1. The first problem is an easy exercise followed by a reading assignment:

   (a) Given the bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] = 0, \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha,\beta}\,, \tag{1}$$

   calculate the commutators

$$[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma], \quad [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta], \quad [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta], \quad \text{and} \quad [\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu]. \tag{2}$$

   (b) And now read my notes on second quantization of bosons, especially the second half (pages 15–29) where I prove all the lemmas. And by the way, some of the lemmas use the commutators from part (a) of this problem.

⋆ The rest of this homework (problems 2–5) concern the *massive* relativistic vector field $A^\mu(x)$ your have already seen in the previous homework (homework set#1, problem 1). The classical Lagrangian density (in $\hbar = c = 1$ units) for this field is

$$\mathcal{L} = -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2\, A_\mu A^\mu - A^\mu J_\mu \tag{3}$$

where the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Because of the mass term, the Lagrangian (3) is *not* gauge invariant. However, we *assume* that the current $J^\mu(x)$ is conserved, $\partial_\mu J^\mu(x) = 0$.

2. In this problem, we develop the classical Hamiltonian formalism for the $A^\mu(x)$ field. Our first step is to identify the canonically conjugate "momentum" fields.

   (a) Show that $\partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\mathbf{E}$ but $\partial\mathcal{L}/\partial\dot{A}_0 \equiv 0$.

In other words, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H = \int d^3\mathbf{x}\left(-\dot{\mathbf{A}}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) - \mathcal{L}\right). \tag{4}$$

(b) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$, and A_0 fields, and their *space* derivatives,

$$H = \int d^3\mathbf{x}\left\{\tfrac{1}{2}\mathbf{E}^2 + A_0\left(J_0 - \nabla\cdot\mathbf{E}\right) - \tfrac{1}{2}m^2A_0^2 + \tfrac{1}{2}\left(\nabla\times\mathbf{A}\right)^2 + \tfrac{1}{2}m^2\mathbf{A}^2 - \mathbf{J}\cdot\mathbf{A}\right\}. \tag{5}$$

Because the A_0 field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x},t)$ to the values of other fields *at the same time* t, specifically

$$\begin{pmatrix}\text{variational}\\ \text{derivative}\end{pmatrix} \quad \frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \left.\frac{\partial\mathcal{H}}{\partial A_0}\right|_{\mathbf{x}} - \nabla\cdot\left.\frac{\partial\mathcal{H}}{\partial(\nabla A_0)}\right|_{\mathbf{x}} = 0. \tag{6}$$

At the same time, the vector fields $\mathbf{A}$ and $\mathbf{E}$ satisfy the Hamiltonian equations of motion,

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\left.\frac{\delta H}{\delta\mathbf{E}(\mathbf{x})}\right|_t \equiv -\left[\frac{\partial\mathcal{H}}{\partial\mathbf{E}} - \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{E})}\right]_{(\mathbf{x},t)}, \tag{7}$$

$$\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) = +\left.\frac{\delta H}{\delta\mathbf{A}(\mathbf{x})}\right|_t \equiv +\left[\frac{\partial\mathcal{H}}{\partial\mathbf{A}} - \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{A})}\right]_{(\mathbf{x},t)}. \tag{8}$$

(c) Write down the explicit form of all these equations.

(d) Verify that the equations you have just written down are equivalent to the relativistic Euler–Lagrange equations for the $A^\mu(x)$, namely

$$(\partial^\mu\partial_\mu + m^2)A^\nu = \partial^\nu(\partial_\mu A^\mu) + J^\nu \tag{9}$$

and hence for a conserved current J^μ,

$$\partial_\mu A^\mu(x) = 0 \quad\text{and}\quad (\partial^\nu\partial_\nu + m^2)A^\mu = 0, \tag{10}$$

cf. homework #1.

3. Next, let's quantize the massive vector field. Since classically the $-\mathbf{E}(\mathbf{x})$ fields are canonically conjugate momenta to the $\mathbf{A}(\mathbf{x})$ fields, the corresponding quantum fields $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ satisfy the canonical equal-time commutation relations

$$\begin{aligned}
[\hat{A}_i(\mathbf{x},t), \hat{A}_j(\mathbf{y},t)] &= 0, \\
[\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= 0, \\
[\hat{A}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= -i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})
\end{aligned} \tag{11}$$

(in the $\hbar = c = 1$ units). The currents also become quantum fields $\hat{J}^\mu(\mathbf{x},t)$, but they are composed of some kind of charged degrees of freedom independent from the vector fields in question. Consequently, *at equal times* the currents $\hat{J}^\mu(\mathbf{x},t)$ commute with both the $\hat{\mathbf{E}}(\mathbf{y},t)$ and the $\hat{\mathbf{A}}(\mathbf{y},t)$ fields.

