This homework has 4 problems. Problems 1 and 2 are about the non-abelian gauge theories. Problems 3 and 4 are about the Lorentz symmetry and its generators. Altogether, it's a pretty large homework set, so start working early.

1. In class, I have focused on the *fundamental multiplet* of the local $SU(N)$ symmetry, *i.e.*, a set of N fields (complex scalars or Dirac fermions) which transform as a complex N–vector,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'^i(x) \;= \sum_j U^i{}_j(x)\Psi^j(x), \quad i,j = 1,2,\ldots,N \tag{1}$$

where $U(x)$ is an x–dependent unitary $N \times N$ matrix, $\det U(x) \equiv 1$. Now consider $N^2 - 1$ real fields $\Phi^a(x)$ forming an *adjoint multiplet*: In matrix form

$$\Phi(x) \;=\; \sum_a \Phi^a(x) \times \frac{\lambda^a}{2} \tag{2}$$

is a traceless hermitian $N \times N$ matrix which transforms under the local $SU(N)$ symmetry as

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x). \tag{3}$$

Note that this transformation law preserves the $\Phi^\dagger = \Phi$ and the $\mathrm{tr}(\Phi) = 0$ conditions.

The covariant derivatives D_μ act on an adjoint multiplet according to

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \,+\, i[\mathcal{A}_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu\Phi(x) \,+\, i\mathcal{A}_\mu(x)\Phi(x) \,-\, i\Phi(x)\mathcal{A}_\mu(x), \tag{4}$$

or in components

$$D_\mu\Phi^a(x) = \;\partial_\mu\Phi_a(x) \;-\; f^{abc}\mathcal{A}^b_\mu(x)\Phi^c(x). \tag{5}$$

(a) Verify that these derivatives are indeed covariant under the finite gauge transforms (3).

(b) Verify the Leibniz rule for the covariant derivatives of matrix products. Let $\Phi(x)$ and $\Xi(x)$ be two adjoint multiplets while $\Psi(x)$ is a fundamental multiplet and $\Psi^\dagger(x)$ is its hermitian conjugate (row vector of Ψ_i^*). Show that

$$\begin{aligned} D_\mu(\Phi\Xi) &= (D_\mu\Phi)\Xi \;+\; \Phi(D_\mu\Xi), \\ D_\mu(\Phi\Psi) &= (D_\mu\Phi)\Psi \;+\; \Phi(D_\mu\Psi), \\ D_\mu(\Psi^\dagger\Xi) &= (D_\mu\Psi^\dagger)\Xi \;+\; \Psi^\dagger(D_\mu\Xi). \end{aligned} \tag{6}$$

(c) Show that for an adjoint multiplet $\Phi(x)$,

$$[D_\mu, D_\nu]\Phi(x) \;=\; i[\mathcal{F}_{\mu\nu}(x), \Phi(x)] \;=\; ig[F_{\mu\nu}(x), \Phi(x)] \tag{7}$$

or in components $[D_\mu, D_\nu]\Phi^a(x) = -gf^{abc}F^b_{\mu\nu}(x)\Phi^c(x)$.

- In my notations A_μ and $F_{\mu\nu}$ are the canonically normalized potential and tension fields, while $\mathcal{A}_\mu = gA_\mu$ is the connection in the covariant derivative and $\mathcal{F}_{\mu\nu} = gF_{\mu\nu}$ is the curvature of that connection.

In class, I have argued (using covariant derivatives) that the tension fields $\mathcal{F}_{\mu\nu}(x)$ themselves transform according to eq. (3). In other words, the $\mathcal{F}^a_{\mu\nu}(x)$ form an adjoint multiplet of the $SU(N)$ symmetry group.

(d) Verify the $\mathcal{F}'_{\mu\nu}(x) = U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$ transformation law directly from the definition $\mathcal{F}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu]$ and the non-abelian gauge transform of the $\mathcal{A}_\mu$ fields.

