1. Let's start with the plane-wave solutions of the Dirac equation, $\Psi_\alpha(x) = u_\alpha \times e^{-ipx}$ and $\Psi_\alpha(x) = v_\alpha \times e^{+ipx}$ for some x-independent Dirac spinors $u_\alpha(p,s)$ and $v_\alpha(p,s)$.

   (a) Check that these waves indeed solve the Dirac equation provided $p^2 = m^2$ while

$$(\not{p} - m)u(p,s) \;=\; 0, \quad (\not{p} + m)v(p,s) \;=\; 0 \tag{1}$$

   where $\not{p}$ is the Dirac slash notation for the $\gamma^\mu p_\mu$. Likewise, for any Lorentz vector a^μ, we may write $\not{a}$ to denote $\gamma^\mu a_\mu$.

   By convention, we always take $E = p^0 = +\sqrt{\mathbf{p}^2 + m^2}$ — that's why we have separate positive-frequency waves $e^{-ipx}u_\alpha$ and negative-frequency waves $e^{+ipx}v_\alpha$ — while the spinor coefficients $u(p,s)$ and $v(p,s)$ are normalized to

$$u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'}\,. \tag{2}$$

   In this problem we shall write down explicit formulae for these spinors in the Weyl convention for the γ^μ matrices.

   (b) Show that for $\mathbf{p} = 0$,

$$u(\mathbf{p} = \mathbf{0}, s) \;=\; \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \tag{3}$$

   where ξ_s is a two-component $SO(3)$ spinor encoding the electron's spin state. The ξ_s are normalized to $\xi_s^\dagger \xi_{s'} = \delta_{s,s'}$.

   (c) For other momenta, $u(p,s) = M_D(\text{boost}) \times u(\mathbf{p} = 0, s)$ for the boost that turns $(m, \vec{0})$ into p^μ. Use eq. (HW6.11) — *i.e.*, eq. (11) from the previous homework set#6 — to show that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E - \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E + \mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi_s \end{pmatrix}. \tag{4}$$

(d) Use similar arguments to show that

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \;=\; \begin{pmatrix} +\sqrt{p_\mu\sigma^\mu}\,\eta_s \\ -\sqrt{p_\mu\bar{\sigma}^\mu}\,\eta_s \end{pmatrix} \tag{5}$$

where η_s are two-component $SO(3)$ spinors normalized to $\eta_s^\dagger\eta_{s'} = \delta_{s,s'}$.

Physically, the η_s should have opposite spins from the ξ_s — the holes in the Dirac sea have opposite spins (as well as p^μ) from the missing negative-energy particles. Mathematically, this requires $\eta_s^\dagger\mathbf{S}\eta_s = -\xi_s^\dagger\mathbf{S}\xi_s$; we may solve this condition by letting $\eta_s = \sigma_2\xi_s^* = \pm i\xi_{-s}^*$.

(e) Check that $\eta_s = \sigma_2\xi_s^* = \pm i\xi_{-s}^*$ indeed provides for the $\eta_s^\dagger\mathbf{S}\eta_s = -\xi_s^\dagger\mathbf{S}\xi_s$, then show that this leads to

$$v(p,s) \;= \gamma^2 u^*(p,s) \quad\text{and}\quad u(p,s) \;=\; \gamma^2 v^*(p,s). \tag{6}$$

(f) Show that for the ultra-relativistic electrons or positrons of definite helicity $\lambda = \pm\frac{1}{2}$, the Dirac plane waves become *chiral* — *i.e.*, dominated by one of the two irreducible Weyl spinor components $\psi_L(x)$ or $\psi_R(x)$ of the Dirac spinor $\Psi(x)$, while the other component becomes negligible. Specifically,

$$\begin{aligned} u(p,-\tfrac{1}{2}) &\;\approx\; \sqrt{2E}\begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, \qquad & u(p,+\tfrac{1}{2}) &\;\approx\; \sqrt{2E}\begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v(p,-\tfrac{1}{2}) &\;\approx\; -\sqrt{2E}\begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, \qquad & v(p,+\tfrac{1}{2}) &\;\approx\; \sqrt{2E}\begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \tag{7}$$

Note that for the electron waves the helicity agrees with the chirality — they are both left or both right, — but for the positron waves the chirality is opposite from the helicity.

