1. Let's start with a bunch of reading assignments.

   (a) First, read carefully my notes on the Golden Rule and the phase space factors.

   Note: you will need the two-particle phase space factor in problems 2–4 of this homework, so please pay attention. Also, a later homework will involve a three-particle phase space, so do not forget the general formulae for n final-state particles.

   (b) If you have trouble following my derivation of the Fermi's Golden Rule, read the online notes of professor Wacker on the subject. Alternatively, read §5.5 of J. J. Sakurai's *Modern Quantum Mechanics* on time-dependent perturbation theory, and focus on the constant perturbation and the Fermi golden rule.

   (c) Finally, read §4.5 of the *Peskin and Schroeder* textbook for an alternative explanation of the phase space factor for the scattering cross-sections.

2. Next, a warm-up exercise. Consider two species of scalar fields, Φ and ϕ, with a cubic coupling to each other,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{M^2}{2}\Phi^2 + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\mu}{2}\Phi\phi^2. \tag{1}$$

   (a) Write down the vertices and the propagators for the Feynman rules for this theory.

   (b) Suppose $M > 2m$, so a single Φ particle may decay to two ϕ particles. Calculate the rate Γ of this decay (in the rest frame of the original Φ) to lowest order in perturbation theory.

3. Now consider N scalar fields ϕ_i of the same mass m and with $O(N)$ symmetric quartic couplings to each other,

$$\mathcal{L} = \frac{1}{2}\sum_i(\partial_\mu\phi_i)^2 - \frac{m^2}{2}\sum_i\phi_i^2 - \frac{\lambda}{8}\left(\sum_i\phi_i^2\right)^2. \tag{2}$$

   (a) Write down the Feynman propagators and the vertices for this theory. Then write down the tree-level scattering amplitude $\mathcal{M}(\phi_i+\phi_j\to\phi_k+\phi_\ell)$ for general $i,j,k,\ell=1,\ldots,N$.

(b) Calculate the tree-level scattering amplitudes $\mathcal{M}$, the partial cross-sections $d\sigma/d\Omega_{\rm cm}$ (in the center-of-mass frame), and the total cross-sections for the following 3 processes:

(i) $\phi_1+\phi_2\to\phi_1+\phi_2$.

(ii) $\phi_1+\phi_1\to\phi_2+\phi_2$.

(iii) $\phi_1+\phi_1\to\phi_1+\phi_1$.

4. Finally, for a harder exercise consider the *linear sigma model*. This model comprises N massless scalar or pseudoscalar fields π_i and one massive scalar field σ with both quartic and cubic couplings to the pions, specifically

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sum\nolimits_i(\partial_\mu\pi_i)^2 \;+\; \frac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{\lambda}{8}\left(\sum\nolimits_i\pi_i^2 \;+\; \sigma^2 \;+\; 2f\sigma\right)^2\\ &= \frac{1}{2}\sum_i(\partial_\mu\pi_i)^2 \;+\; \frac{1}{2}(\partial_\mu\sigma)^2 \;-\; \frac{\lambda f^2}{2}\times\sigma^2\\ &\qquad -\;\frac{\lambda f}{2}\times\left(\sigma^3+\sigma\sum\nolimits_i\pi_i^2\right) \;-\; \frac{\lambda}{8}\left(\sum\nolimits_i\pi_i^2\;+\;\sigma^2\right)^2\end{aligned} \tag{3}$$

Both the masslessness of the π_i fields and the specific relations between the quartic couplings, the cubic couplings, and the sigma's mass $M_\sigma^2=\lambda f^2$ in this model stem from the *spontaneous breaking down* of the $O(N+1)$ symmetry, which I shall explain in class later this semester. I shall also explain the relation of this model to the approximate chiral symmetry of QCD and hence to the real-life pi-mesons and their low-energy scattering amplitudes.

But in this homework, you should simply take the Lagrangian (3) as it is, and explore its implications for the scattering of π particles.

(a) Write down all the vertices and all the propagators for the Feynman rules for this theory.

(b) Draw *all* the tree diagrams and calculate the tree-level scattering amplitudes of two pions to two pions, $\mathcal{M}_{\rm tree}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$.

(c) Show that due to specific relations between the quartic and the cubic couplings in the Lagrangian (3), in the low-energy limit $E_{\rm tot} \ll M_\sigma$, all the amplitudes $\mathcal{M}_{\rm tree}(\pi^j + \pi^k \to \pi^\ell + \pi^m)$ become small as $O(E_{\rm tot}^2/M_\sigma^2)$ or smaller.

(d) Use Mandelstam's variables s, t, u to show that when any of the incoming or outgoing pions' energy becomes small (while the other pions' energies are $O(M_\sigma)$), the scattering amplitudes become small as $O(E_{\rm small}/M_\sigma)$ or smaller.

Later in class, we shall learn that this behavior stems from the *Goldstone–Nambu theorem.*

(e) Write down specific tree-level amplitudes, partial cross-sections (in the CM frame), and total cross-sections for the processes

(i) $\pi^1 + \pi^2 \to \pi^1 + \pi^2$

(ii) $\pi^1 + \pi^1 \to \pi^2 + \pi^2$

(iii) $\pi^1 + \pi^1 \to \pi^1 + \pi^1$

in the low-energy limit $E_{\rm cm} \ll M_\sigma$.