(e) Verify the covariant differential identity for the non-abelian tension fields $\mathcal{F}_{\mu\nu}(x)$:

$$D_\lambda\mathcal{F}_{\mu\nu} \;+\; D_\mu\mathcal{F}_{\nu\lambda} \;+\; D_\nu\mathcal{F}_{\lambda\mu} \;=\; 0. \tag{8}$$

Note the covariant derivatives (4) in this equation.

Finally, consider the $SU(N)$ Yang–Mills theory — the non-abelian gauge theory that does not have any fields except $\mathcal{A}^a(x)$ and $\mathcal{F}^a(x)$; its Lagrangian is

$$\mathcal{L}_{\rm YM} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;=\; -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu}. \tag{9}$$

(f) Show that the Euler–Lagrange field equations for the Yang–Mills theory can be written in covariant form as $D_\mu\mathcal{F}^{\mu\nu}=0$.
Hint: first show that for an infinitesimal variation $\delta\mathcal{A}_\mu(x)$ of the non-abelian gauge fields, the tension fields vary according to $\delta\mathcal{F}_{\mu\nu}(x)=D_\mu\delta\mathcal{A}_\nu(x)-D_\nu\delta\mathcal{A}_\mu(x)$.

2. Continuing the previous problem, consider an $SU(N)$ gauge theory in which N^2-1 vector fields $A^a_\mu(x)$ interact with some "matter" fields $\phi_\alpha(x)$,

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \mathcal{L}_{\rm mat}(\phi,D_\mu\phi). \tag{10}$$

For the moment, let me keep the matter fields completely generic — they can be scalars, or vectors, or spinors, or whatever, and form any kind of a multiplet of the local $SU(N)$ symmetry as long as such multiplet is complete and non-trivial. All we need to know right now is that there are well-defined covariant derivatives $D_\mu\phi$ that depend on the gauge fields A^a_μ, which give rise to the currents

$$J^{a\mu} \;=\; -\frac{\partial\mathcal{L}_{\rm mat}}{\partial A^a_\mu} \;=\; -\sum_\phi \frac{\partial\mathcal{L}_{\rm mat}}{\partial(D_\mu\phi)}\times ig\hat{T}^a\phi. \tag{11}$$

Collectively, these N^2-1 currents should form an adjoint multiplet of the local $SU(N)$ symmetry, meaning

$$J_\mu(x) \;\stackrel{\rm def}{=}\; \sum_a J^a_\mu(x)\,\frac{\lambda^a}{2} \quad\text{transforms to}\quad J'_\mu(x) \;=\; U(x)J_\mu(x)U^\dagger(x). \tag{12}$$

(a) Show that in this theory the equation of motion for the A^a_μ fields are $D_\mu F^{a\mu\nu}=J^{a\nu}$ and that consistency of these equations requires the currents to be *covariantly* *conserved,*

$$D_\mu J^\mu \;=\; \partial_\mu J^\mu \;+\; i[\mathcal{A}_\mu, J^\mu] \;=\; 0, \tag{13}$$

or in components, $\partial_\mu J^{a\mu} - f^{abc}\mathcal{A}^b_\mu J^{c\mu} = 0$.

Note: a covariantly conserved current does *not* lead to a conserved charge,

$$\frac{d}{dt}\int d^3\mathbf{x}\, J^{a0}(\mathbf{x},t) \;\neq\; 0. \tag{14}$$

Now consider a simple example of matter fields — a fundamental multiplet $\Psi(x)$ of N scalar fields $\Psi^i(x)$, with a Lagrangian

$$\mathcal{L}_{\mathrm{mat}} \;=\; D_\mu\Psi^\dagger D^\mu\Psi \;-\; m^2\Psi^\dagger\Psi \;-\; \frac{\lambda}{4}\big(\Psi^\dagger\Psi\big)^2, \qquad \mathcal{L}_{\mathrm{net}} \;=\; \mathcal{L}_{\mathrm{mat}} \;-\; \frac{1}{2g^2}\,\mathrm{tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big), \tag{15}$$

and hence the $SU(N)$ currents are

$$J^{a\mu} \;=\; -g\,\mathrm{Im}\Big(\Psi^\dagger\lambda^a D^\mu\Psi\Big) \;=\; -\frac{ig}{2}\big(D^\mu\Psi^\dagger\big)\lambda^a\Psi \;+\; \frac{ig}{2}\Psi^\dagger\lambda^a D^\mu\Psi. \tag{16}$$