The classical $A^0(\mathbf{x},t)$ field does not have a canonical conjugate and its equation of motion (6) does not involve time derivatives. The quantum $\hat{A}^0(\mathbf{x},t)$ field satisfies a similar time-independent constraint

$$m^2\hat{A}^0(\mathbf{x},t) = \hat{J}^0(\mathbf{x},t) - \nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t), \tag{12}$$

but from the Hilbert space point of view this is an operatorial identity rather than an equation of motion. Consequently, the commutation relations of the scalar potential field follow this identity and eqs. (11); in particular, at equal times the $\hat{A}^0(\mathbf{x},t)$ commutes with the $\hat{\mathbf{E}}(\mathbf{y},t)$ but does not commute with the $\hat{\mathbf{A}}(\mathbf{y},t)$.

Finally, the Hamiltonian operator follows from the classical Hamiltonian (5), namely

$$\begin{aligned}
\hat{H} &= \int\! d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \hat{A}_0\left(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}}\right) - \tfrac{1}{2}m^2\hat{A}_0^2 + \tfrac{1}{2}\left(\nabla\times\hat{\mathbf{A}}\right)^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right\} \\
&= \int\! d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \frac{1}{2m^2}\left(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}}\right)^2 + \tfrac{1}{2}\left(\nabla\times\hat{\mathbf{A}}\right)^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right\}
\end{aligned} \tag{13}$$

where the second line follows from the first line via eq. (12).

Your task is to calculate the commutators $[\hat{A}_i(\mathbf{x},t), \hat{H}]$ and $[\hat{E}_i(\mathbf{x},t), \hat{H}]$ and write down the Heisenberg equations for the quantum vector fields. Make sure those equations are similar to the classical Hamilton equations (7) and (8).

4. The quantum massive vector field is dual to theory of an arbitrary number of massive relativistic particles of spin 1. The duality works similarly to the scalar field explained in class, but involves some extra technical details we are going to work out in this problem. For simplicity, we shall focus on the free vector field, without any current sourcing it, thus let $\hat{J}^\mu(x) \equiv 0$. Also, in this problem we shall work in the Schrödinger rather than Heisenberg picture of the quantum mechanics.

In general, a QFT has a creation operator $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and an annihilation operator $\hat{a}_{\mathbf{k},\lambda}$ for each plane wave with momentum $\mathbf{k}$ and polarization λ. The massive vector fields have 3 independent polarizations corresponding to 3 orthogonal unit 3–vectors. One may use any basis of 3 such vectors $\mathbf{e}_\lambda(\mathbf{k})$, and it's often convenient to make them $\mathbf{k}$–dependent and complex; in the complex case, orthogonality+unit length mean

$$\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}^*_{\lambda'}(\mathbf{k}) \;=\; \delta_{\lambda,\lambda'}\,. \tag{14}$$

Of particular convenience is the helicity basis of eigenvectors of the vector product $i\mathbf{k}\times$, namely

$$i\mathbf{k}\times\mathbf{e}_\lambda(\mathbf{k}) \;=\; \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k}), \qquad \lambda = -1, 0, +1. \tag{15}$$

By convention, the phases of the complex helicity eigenvectors are chosen such that

$$\mathbf{e}_0(\mathbf{k}) \;=\; \frac{\mathbf{k}}{|\mathbf{k}|}\,, \quad \mathbf{e}^*_{\pm1}(\mathbf{k}) \;=\; -\mathbf{e}_{\mp1}(\mathbf{k}), \quad \mathbf{e}_\lambda(-\mathbf{k}) \;=\; -\mathbf{e}^*_\lambda(+\mathbf{k}), \tag{16}$$

for example, for $\mathbf{k}$ pointing in the positive z direction

$$\mathbf{e}_{+1}(\mathbf{k}) \;=\; \tfrac{1}{\sqrt{2}}\big(+1,+i,0\big), \quad \mathbf{e}_{-1}(\mathbf{k}) \;=\; \tfrac{1}{\sqrt{2}}\big(-1,+i,0\big), \quad \mathbf{e}_0(\mathbf{k}) \;=\; (0,0,1). \tag{17}$$

