The first two problems (1 and 2) of this homework set are about the $SO(N)$ symmetry of the quantum theory of N scalar fields. The other two problems (3 and 4) are about the stress-energy tensor of the electromagnetic fields.

1. Consider N interacting real scalar fields $\Phi_1, \ldots, \Phi_N$ with the $O(N)$ symmetric Lagrangian

$$\mathcal{L} \;=\; \frac{1}{2}\sum_{a=1}^{N}\bigl(\partial_\mu\Phi_a\bigr)^2 \;-\; \frac{m^2}{2}\sum_{a=1}^{N}\Phi_a^2 \;-\; \frac{\lambda}{24}\left(\sum_{a=1}^{N}\Phi_a^2\right)^2. \tag{1}$$

By the Noether theorem, the continuous $SO(N)$ subgroup of the $O(N)$ symmetry gives rise to $\frac{1}{2}N(N-1)$ conserved currents

$$J^\mu_{ab}(x) \;=\; -J^\mu_{ba}(x) = \;\Phi_a(x)\,\partial^\mu\Phi_b(x) - \;\Phi_b(x)\,\partial^\mu\Phi_a(x). \tag{2}$$

In the quantum field theory, these currents become operators

$$\begin{aligned} \hat{\mathbf{J}}_{ab}(\mathbf{x},t) \;&=\; -\hat{\mathbf{J}}_{ba}(\mathbf{x},t) \;=\; -\hat\Phi_a(\mathbf{x},t)\nabla\hat\Phi_b(\mathbf{x},t) \;+\; \hat\Phi_b(\mathbf{x},t)\nabla\hat\Phi_a(\mathbf{x},t),\\ \hat{J}^0_{ab}(\mathbf{x},t) \;&=\; -\hat{J}^0_{ba}(\mathbf{x},t) \;=\; \hat\Phi_a(\mathbf{x},t)\hat\Pi_b(\mathbf{x},t) \;-\; \hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\mathbf{x},t). \end{aligned} \tag{3}$$

This problem is about the net charge operators

$$\hat{Q}_{ab}(t) \;=\; -\hat{Q}_{ba}(t) \;=\int\! d^3\mathbf{x}\,\hat{J}^0_{ab}(\mathbf{x},t) \;=\int\! d^3\mathbf{x}\left(\hat\Phi_a(\mathbf{x},t)\hat\Pi_b(\mathbf{x},t) \;-\; \hat\Phi_b(\mathbf{x},t)\hat\Pi_a(\mathbf{x},t)\right). \tag{4}$$

(a) Write down the equal-time commutation relations for the quantum $\hat\Phi_a$ and $\hat\Pi_a$ fields. Also, write down the Hamiltonian operator for the interacting fields.

(b) Show that

$$\begin{aligned} \Bigl[\hat{Q}_{ab}(t), \hat\Phi_c(\mathbf{x},\text{same }t)\Bigr] \;&=\; -i\delta_{bc}\hat\Phi_a(\mathbf{x},t) \;+\; i\delta_{ac}\hat\Phi_b(\mathbf{x},t),\\ \Bigl[\hat{Q}_{ab}(t), \hat\Pi_c(\mathbf{x},\text{same }t)\Bigr] \;&=\; -i\delta_{bc}\hat\Pi_a(\mathbf{x},t) \;+\; i\delta_{ac}\hat\Pi_b(\mathbf{x},t), \end{aligned} \tag{5}$$

(c) Show that the all the $\hat{Q}_{ab}$ commute with the Hamiltonian operator $\hat{H}$. In the Heisenberg picture, this makes all the charge operators $\hat{Q}_{ab}$ time independent.

(d) Verify that the $\hat{Q}_{ab}$ obey commutation relations of the $SO(N)$ generators,

$$\left[\hat{Q}_{ab}, \hat{Q}_{cd}\right] \;=\; -i\delta_{[c[b}\hat{Q}_{a]d]} \;\equiv\; -i\delta_{bc}\hat{Q}_{ad} \;+\; i\delta_{ac}\hat{Q}_{bd} \;+\; i\delta_{bd}\hat{Q}_{ac} \;-\; i\delta_{ad}\hat{Q}_{bc}\,. \tag{6}$$

2. Continuing the previous problem, let's turn off the interactions (*i.e.*, take $\lambda = 0$) and focus on the free fields.

(a) Expand al the fields into linear combinations of the creation and annihilation operators $\hat{a}^\dagger_{\mathbf{p},a}$ and $\hat{a}_{\mathbf{p},a}$ ($a = 1, \ldots, N$), then show that in terms of these operators the charges (4) become

$$\hat{Q}_{ab} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\left(-i\hat{a}^\dagger_{\mathbf{p},a}\hat{a}_{\mathbf{p},b} \;+\; i\hat{a}^\dagger_{\mathbf{p},b}\hat{a}_{\mathbf{p},a}\right). \tag{7}$$