(b) Derive these $SU(N)$ currents. Then use the matrix identity

$$\sum_a \big(\lambda^a\big)^i{}_j \big(\lambda^a\big)^k{}_\ell \;=\; 2\delta^i{}_\ell\delta^k{}_j \;-\; \frac{2}{N}\delta^i{}_j\delta^k{}_\ell \tag{17}$$

to show that in the matrix form

$$\begin{aligned} J_\mu \;\stackrel{\mathrm{def}}{=}\; \sum_a J^a_\mu \times \tfrac{1}{2}\lambda^a \;=\;& \frac{ig}{2}\Big((D_\mu\Psi)\otimes\Psi^\dagger \;-\; \Psi\otimes D_\mu\Psi^\dagger\Big) \\ &-\; \frac{ig}{2N}\Big(\Psi^\dagger D_\mu\Psi \;-\; (D_\mu\Psi^\dagger)\Psi\Big) \times \mathbf{1}_{N\times N} \end{aligned} \tag{18}$$

where $(D_\mu\Psi)\otimes\Psi^\dagger$ denotes $N\times N$ matrix with elements $(D_\mu\Psi)^i \times \Psi^*_j$, and likewise for the $\Psi\otimes D_\mu\Psi^\dagger$.

(c) Verify that under the $SU(N)$ gauge transforms, the currents (16) transform into each other as members of the adjoint multiplet, *i.e.*, the matrix (18) transforms according to eq. (12).

(d) Finally, verify the covariant conservation $D_\mu J^{a\mu}$ of these currents when the scalar fields $\Psi^i(x)$ and $\Psi^\dagger_i(x)$ obey their equations of motion.

3. Now consider a different subject, namely the continuous Lorentz group $SO^+(3,1)$ and its generators $\hat{J}^{\mu\nu} = -\hat{J}^{\nu\mu}$. In 3D terms, the six independent $\hat{J}^{\mu\nu}$ generators comprise the 3 components of the angular momentum $\hat{J}^i = \frac{1}{2}\epsilon^{ijk}\hat{J}^{jk}$ — which generate the rotations of space — plus 3 generators $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ of the Lorentz boosts.

(a) In 4D terms, the commutation relations of the Lorentz generators are

$$\left[\hat{J}^{\alpha\beta}, \hat{J}^{\mu\nu}\right] = ig^{\beta\mu}\hat{J}^{\alpha\nu} - ig^{\alpha\mu}\hat{J}^{\beta\nu} - ig^{\beta\nu}\hat{J}^{\alpha\mu} + ig^{\alpha\nu}\hat{J}^{\beta\mu}. \tag{19}$$

Show that in 3D terms, these relations become

$$\left[\hat{J}^i, \hat{J}^j\right] = i\epsilon^{ijk}\hat{J}^k, \quad \left[\hat{J}^i, \hat{K}^j\right] = i\epsilon^{ijk}\hat{K}^k, \quad \left[\hat{K}^i, \hat{K}^j\right] = -i\epsilon^{ijk}\hat{J}^k. \tag{20}$$

The Lorentz symmetry dictates the commutation relations of the $\hat{J}^{\mu\nu}$ with any operators comprising a Lorentz multiplet. In particular, for any Lorentz vector $\hat{V}^\mu$

$$\left[\hat{V}^\lambda, \hat{J}^{\mu\nu}\right] = ig^{\lambda\mu}\hat{V}^\nu - ig^{\lambda\nu}\hat{V}^\mu. \tag{21}$$

(b) Spell out these commutation relations in 3D terms, then use them to show that the Lorentz boost generators $\hat{\mathbf{K}}$ do not commute with the Hamiltonian $\hat{H}$.