(a) As a first step towards constructing the $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ operators, we Fourier transform the vector fields $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ and then decompose the vectors $\hat{\mathbf{A}}_\mathbf{k}$ and $\hat{\mathbf{E}}_\mathbf{k}$ into helicity components,

$$\begin{aligned}
\hat{\mathbf{A}}(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{A}_{\mathbf{k},\lambda}\,, \qquad \hat{A}_{\mathbf{k},\lambda} \;=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}(\mathbf{x}),\\
\hat{\mathbf{E}}(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{E}_{\mathbf{k},\lambda}\,, \qquad \hat{E}_{\mathbf{k},\lambda} \;=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{E}}(\mathbf{x}).
\end{aligned} \tag{18}$$

Show that $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$ and $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$. Also, convert the commutations relations (11) for the $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ fields to the commutation relations for the mode operators $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$.

(b) Show that for the free fields with $\hat{J}^\mu(x) \equiv 0$, the Hamiltonian operator (13) expands into the mode operators as

$$\begin{aligned} \hat{H} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \left(\frac{C_{\mathbf{k},\lambda}}{2} \hat{E}^\dagger_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda} \;+\; \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}} \hat{A}^\dagger_{\mathbf{k},\lambda} \hat{A}_{\mathbf{k},\lambda} \right) \\ \text{where} \quad \omega_{\mathbf{k}} &= \sqrt{\mathbf{k}^2 + m^2}, \\ \text{and} \quad C_{\mathbf{k},\lambda} &= \begin{cases} \omega^2_{\mathbf{k}}/m^2 & \text{for } \lambda = 0, \\ 1 & \text{for } \lambda = \pm 1. \end{cases} \end{aligned} \tag{19}$$

(c) Define the annihilation and the creation operators according to

$$\hat{a}_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}} \hat{A}_{\mathbf{k},\lambda} \;-\; iC_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}, \qquad \hat{a}^\dagger_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}} \hat{A}^\dagger_{\mathbf{k},\lambda} \;+\; iC_{\mathbf{k},\lambda} \hat{E}^\dagger_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}, \tag{20}$$

and verify that they satisfy the relativistically normalized bosonic commutation relations. (At equal times or in the Schrödinger picture.)

(d) Re-express the field-mode operators $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ in terms of the annihilation and creation operators (20), plug them into the Hamiltonian (19), and show that

$$\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3\, 2\omega_{\mathbf{k}}} \sum_\lambda \omega_{\mathbf{k}}\, \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} \;+\; \text{const.} \tag{21}$$

Note: by a constant here I mean a c-number rather than an operator. It happens to be a badly divergent number, but that's OK.

(e) Argue that the Hamiltonian (21) describes an arbitrary number of identical bosons, each boson being a massive relativistic particle of mass m and having 3 distinct polarization states labeled by $\lambda = -1, 0, +1$.

5. Finally, let's go back to the Heisenberg picture of the quantum mechanics and consider the time-dependence of the quantum vector field.

(a) Use the results of problem 4 to show that

$$\hat{\mathbf{A}}(\mathbf{x},t) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{+i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(t) \;+\; e^{-i\mathbf{k}\cdot\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(t)\right) \tag{22}$$

(b) Solve the Heisenberg for the creation and annihilation operators, plug the solutions into eq. (22) and show that it becomes

$$\hat{\mathbf{A}}(\mathbf{x},t) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{-ikx}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \tag{23}$$

(c) Write down similar expansion for the electric field $\hat{\mathbf{E}}(\mathbf{x},t)$ and the scalar potential $\hat{A}^0(\mathbf{x},t)$; use eq. (12) for the latter.

(d) Combine the results of parts (b) and (c) into a relativistic formula

$$\hat{A}_\mu(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\sum_\lambda\left(e^{-ikx}f_\mu(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx}f^*_\mu(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}} \tag{24}$$

for the 4–vector field $\hat{A}^\mu(x)$, and write down explicit formulae for the polarization 4–vectors $f^\mu(\mathbf{k},\lambda)$.

(e) Check that these polarization vectors obey $k_\mu f^\mu(\mathbf{k},\lambda) = 0$ and use that to show that the quantum vector field $\hat{A}^\mu(x)$ obeys the classical equation of motion $\partial_\mu\hat{A}^\mu(x) = 0$. Also, show that the $\hat{A}^\mu(x)$ obeys the other classical equation of motion, $(\partial^2 + m^2)\hat{A}^\mu(x) = 0$.