In the previous homework (set#6, problem#4), we saw that for $m = 0$ the LH and the RH Weyl spinor fields decouple from each other. Now this exercise show us which particle modes comprise each Weyl spinor: *The $\psi_L(x)$ and its hermitian conjugate $\psi_L^\dagger(x)$ contain the left-handed fermions and the right-handed antifermions, while the $\psi_R(x)$ and the $\psi_R^\dagger(x)$ contain the right-handed fermions and the left-handed antifermions.*

2. In problem 1 we have worked in the Weyl convention for the Dirac matrices and Dirac spinors. In this problem we are going to establish some convention-independent properties of these Dirac spinors, — although you may use the Weyl convention formulae from problem 1 to verify them. We shall use these properties in class when we get to the Quantum Electrodynamics (QED).

   (a) Dirac spinors $u(p,s)$ and $v(p,s)$ are normalized to

$$u^\dagger(p,s)u(p,s') \;=\; 2E_p\delta_{s,s'}\,, \quad v^\dagger(p,s)v(p,s') \;=\; 2E_p\delta_{s,s'}\,. \tag{2}$$

   Show that the combinations $\bar{u}u$ and $\bar{v}v$ have a different normalization, namely

$$\bar{u}(p,s)u(p,s') \;=\; +2m\delta_{s,s'}\,, \quad \bar{v}(p,s)v(p,s') \;=\; -2m\delta_{s,s'}\,. \tag{8}$$

   (b) There are only two independent $SO(3)$ spinors, hence $\sum_s \xi_s\xi_s^\dagger = \sum_s \eta_s\eta_s^\dagger = \mathbf{1}_{2\times 2}$. Use this fact to show that

$$\sum_{s=1,2} u_\alpha(p,s)\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s)\bar{v}_\beta(p,s) \;=\; (\not{p}-m)_{\alpha\beta}\,. \tag{9}$$

3. In class we have studied the charge conjugation symmetry $\mathbf{C}$ in some detail, but we spent much less time on other discrete symmetries. In this problem, we focus on the *parity* $\mathbf{P}$, an im-proper Lorentz symmetry which reflects the space but not the time, $(\mathbf{x},t)\to(-\mathbf{x},+t)$. This symmetry acts on the Dirac spinor fields according to

$$\widehat{\Psi}'(-\mathbf{x},+t) \;=\; \pm\gamma^0\widehat{\Psi}(+\mathbf{x},+t) \tag{10}$$

   where the overall $\pm$ sign is the *intrinsic parity* of the fermion species described by the $\widehat{\Psi}$ field.

   (a) Verify that the Dirac equation transforms covariantly under (10) and that the Dirac Lagrangian is invariant (apart from $\mathcal{L}(\mathbf{x},t)\to\mathcal{L}(-\mathbf{x},t)$).

In the Fock space, eq. (10) becomes

$$\widehat{\mathbf{P}}\widehat{\Psi}(\mathbf{x},t)\widehat{\mathbf{P}} \;=\; \pm\gamma^0\widehat{\Psi}(-\mathbf{x},t) \tag{11}$$

for some unitary operator $\widehat{\mathbf{P}}$ that squares to one. Let's find how this operator acts on the particles and their states.

(b) First, check the plane-wave solutions $u(\mathbf{p},s)$ and $v(\mathbf{p},s)$ from problem 1, and show that $u(-\mathbf{p},s)=+\gamma^0 u(\mathbf{p},s)$ while $v(-\mathbf{p},s)=-\gamma^0 v(\mathbf{p},s)$.

(c) Now show that eq. (11) implies

$$\begin{aligned} \widehat{\mathbf{P}}\,\hat{a}_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \pm\hat{a}_{-\mathbf{p},+s}\,, & \widehat{\mathbf{P}}\,\hat{a}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \pm\hat{a}^\dagger_{-\mathbf{p},+s}\,,\\ \widehat{\mathbf{P}}\,\hat{b}_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \mp\hat{b}_{-\mathbf{p},+s}\,, & \widehat{\mathbf{P}}\,\hat{b}^\dagger_{\mathbf{p},s}\,\widehat{\mathbf{P}} &= \mp\hat{b}^\dagger_{-\mathbf{p},+s}\,, \end{aligned} \tag{12}$$

and hence

$$\widehat{\mathbf{P}}\,|F(\mathbf{p},s)\rangle \;=\; \pm\,|F(-\mathbf{p},+s)\rangle \quad\text{and}\quad \widehat{\mathbf{P}}\,\big|\overline{F}(\mathbf{p},s)\big\rangle \;=\; \mp\,\big|\overline{F}(-\mathbf{p},+s)\big\rangle\,. \tag{13}$$

Note that the fermion F and the antifermion $\overline{F}$ have opposite intrinsic parities!

