse gluons. For simplicity, work in the center-of-mass frame and use linear polarizations for each gluon, either $\|$ to the plane of scattering or $\perp$ to it. For the set of 4 gluons there are 16 choices of such polarizations, but the symmetries forbid some combinations and relate other combinations to each other.

(g) Spell out which polarized $gg \to gg$ processes are forbidden and which are allowed. Write down the symmetry relations between the allowed processes. How many of them are independent?

(h) Calculate the amplitudes (2) and the partial cross-section for the simplest choice of polarizations: all 4 gluons are polarized $\perp$ to the scattering plane.

($\star$) Optional exercise, for extra credit:

Calculate the partial cross-sections for the other independent polarizations.

Warning: such amplitudes involve much messier algebra than the all–$\perp$ case (h), so use Mathematica or calculate them numerically as functions of the scattering angle θ. If you try to calculate them by hand, you are liable to make more algebraic mistakes then you can fix during the time available for this exam.

2. Three exams ago — in the Fall 2022 midterm — you saw topologically massive gauge fields in $2+1$ spacetime dimensions. Let me refresh your memory of how they work: In the abelian case, the Lagrangian is

$$\mathcal{L} \;=\; -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \frac{M}{4}\,\epsilon_{\lambda\mu\nu}A^{\lambda}F^{\mu\nu} \tag{9}$$

where the second term — called the *topological mass term* — is not gauge invariant, but its variance is a total derivative, so the action $\int d^3x\,\mathcal{L}$ is gauge invariant. Also, the topological mass term breaks the parity symmetry, so the massive photons have nontrivial $SO(2)$ spins: $m_s = +1$ (and only $m_s = +1$) for $M > 0$, or $m_s = -1$ (and only $m_s = -1$) for $M < 0$.

The non-abelian version of the topological mass term is the *Chern–Simons* term

$$\frac{k}{4\pi}\,\epsilon^{\lambda\mu\nu}\,\mathrm{tr}\left(\mathcal{A}_{\lambda}\,\partial_{\mu}\mathcal{A}_{\nu} \;+\; \frac{2i}{3}\,\mathcal{A}_{\lambda}\mathcal{A}_{\mu}\mathcal{A}_{\nu}\right) \tag{10}$$

where $\mathcal{A}_{\mu}(a) = gA^{a}_{\mu}(x)T^{a}$ is the (non-canonical) matrix-valued gauge field. In terms of this field and its tension $\mathcal{F}_{\mu\nu}(x) = gF^{a}_{\mu\nu}(x)$, the net Lagrangian of the *topologically massive Yang–Mills theory* in 3D is

$$\begin{aligned}\mathcal{L} \;=\; \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{CS}} \;&=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \frac{k}{4\pi}\,\epsilon^{\lambda\mu\nu}\,\mathrm{tr}\left(\mathcal{A}_{\lambda}\,\partial_{\mu}\mathcal{A}_{\nu} \;+\; \frac{2i}{3}\,\mathcal{A}_{\lambda}\mathcal{A}_{\mu}\mathcal{A}_{\nu}\right)\\ &=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \frac{k}{8\pi}\,\epsilon^{\lambda\mu\nu}\,\mathrm{tr}\left(\mathcal{A}_{\lambda}\,\mathcal{F}_{\mu\nu} \;-\; \frac{2i}{3}\,\mathcal{A}_{\lambda}\mathcal{A}_{\mu}\mathcal{A}_{\nu}\right).\end{aligned} \tag{11}$$

The mass of the gluons in this theory is $M = kg^2/4\pi$; note that g^2 has dimensionality of mass in 3D, so the k coefficient — called the *Chern–Simons level* — is dimensionless. In fact, k must be integer (positive, negative, or zero) to assure the gauge invariance of the e^{iS} — and hence of the path integral of the quantum theory — despite the gauge dependence of the Chern–Simons term itself. Indeed, you should have seen 3 exams ago that the YM + CS action $\int d^3x\,\mathcal{L}$ is invariant under *infinitesimal* gauge transforms, but under a finite gauge transform $U(x)$ it changes by

$$\Delta S \;=\; \frac{-k}{12\pi}\int d^3x\,\epsilon^{\lambda\mu\nu}\;\mathrm{tr}\Big(U^{-1}\partial_{\lambda}U\cdot U^{-1}\partial_{\mu}U\cdot U^{-1}\partial_{\nu}U\Big). \tag{12}$$

The integral here depends only on the topological properties of $U(x)$ and its values are always integer $\times\, 24\pi^2$, hence for an integer k — and only for an integer k — e^{iS} is gauge invariant.

In this exam, we shall focus on the quantum origin of the Chern–Simons term rather than its effects on the semi-classical gauge fields. Specifically, we are going to *induce* an effective Chern–Simons term for the gluons in the 3D QCD from the loop diagrams involving massive quarks. For simplicity, let's start with the 3D $SU(N)$ gauge theory with a single fundamental multiplet $\mathbf{N}$ of quarks (*i.e.*, N colors, one flavor) and no tree-level CS term, thus

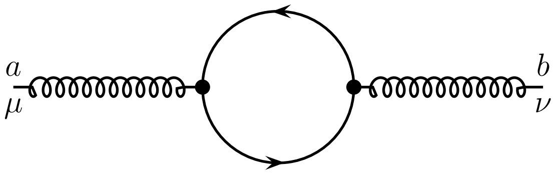
$$\mathcal{L}_{\rm phys} \;=\; \frac{-1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\big) \;+\; \overline{\Psi}(i\not{D}-m)\Psi. \tag{13}$$

For the sake of definiteness, assume $m>0$: it matters in 3D (or in any other odd spacetime dimension) because the sign of a Dirac fermion's mass breaks the Parity symmetry.