For $N = 2$, the $SO(2)$ symmetry becomes the $U(1)$ phase symmetry one complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and its conjugate $\Phi^* = (\Phi_1 - i\Phi_2)/\sqrt{2}$,

$$\Phi(x) \;\to\; e^{-i\theta}\Phi(x), \quad \Phi^*(x) \;\to\; e^{+i\theta}\Phi^*(x). \tag{8}$$

In the Fock space, the corresponding quantum fields $\hat{\Phi}(x)$ and $\hat{\Phi}^\dagger(x)$ give rise to particles and anti-particles of opposite charges; the creation and annihilation operators for such particles and antiparticles are

$$\begin{aligned}
\hat{a}_\mathbf{p} &= \frac{\hat{a}_{\mathbf{p},1} + i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are particle annihilation operators,}\\
\hat{b}_\mathbf{p} &= \frac{\hat{a}_{\mathbf{p},1} - i\hat{a}_{\mathbf{p},2}}{\sqrt{2}} && \text{are antiparticle annihilation operators,}\\
\hat{a}^\dagger_\mathbf{p} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} - i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} && \text{are particle creation operators,}\\
\hat{b}^\dagger_\mathbf{p} &= \frac{\hat{a}^\dagger_{\mathbf{p},1} + i\hat{a}^\dagger_{\mathbf{p},2}}{\sqrt{2}} && \text{are antiparticle creation operators.}
\end{aligned} \tag{9}$$

(b) Show that in terms of the operators (9),

$$\hat{Q}_{21} \;=\; -\hat{Q}_{12} \;=\; \hat{N}_{\text{particles}} \;-\; \hat{N}_{\text{antiparticles}} \;=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\left(\hat{a}^\dagger_\mathbf{p}\hat{a}_\mathbf{p} \;-\; \hat{b}^\dagger_\mathbf{p}\hat{b}_\mathbf{p}\right). \tag{10}$$

(c) In terms of $\hat{\Phi}$ and $\hat{\Phi}^\dagger$, the commutation relations (5) become

$$[\hat{Q}_{21}, \hat{\Phi}(x)] \;=\; -\hat{\Phi}(x), \quad [\hat{Q}_{21}, \hat{\Phi}^\dagger(x)] \;=\; +\hat{\Phi}^\dagger(x). \tag{11}$$

Verify these commutators, then use the Hadamard Lemma

$$\begin{aligned} e^{\hat{A}}\hat{B}e^{-\hat{A}} &= \sum_{n=0}^{\infty} \frac{1}{n!}[\hat{A}, \ldots, [\hat{A}, \hat{B}] \cdots]_{n\,\text{times}} \\ &= B \;+\; [\hat{A}, \hat{B}] \;+\; \tfrac{1}{2}[\hat{A}, [\hat{A}, \hat{B}]] \;+\; \tfrac{1}{6}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] \;+\; \cdots \end{aligned} \tag{12}$$

to show that the charge $\hat{Q}_{21}$ *generates* the phase symmetry (8) according to

$$\begin{aligned} \exp(+i\theta\hat{Q}_{21})\hat{\Phi}(x)\exp(-i\theta\hat{Q}_{21}) &= e^{-i\theta}\hat{\Phi}(x), \\ \exp(+i\theta\hat{Q}_{21})\hat{\Phi}^\dagger(x)\exp(-i\theta\hat{Q}_{21}) &= e^{+i\theta}\hat{\Phi}^\dagger(x). \end{aligned} \tag{13}$$

Now let's go back to $N > 2$ and show that the charges $\hat{Q}_{ab}$ generate the $SO(N)$ symmetry of the quantum fields. Any $SO(N)$ rotation matrix R can be written as a matrix exponential of an antisymmetric matrix, $R = \exp(A)$ for $A^\top = -A$. For this matrix A, let's define a unitary operator in the Fock space

$$\hat{U} \;=\; \exp\left(-\frac{i}{2}\sum_{ab} A_{ab}\hat{Q}_{ab}\right). \tag{14}$$

(d) Verify that this operator is indeed unitary for any real antisymmetric matrix A. Hint: check and use the hermiticity of the generators $\hat{Q}_{ab}$.

(e) Show that $\hat{U}$ implements the $SO(N)$ rotation R in the scalar field space,

$$\hat{U}\hat{\Phi}_a(x)\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{\Phi}_b\,. \tag{15}$$

Hint: use the commutation relations (5) and the Hadamard lemma (12).

(f) Argue that $[\hat{Q}_{ab},\hat{H}]=0$ and eq. (15) for the action of the $\hat{U}$ symmetries on the quantum fields together imply simlar transformation laws for the creation and the annihilation operators

$$\hat{U}\hat{a}_{\mathbf{p},a}\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{a}_{\mathbf{p},b} \quad\text{and}\quad \hat{U}\hat{a}^\dagger_{\mathbf{p},a}\hat{U}^\dagger \;=\; \sum_b R_{ab}\hat{a}^\dagger_{\mathbf{p},b}\,. \tag{16}$$

(g) Finally, show that when $\hat{U}$ acts on a multiparticle state, it rotates the species index of each particle by R,

$$\hat{U}\,|n:(\mathbf{p}_1,a_1),\ldots,(\mathbf{p}_n,a_n)\rangle \;=\; \sum_{b_1,\ldots,b_n} R_{a_1,b_1}\cdots R_{a_n,b_n}\;|n:(\mathbf{p}_1,b_1),\ldots,(\mathbf{p}_n,b_n)\rangle\,. \tag{17}$$

Note: for simplicity assume that all particles have different momenta, $\mathbf{p}_1\neq\mathbf{p}_2$, *etc.*, then use part (j).