(c) Show that even in the non-relativistic limit, the Galilean boosts $t' = t$, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and their generators $\hat{\mathbf{K}}_G$ do not commute with the Hamiltonian operator of a QFT or a quantum system of several particles.

Note: Only the *time-independent* symmetries commute with the Hamiltonian. But when the action of a symmetry is manifestly time dependent — like a Galilean boost $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$, or a Lorentz boost — the symmetry operators do not commute with the time evolution and hence with the Hamiltonian.

4. Next, consider the little group $G(p)$ of Lorentz symmetries preserving some momentum 4–vector p^μ. For the moment, allow the p^μ to be time-like, light-like, or even space-like — anything goes as long as $p \neq 0$.

   (a) Show that the little group $G(p)$ is generated by the 3 components of the vector

   $$\hat{\mathbf{R}} \;=\; p^0\hat{\mathbf{J}} \;+\; \mathbf{p}\times\hat{\mathbf{K}} \tag{22}$$

   after a suitable component-by-component rescaling.

   Suppose the momentum p^μ belongs to a massive particle, thus $p^\mu p_\mu = m^2 > 0$. For simplicity, assume the particle moves in z direction with velocity β, thus $p^\mu = (E, 0, 0, p)$ for $E = \gamma m$ and $p = \beta\gamma m$. In this case, the properly normalized generators of the little group $G(p)$ are the

   $$\begin{aligned} \widetilde{J}^x \;&=\; \frac{1}{m}\,\hat{R}^x \;=\; \gamma\hat{J}^x \;-\; \beta\gamma\hat{K}^y, \\ \widetilde{J}^y \;&=\; \frac{1}{m}\,\hat{R}^y \;=\; \gamma\hat{J}^y \;+\; \beta\gamma\hat{K}^x, \\ \widetilde{J}^z \;&=\; \frac{1}{\gamma m}\,\hat{R}^z \;=\; \hat{J}^z, \quad \text{the helicity.} \end{aligned} \tag{23}$$

   (b) Show that these generators have angular-momentum-like commutators with each other, $[\widetilde{J}^i, \widetilde{J}^j] = i\epsilon^{ijk}\widetilde{J}^k$. Consequently, the little group $G(p)$ is isomorphic to the rotation group $SO(3)$.

   Now suppose the momentum p^μ belongs to a massless particle, $p^\mu p_\mu = 0$. Again, assume for simplicity that the particle moves in the z direction, thus $p^\mu = (E, 0, 0, E)$. In this case, we cannot normalize the generators of the little group as in eq. (23); instead, let's normalize them according to

   $$\hat{\mathbf{I}} \;=\; \frac{1}{E}\,\hat{\mathbf{R}} \;=\; \hat{\mathbf{J}} \;+\; \vec{\beta}\times\hat{\mathbf{K}}, \tag{24}$$

   or in components,

   $$\hat{I}^x \;=\; \hat{J}^x - \hat{K}^y, \quad \hat{I}^y \;=\; \hat{J}^y + \hat{K}^x, \quad \hat{I}^z \;=\; \hat{J}^z. \tag{25}$$

   (c) Show that these generators obey similar commutation relations to the $\hat{p}^x$, $\hat{p}^y$, and

$\hat{J}^z$ operators, namely

$$[\hat{J}^z, \hat{I}^x] \;=\; +i\hat{I}^y, \quad [\hat{J}^z, \hat{I}^y] \;=\; -i\hat{I}^x, \quad [\hat{I}^x, \hat{I}^y] \;=\; 0. \tag{26}$$

Consequently, the little group $G(p)$ is isomorphic to the ISO(2) group of *rotations and translations* in the xy plane.

(d) Finally, show that for a tachyonic momentum with $p^\mu p_\mu < 0$, the properly normalized generators of the little group have similar commutation relations to the $\hat{K}^x$, $\hat{K}^y$, and $\hat{J}^z$ operators. Consequently, the little group $G(p)$ is isomorphic to the $SO^+(2,1)$, the continuous Lorentz group in $2+1$ spacetime dimensions.