Beyond the tree level, parity violation in the quark sector yields parity-violating gluonic amplitudes via quark loop diagrams, so the gluon sector of the theory also becomes parity violating. Technically, this works through 3D Dirac matrices obeying $\gamma^0\gamma^1\gamma^2 = +i$, and similar relations in other odd dimensions. Consequently, 3D Dirac traces are different from their 4D analogues:

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= 2g^{\mu\nu},\\
\mathrm{tr}(\gamma^\lambda\gamma^\mu\gamma^\nu) &= 2i\epsilon^{\lambda\mu\nu} \;\neq\; 0,\\
\mathrm{tr}(\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu) &= 2(g^{\kappa\lambda}g^{\mu\nu} - g^{\kappa\mu}g^{\lambda\nu} + g^{\kappa\nu}g^{\lambda\mu}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{14}$$

(a) Evaluate the one loop diagram



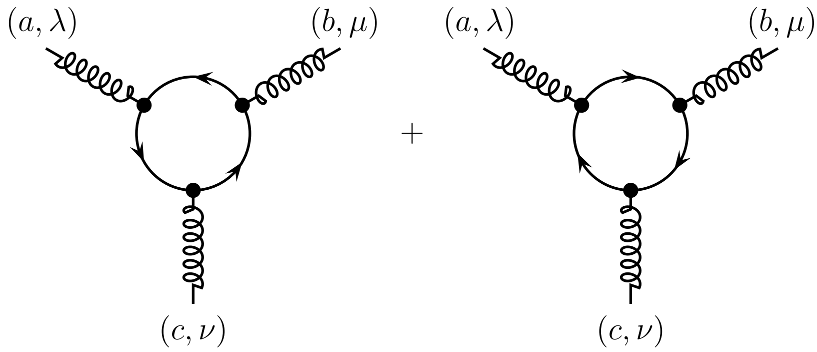
(15)

and show that for *small* gluon momenta $|k| \ll m$, it yields

$$\Sigma^{\mu a,\nu b}_{\psi\,\mathrm{loop}}(p) \;=\; \frac{g^2\delta^{ab}}{8\pi}\left(-ik_\lambda\epsilon^{\lambda\mu\nu} \;+\; \frac{k^\mu k^\nu - g^{\mu\nu}k^2}{3m} \;+\; O\left(\frac{k^3}{m^2}\right)\right). \tag{16}$$

(b) Similarly, show that for three external gluons with small momenta (compared to the fermion's mass m), the one-loop 1PI amplitude is

$$iV^{abc}_{\lambda\mu\nu} \ = \qquad + \tag{17}$$



$$= \ -\frac{ig^3}{8\pi}\, f^{abc}\epsilon_{\lambda\mu\nu} \ + \ O\left(\frac{k}{m}\right).$$

(c) Show that for quark loops with four or more external gluons with small momenta, all the one-quark-loop amplitudes are suppressed by negative powers of the quark mass m.

Now consider the Functional Integral for the $d = 3$ QCD. Let us integrate $\iint D[\Psi(x)]$ $\iint D[\overline{\Psi}(x)]$ over the quark fields for fixed gauge fields $A^a_\mu(x)$. The result of this integration is an effective quantum theory of the gauge fields with Minkowski-space action

$$S[\mathcal{A}_\mu] \ = \ S_{\rm YM}[\mathcal{A}_\mu] \ + \ \Delta S[\mathcal{A}_\mu] \tag{18}$$

where

$$i\Delta S[\mathcal{A}_\mu] \ = \ \log {\rm Det}(i\not{D} - m) \ = \ {\rm Tr}\log(i\not{D} - m). \tag{19}$$

(d) Expand the ΔS here into Feynman diagrams, then use the results of questions (a–c) to show that in the large quark mass limit $m \to +\infty$,

$$\Delta S \ = \ \frac{1}{8\pi}\int d^3x \left\{ \epsilon^{\lambda\mu\nu}\,{\rm tr}\left(\mathcal{A}_\lambda\,\partial_\mu\mathcal{A}_\nu \ + \ \tfrac{2i}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu\right) \ + \ O\left(\frac{1}{m}\right)\right\}. \tag{20}$$

Hence, the effective low-energy quantum theory for the gluons is precisely the topologically massive Yang–Mills theory (11) with Chern–Simons level $k = +\frac{1}{2}$.

Since the half-integral Chern–Simons level $k = +\frac{1}{2}$ would break the gauge invariance of the quantum theory, let's consider a more general example. Namely, 3D QCD with several flavors of massive quarks, some with $m_f > 0$ and some with $m_f < 0$ (in 3D, this makes a difference). Let's also have a tree-level Chern–Simons level k_0.

(e) Show that when we integrate out all the quarks, we end up with the net Chern–Simons level

$$k \;=\; k_0 \;+\; \frac{\#(m_f>0)\;-\;\#(m_f<0)}{2}\,. \tag{21}$$

Note: consistency of the quantum theory requires *an integer net CS level* k rather than an integer tree-level k_0. Consequently, in theories with even N_f, the tree-level k_0 should be integer, but the theories with odd N_f should have half-integer $k_0 \in \mathbf{Z}+\frac{1}{2}$.