3. Now let's turn our attention to the stress-energy tensor. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noether}} \;=\sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{18}$$

For the scalar fields, real or complex, this Noether stress-energy tensor is properly symmetric, $T^{\mu\nu}_{\text{Noether}} = T^{\nu\mu}_{\text{Noether}}$. But for the vector, tensor, spinor, *etc.*, fields, the Noether stress-energy tensor (18) comes out asymmetric, so to make it properly symmetric one adds a total-divergence term of the form

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noether}} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu\,\nu}, \tag{19}$$

where $\mathcal{K}^{\lambda\mu\,\nu}\equiv-\mathcal{K}^{\mu\lambda\,\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu,\partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{20}$$

where A_μ is a real vector field and $F_{\mu\nu}\stackrel{\text{def}}{=}\partial_\mu A_\nu-\partial_\nu A_\mu$.

(a) Write down $T^{\mu\nu}_{\text{Noether}}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(b) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\text{EM}} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{21}$$

Show that this expression indeed has form (19) for

$$\mathcal{K}^{\lambda\mu,\nu} \;=\; -F^{\lambda\mu}A^{\nu} \;=\; -\mathcal{K}^{\mu\lambda,\nu}. \tag{22}$$

(c) Write down the components of the stress-energy tensor (21) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density, and stress.

Next, consider the electromagnetic fields coupled to the electric current J^{μ} of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P^{μ}_{EM} and P^{μ}_{mat}. Consequently, we should have

$$\partial_{\mu}T^{\mu\nu}_{\text{net}} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\text{net}} \;=\; T^{\mu\nu}_{\text{EM}} \;+\; T^{\mu\nu}_{\text{mat}} \tag{23}$$

but generally $\partial_{\mu}T^{\mu\nu}_{\text{EM}} \neq 0$ and $\partial_{\mu}T^{\mu\nu}_{\text{mat}} \neq 0$.

(d) Use Maxwell's equations to show that

$$\partial_{\mu}T^{\mu\nu}_{\text{EM}} \;=\; -F^{\nu\lambda}J_{\lambda} \tag{24}$$

(in $c = 1$ units), and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_{λ} according to

$$\partial_{\mu}T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda}J_{\lambda}. \tag{25}$$

(e) Rewrite eq. (24) in non-relativistic notations and explain its physical meaning in terms of the electromagnetic energy, momentum, work, and forces.

4. Continuing problem 3, consider the EM fields coupled to a specific model of charged matter, namely a complex scalar field $\Phi(x) \neq \Phi^*(x)$ of electric charge $q \neq 0$. Altogether, the net Lagrangian for the A^μ, Φ, and Φ^* fields is

$$\mathcal{L}_{\rm net} \;=\; D^\mu\Phi^* D_\mu\Phi \;-\; m^2\Phi^*\Phi \;-\; \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{26}$$

where

$$D_\mu\Phi \;=\; (\partial_\mu \,+\, iqA_\mu)\Phi \quad\text{and}\quad D_\mu\Phi^* \;=\; (\partial_\mu \,-\, iqA_\mu)\Phi^* \tag{27}$$

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^\mu \;\stackrel{\rm def}{=}\; -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{28}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole system and show that

$$T^{\mu\nu}_{\rm net} \;\equiv\; T^{\mu\nu}_{\rm EM} \,+\, T^{\mu\nu}_{\rm mat} \;=\; T^{\mu\nu}_{\rm Noether} \,+\, \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{29}$$

where $T^{\mu\nu}_{\rm EM}$ is exactly as in eq. (21) for the free EM fields, the improvement tensor $\mathcal{K}^{\lambda\mu\nu} = -\mathcal{K}^{\mu\lambda\nu}$ is also exactly as in eq. (22), and

$$T^{\mu\nu}_{\rm mat} \;=\; D^\mu\Phi^*\,D^\nu\Phi \;+\; D^\nu\Phi^*\,D^\mu\Phi \;-\; g^{\mu\nu}\big(D_\lambda\Phi^*\,D^\lambda\Phi \;-\; m^2\Phi^*\,\Phi\big). \tag{30}$$

Note: although the improvement tensor $\mathcal{K}^{\lambda\mu\nu}$ for the EM + matter system is the same as for the free EM fields, in presence of an electric current J^μ its derivative $\partial_\lambda\mathcal{K}^{\lambda\mu\nu}$ contains an extra $J^\mu A^\nu$ term. Pay attention to this term — it is important for obtaining the gauge-invariant stress-energy tensor (30) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{31}$$

to show that

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu\lambda}J_\lambda \tag{32}$$

exactly as in eq. (25), and therefore the *net* stress-energy tensor (29) is conserved, *cf.* problem **3**(d).