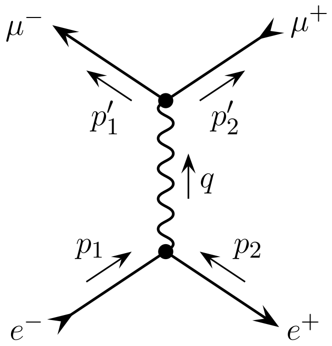
1. First, a reading assignment: §4.7 of the *Peskin&Schroeder* textbook about the Feynman rules of the Yukawa theory. Find out where the sign rules for the fermionic lines come from. Also find out the origin of the Yukawa potential $V(r) \propto e^{-mr}/r$. (There is also a much shorter explanation of the Yukawa theory and the Yukawa potential on the last three pages of my notes on QED Feynman rules.)

2. Second, a simple QED problem: pair production of muons in electron-positron collisions, $e^-e^+ \to \mu^-\mu^+$. As I explained in class, there is only one tree diagram for this process,



which yields the amplitude

$$\left\langle \mu^-,\mu^+ \right| \mathcal{M} \left| e^-,e^+ \right\rangle \;=\; \frac{e^2}{s} \times \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-). \tag{1}$$

In class I have focused on the un-polarized pair-production cross-section — see my notes on the subject, — but in this exercise you should focus on the polarized amplitudes for definite helicities of all 4 particles involved.

For simplicity, let us assume that all the particles are ultra-relativistic so that their Dirac spinors $u(e^-)$, $v(e^+)$, $u(\mu^-)$, $v(\mu^+)$ all have definite chiralities,

$$\begin{aligned} u_L &\approx \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \qquad & u_R &\approx \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix}, \\ v_L &\approx -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \qquad & v_R &\approx \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}. \end{aligned} \tag{2}$$

cf. homework set#7, eq. (7).

(a) Show that in the approximation (2),

$$\bar{v}(e_L^+)\gamma_\nu u(e_L^-) \;=\; \bar{v}(e_R^+)\gamma_\nu u(e_R^-) \;=\; 0, \tag{3}$$

which means there is no muon pairs production unless the initial electron and positron have *opposite helicities*.

This fact is of great practical importance for the electron-positron colliders. Any kind of particle production which proceeds through a virtual vector particle — a photon, or Z^0, or even something not yet discovered — would have the $\bar{v}(e^+)\gamma_\nu u(e^-)$ factor in the amplitude, so the electron and the positron must have have opposite helicities, or they would not annihilate each other and make pairs.

Now suppose we have a longitudinally polarized electron beam — say $\lambda = +\frac{1}{2}$ only — but the positron beam is un-polarized. Because of eq. (5), only the left-handed positrons would collide with the right-handed electrons and produce pairs, while the left-handed positrons would do something else. Likewise, if the electron beam has the $\lambda = -\frac{1}{2}$ polarization, then only the right-handed positrons would collide with our left-handed electrons and make pairs, while the left-handed positrons would do something else. Thus, as far as the pair-production is concerned, the positron beam could just as well be longitudinally polarized with $\lambda(e^+) = -\lambda(e^-)$.

In other words, if we want to study polarization effects in fermion pair production, it's enough to longitudinally polarize just the electron beam. We do not need to polarize the positron beam — which is much harder to do — because the electrons of a definite helicity would automatically select positrons of the opposite helicity.

(b) Show that the μ^- and the μ^+ produced in electron-positron collision must also have *opposite helicities* because otherwise

$$\bar{u}(\mu_L^-)\gamma^\nu v(\mu_L^+) \;=\; \bar{u}(\mu_R^-)\gamma^\nu v(\mu_R^+) \;=\; 0. \tag{4}$$

(c) Let's work in the center-of-mass frame where the initial e^- and e^+ collide along the z axis, $p_1^\nu = (E, 0, 0, +E)$, $p_2^\nu = (E, 0, 0, -E)$. Calculate the 4–vector $\bar{v}(e^+)\gamma^\nu u(e^-)$

in this frame and show that

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_R) \;=\; 2E\times(0,-i,+1,0)_\nu\,, \qquad \bar{v}(e^+_R)\gamma_\nu u(e^-_L) \;=\; 2E\times(0,+i,+1,0)_\nu\,. \tag{5}$$

