1. The ordinary quantum mechanics of a single relativistic particle — or any fixed number of relativistic particles — violates the relativistic causality by allowing particles to move faster than light. In this problem, we shall see how this works for the simplest case of a single free relativistic spinless particle with the Hamiltonian

$$\hat{H} \;=\; +\sqrt{m^2+\hat{\mathbf{P}}^2} \tag{1}$$

(in the $c=\hbar=1$ units). By general rules of quantum mechanics, the amplitude $U(x\to y)$ for this particle to propagate from point $\mathbf{x}$ at time x^0 to point $\mathbf{y}$ at time y^0 obtains from the Hamiltonian (1) as

$$U(x\to y) \;=\; \big\langle \mathbf{y},y^0\big|\mathbf{x},x^0\big\rangle^{\text{Heisenberg}}_{\text{picture}} \;=\; \langle\mathbf{y}|\exp\bigl(-i(y^0-x^0)\hat{H}\bigr)\,|\mathbf{x}\rangle^{\text{Schroedinger}}_{\text{picture}}\,. \tag{2}$$

(a) Use momentum basis for the Hamiltonian (1) to evaluate the coordinate-basis evolution kernel (2) as

$$U(x\to y) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\exp\Bigl(i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})-i\omega(\mathbf{k})\times(y^0-x^0)\Bigr) \tag{3}$$

$$\text{for}\quad \omega(\mathbf{k}) \;\stackrel{\text{def}}{=}\; +\sqrt{m^2+\mathbf{k}^2}\,, \tag{4}$$

then reduce the 3D momentum integral to the one-dimensional integral

$$U(x\to y) \;=\; \frac{-i}{4\pi^2\,r}\int\limits_{-\infty}^{+\infty}\!dk\,k\,\exp\bigl(irk-it\omega(k)\bigr) \tag{5}$$

where $r=|\mathbf{y}-\mathbf{x}|$ and $t=y^0-x^0$.

We are particularly interested in the asymptotic behavior of the integral (5) in the limit of $r\to\infty$, $t\to\infty$, fixed t/r ratio. The best method for obtaining the asymptotic behavior of such integrals — or more general integrals of the form

$$\int\!dx\,f(x)\times\exp\bigl(-Ag(x)\bigr),\qquad A\,\to\,\infty \tag{6}$$

is the *saddle-point method* (AKA the mountain-pass method).

(b) If you are not familiar with the saddle-point method, read my notes on it.

Those notes were originally written for a QM class, so they include the Airy function example and the relation of the Airy functions to the WKB approximation. You do not need the WKB or the Airy functions for this homework, just the saddle-point method itself, so focus on the first 6 pages of my notes, the rest is optional.

(c) Now use the saddle point method to evaluate the integral (5) in the limit of $r \to \infty$, $t \to \infty$, while the ratio r/t stays fixed. Specifically, let $(r/t) < 1$ so we stay inside the future light cone.

Show that in this limit, the evolution kernel (5) becomes

$$U(x \to y) \;\approx\; \left(\frac{-iM}{2\pi}\right)^{3/2} \times \frac{t}{(t^2 - r^2)^{5/4}} \times \exp(-iM\sqrt{t^2 - r^2}). \tag{7}$$

(d) Finally, take a similar limit but go outside the light cone, thus fixed $(r/t) > 1$ while $r, t \to +\infty$. Show that in this limit, the kernel becomes

$$U(x \to y) \;\approx\; \frac{iM^{3/2}}{(2\pi)^{3/2}} \times \frac{t}{(r^2 - t^2)^{5/4}} \times \exp(-M\sqrt{r^2 - t^2}). \tag{8}$$

Hint: for $r > t$ the saddle point is at complex k.

Eq. (8) shows that the propagation amplitude $U(x \to y)$ diminishes exponentially outside the light cone, *but it does not vanish!* Thus, given a particle localized at point $\mathbf{x}$ at the time x^0, at a later time $y^0 = x^0 + t$ the wave function is *mostly* limited to the future light cone $r < t$, *but there is an exponential tail outside the light cone.* In other words, the probability of superluminal motion is exponentially small but non-zero.

Obviously, such superluminal propagation cannot be allowed in a consistently relativistic theory. And that's why relativistic quantum mechanics of a single particle is inconsistent. Likewise, relativistic quantum mechanics of any fixed number of particles does not work, except as an approximation.

In the quantum field theory, this paradox is resolved by allowing for creation and annihilation of particles. Quantum field operators acting at points x and y outside each others' future lightcones can either create a particle at x and then annihilate it at y, or else annihilate it at y and then create it at x, and we saw in class (*cf.* my notes) that the two effects *precisely* cancel each other. Altogether, there is no net propagation outside the light cone, and that's how the relativistic QFT is perfectly causal while the relativistic QM is not.