4. Consider a bound state of a charged Dirac fermion F and the corresponding antifermion, for example a $q\bar{q}$ meson or a positronium "atom" (a hydrogen-atom-like bound state of e^- and e^+). For simplicity, let this bound state have zero net momentum. In the Fock space of fermions and antifermions, such a bound state appears as

$$|B(\mathbf{p}_{\rm tot}=0)\rangle \;=\; \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\sum_{s_1,s_2}\psi(\mathbf{p}_{\rm red},s_1,s_2)\times\hat{a}^\dagger(+\mathbf{p}_{\rm red},s_1)\,\hat{b}^\dagger(-\mathbf{p}_{\rm red},s_2)\,|0\rangle \tag{14}$$

for some wave-function ψ of the reduced momentum and of the two spins.

Suppose this bound state has a definite orbital angular momentum L — which controls the symmetry of the wave function ψ with respect to $\mathbf{p}_{\rm red}\to-\mathbf{p}_{\rm red}$ — and a definite net spin S — which controls the symmetry of ψ under $s_1\leftrightarrow s_2$. Turns out that the L and the S of the bound state also determine its C-parity and P-parity.

(a) Show that $C=(-1)^{L+S}$.

(b) Show that $P=(-1)^{L+1}$.

Now let's apply these results to the positronium — a hydrogen-atom-like bound state of a positron e^+ and an electron e^-. The ground state of positronium is hydrogen-like 1S ($n = 1$, $L = 0$), with the net spin which could be either $S = 0$ or $S = 1$.

(c) Explain why the $S = 0$ state annihilates into photons much faster than the $S = 1$ state.
Hint#1: The annihilation rate of positronium into n photons happens in the n^{th} order of QED perturbation theory, so the rate $\propto \alpha^n$ (for $\alpha \approx 1/137$).
Hint#2: Since the EM fields couple linearly to the electric charges and currents (which are reversed by $\widehat{\mathbf{C}}$), each photon has $C = -1$.

5. Consider the bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where Γ is one of 16 matrices discussed in the previous homework#6, problem 3(h). Altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}\tfrac{i}{2}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^\mu\gamma^5\Psi, \quad P = \overline{\Psi}i\gamma^5\Psi. \quad (15)$$

(a) Show that all the bilinears (15) are Hermitian.
Hint: First, show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$.
Note: despite the Fermi statistics, $\left(\Psi^\dagger_\alpha\Psi_\beta\right)^\dagger = +\Psi^\dagger_\beta\Psi_\alpha$.

(b) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (15) under parity and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

Now consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take the classical limit where $\Psi(x)$ and $\Psi^\dagger(x)$ *anticommute* with each other, $\Psi_\alpha\Psi^\dagger_\beta = -\Psi^\dagger_\beta\Psi_\alpha$.

(d) Show that in the Weyl convention, $\mathbf{C}$ turns $\overline{\Psi}\Gamma\Psi$ into $\overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(e) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are C–even and which are C–odd.

6. Finally, a couple of *optional* reading assignments about the time reversal and related symmetries.

   (a) *Modern Quantum Mechanics* by J. J. Sakurai,⋆ §3.10, about the time reversal symmetry in quantum mechanics.
   If you have already read the Sakurai's book before but it has been a while, please read it again.

   (b) Peskin & Schroeder textbook, §3.6, about the discrete symmetries of Dirac spinors. Focus on the subsections about the time reversal symmetry and about the combined **CPT** symmetry.

⋆ The UT Math–Physics–Astronomy library has several hard copies but no electronic copies of the book. However, you can find several pirate scans of the book (in PDF format) all over the web; Google them up if you cannot find a legitimate copy.