(d) In the center-of-mass frame, the muons fly away in opposite directions at some angle θ to the electron / positron directions. Without loss of generality we may assume the muons' momenta being in the xz plane, thus

$$p_1'^\nu \;=\; (E,+E\sin\theta,0,+E\cos\theta), \qquad p_1'^\nu \;=\; (E,-E\sin\theta,0,-E\cos\theta) \tag{6}$$

Calculate the 4–vector $\bar{u}(\mu^-)\gamma_\nu v(\mu^+)$ for the muons and show that

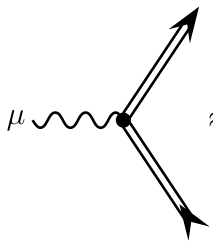
$$\begin{aligned} \bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_L) &= 2E\times(0,-i\cos\theta,-1,+i\sin\theta),\\ \bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_R) &= 2E\times(0,+i\cos\theta,-1,-i\sin\theta). \end{aligned} \tag{7}$$

(e) Now calculate the amplitudes (1) for all possible combinations of particles' helicities, calculate the partial cross-sections, and show that

$$\begin{aligned} \frac{d\sigma(e^-_L+e^+_R\to\mu^-_L+\mu^+_R)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R+e^+_L\to\mu^-_R+\mu^+_L)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s}\times(1+\cos\theta)^2,\\ \frac{d\sigma(e^-_L+e^+_R\to\mu^-_R+\mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R+e^+_L\to\mu^-_L+\mu^+_R)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s}\times(1-\cos\theta)^2,\\ \frac{d\sigma(e^-_L+e^+_L\to\mu^-_{\text{any}}+\mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R+e^+_R\to\mu^-_{\text{any}}+\mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = 0,\\ \frac{d\sigma(e^-_{\text{any}}+e^+_{\text{any}}\to\mu^-_L+\mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_{\text{any}}+e^+_{\text{any}}\to\mu^-_R+\mu^+_R)}{d\Omega_{\text{c.m.}}} = 0. \end{aligned} \tag{8}$$

(f) Finally, sum / average over the helicities and calculate the un-polarized cross-section for the muon pair production.

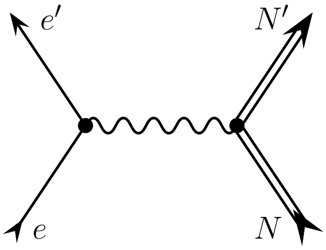
3. Third, another simple QED problem: *Mott scattering* of a relativistic electron off a heavy nucleus of charge $+Ze$ and mass $M_N \gg m_e$. As long as the electron's energy E_e is no larger than a few tens of MeV, we may treat the nucleus as a point source of the electric field, and we may also neglect its recoil. Hence, in the CM frame — which is essentially the nucleus's frame — we may approximate the nucleus-nucleus-photon vertex as



$$\approx\ -iZe \times \begin{cases} 2M_N & \text{for } \mu = 0, \\ 0 & \text{for } \mu = 1,2,3. \end{cases} \tag{9}$$

To be precise, this formula includes the vertex and the external leg factors for the incoming and outgoing nucleus, but it does not include the photon's propagator.

In QED, there is only one tree diagram for the Mott scattering, namely



(10)

(a) Evaluate this diagram and write down the amplitude $\mathcal{M}$ in terms of $q = p' - p$ and $\bar{u}(p', s')\gamma^0 u(p, s)$.

(b) Assume the initial electron beam is un-polarized (*i.e.*, both values of spin s are equally likely) and the detector for the scattered electron does not measure its spin s' but only momentum s'. Show that for such an experiment,

$$\frac{d\sigma}{d\Omega} \;=\; \frac{(Z\alpha)^2}{(\mathbf{q}^2)^2} \times \frac{1}{2}\sum_{s,s'} \left|\bar{u}(p', s')\gamma^0 u(p, s)\right|^2 \tag{11}$$

where $\alpha = e^2/4\pi$ (or in conventional units, $\alpha = e^2/\hbar c$; anyhow, $\alpha \approx 1/137$.)

(c) Sum over the electron spins and show that

$$\frac{1}{2}\sum_{s,s'}\left|\bar{u}(p',s')\gamma^0 u(p,s)\right|^2 \;=\; 2\left(m_e^2 \,+\, EE' \,+\, \mathbf{p}\cdot\mathbf{p}'\right). \tag{12}$$

(d) Finally, assemble all the factors together and derive Mott formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \;=\; \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \times \frac{1-\beta^2\sin^2(\theta/2)}{\gamma^2} \tag{13}$$

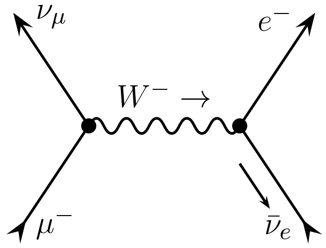
where β is the electron's speed (in $c=1$ units), $\gamma = E/m_e$, and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \;=\; \frac{(Z\alpha)^2}{4m_e^2\beta^4\sin^4(\theta/2)} \tag{14}$$

is the classical Rutherford scattering cross-section (translated into $\hbar = c = 1$ units).