2. Now let's go back to the massive vector field $\hat{A}^\mu(x)$ from the previous homework (set#2, problems 2–5). In that homework you (should have) expanded the free vector field into creation and annihilation operators multiplied by the plane-waves according to

$$\hat{A}^\mu(x) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\Bigl(e^{-ikx}\times f^\mu_{\mathbf{k},\lambda}\times\hat{a}_{\mathbf{k},\lambda}\,+\,e^{+ikx}\times f^{*\mu}_{\mathbf{k},\lambda}\times\hat{a}^\dagger_{\mathbf{k},\lambda}\Bigr)^{k^0=+\omega_\mathbf{k}}. \tag{9}$$

The λ here labels the independent polarizations of a vector particle (for example, the helicities $\lambda=-1,0,+1$), while $f^\mu_{\mathbf{k},\lambda}$ are the polarization vectors obeying $k_\mu f^\mu(\mathbf{k},\lambda)=0$. Specifically, in the helicity basis

$$\begin{aligned}
&\text{for } \lambda=\pm1: \quad f^0_{\mathbf{k},\lambda} = 0, \quad \mathbf{f}_{\mathbf{k},\lambda} = \mathbf{e}_\lambda(\mathbf{k}),\\
&\text{for } \lambda=0: \quad f^0_{\mathbf{k},\lambda} = \frac{|\mathbf{k}|}{m}, \quad \mathbf{f}_{\mathbf{k},\lambda} = \frac{\omega_\mathbf{k}}{m}\frac{\mathbf{k}}{|\mathbf{k}|}.
\end{aligned}\tag{10}$$

In this problem, we are going to calculate the Feynman propagator for the massive vector field (9).

(a) First, a lemma about the polarization 4-vectors (10). Show that these 4-vectors obtain obtain by Lorentz boosting of the purely-spatial vectors $(0,\mathbf{e}_\lambda(\mathbf{k}))$ into the frame of the wave moving with the velocity $\mathbf{v}=\mathbf{k}/\omega_\mathbf{k}$. Also, verify that the $f^\mu_{\mathbf{k},\lambda}$ are normalized to

$$g_{\mu\nu}f^\mu_{\mathbf{k},\lambda}f^{*\nu}_{\mathbf{k},\lambda'} = -\delta_{\lambda,\lambda'}\,. \tag{11}$$

(b) Next, another lemma: show that

$$\sum_\lambda f^\mu_{\mathbf{k},\lambda}f^{*\nu}_{\mathbf{k},\lambda} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\,. \tag{12}$$

(c) Now use these lemmas to calculate the "vacuum sandwich" of two vector fields (9) and show that

$$\begin{aligned}\langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left[\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)e^{-ik(x-y)}\right]_{k^0=+\omega_{\mathbf{k}}}\\ &=\; \left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{m^2}\right)D(x-y)\end{aligned} \tag{13}$$

where

$$D(x-y) \;\stackrel{\text{def}}{=}\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left[e^{-ik(x-y)}\right]^{k^0=+\omega_{\mathbf{k}}}. \tag{14}$$

Please note: no time ordering in the “vacuum sandwich” (13).

(d) Next, consider a free scalar field (of the same mass m as the vector field) and its Feynman propagator $G_F^{\text{scalar}}(x-y)$. Show that

$$\left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{m^2}\right)G_F^{\text{scalar}}(x-y) \;=\; \langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{15}$$

To avoid the δ–function singularity in formulae like (15), the time-ordered product of the vector fields (or rather, just of their $\hat{A}^0$ components) is *modified** according to

$$\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{16}$$

Consequently, the Feynman propagator for the massive vector field is defined using the modified time-ordered product of the two fields,

$$G_F^{\mu\nu}(x-y) \;\stackrel{\text{def}}{=}\; \langle 0|\,\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \tag{17}$$

(e) Show that this propagator obtains as

$$G_F^{\mu\nu}(x-y) \;=\; \int\!\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)\times\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,. \tag{18}$$

$\star$ See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.

The classical action for the free vector field can be written as

$$S = \frac{1}{2}\int d^4x\, A_\mu(x)\, \mathcal{D}^{\mu\nu} A_\nu(x) \tag{19}$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator

$$\mathcal{D}^{\mu\nu} \stackrel{\text{def}}{=} (\partial^2 + m^2) g^{\mu\nu} \; - \; \partial^\mu \partial^\nu . \tag{20}$$

(f) Check that the action (19) is correct, then show that the Feynman propagator (18) is a Green's function of the operator (20),

$$\mathcal{D}_x^{\mu\nu} G^F_{\nu\lambda}(x-y) = +i\delta^\mu_\lambda \delta^{(4)}(x-y). \tag{21}$$

3. Finally, a reading assignment. To help you understand the relations between the continuous symmetries, their generators, the multiplets, and the representations of the generators and of the finite symmetries, read about the rotational symmetry and its generators in chapter 3 of the J. J. Sakurai's book *Modern Quantum Mechanics*.† Please focus on sections 1, 2, 3, second half of section 5 (representations of the rotation operators), and section 10; the other sections 4, 6, 7, 8, and 9 are not relevant to the present class material.

   PS: If you have already read the Sakurai's book before but it has been a while, please read it again.

† The UT Math–Physics–Astronomy library has several hard copies but no electronic copies of the book. However, you can find several pirate scans of the book (in PDF format) all over the web; Google them up if you cannot find a legitimate copy.