4. Finally, a harder problem on spin averaging and also on 3-body phase space. It involves weak interactions rather than QED, but uses the same techniques.

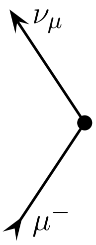
Consider the muon decay into an electron, an electron-flavored antineutrino, and a muon-flavored neutrino, $\mu^- \to e^-\bar{\nu}_e\nu_\mu$. At the tree level of the Standard model, this decay proceeds through a single Feynman diagram



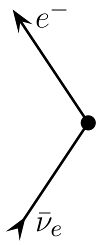
(15)

Since I have not yet explained the Standard Model in class — although I plan to do it after the Thanksgiving break, — let me simply spell out the Feynman rules relevant to this diagram.

- The vertices and the external fermionic legs attached to them are



$$= \bar{u}(\nu_\mu)\left(\frac{-ig_2}{\sqrt{2}}\gamma^\kappa \frac{1-\gamma^5}{2}\right)u(\mu^-),$$



$$= \bar{u}(e^-)\left(\frac{-ig_2}{\sqrt{2}}\gamma^\lambda \frac{1-\gamma^5}{2}\right)v(\bar{\nu}_e), \tag{16}$$

  where g_2 is the $SU(2)_W$ gauge coupling.

- W^- is a massive vector particle, so its propagator is

$$= \frac{-i}{q^2 - M_W^2}\left(g_{\kappa\lambda} - \frac{k_\kappa k_\lambda}{M_W^2}\right) \xrightarrow[|q|\ll M_W]{} \frac{ig_{\kappa\lambda}}{M_W^2}\,. \tag{17}$$

  The approximation here corresponds to the effective Fermi theory of weak interactions. It is valid for all nuclear beta decays as well as weak decays of all particles much lighter than $M_W \approx 80$ GeV. In particular, it is valid for the muon decay in question.

(a) Assemble the muon decay amplitude (in the Fermi theory approximation) to

$$\mathcal{M} \;=\; -\frac{G_F}{\sqrt{2}} \times \left[\bar{u}(\nu_\mu)\gamma^\lambda(1-\gamma^5)u(\mu^-)\right] \times \left[\bar{u}(e^-)\gamma_\lambda(1-\gamma^5)v(\bar{\nu}_e)\right] \tag{18}$$

where

$$G_F \;=\; \frac{g_2^2}{4\sqrt{2}M_W^2} \;\approx\; 1.17\cdot 10^{-5}\,\mathrm{GeV}^{-2} \tag{19}$$

is the Fermi constant of low-energy weak interactions.

(b) Sum the absolute square of the amplitude (18) over the final particle spins, average over the initial muon's spin, and write the result as a product of two Dirac traces,

$$\overline{|\mathcal{M}|^2} \;\stackrel{\text{def}}{=}\; \frac{1}{2}\sum_{\substack{\text{all}\\\text{spins}}} |\mathcal{M}|^2 \;=\; \frac{G_F^2}{4} \times \mathrm{tr}(\text{matrix product \#1}) \times \mathrm{tr}(\text{matrix product \#2}). \tag{20}$$

Note: since the neutrino, the antineutrino, and even the electron are much lighter than the initial muon, you may neglect their masses.

(c) Evaluate the traces in eq. (20).

(d) Sum over the Lorentz indices and show that altogether

$$\overline{|\mathcal{M}|^2} \;=\; 64G_F^2(p_\mu\cdot p_{\bar\nu})\,(p_e\cdot p_\nu). \tag{21}$$

The following lemma is very useful for three-body decays like $\mu^-\to e^-+\nu_\mu+\bar\nu_e$:
For a decay of initial particle of mass M_0 into three final particles of respective masses m_1, m_2, and m_3, the partial decay rate in the rest frame of the original particle is

$$d\Gamma \;=\; \frac{1}{2M_0}\times\overline{|\mathcal{M}|^2}\times\frac{d^3\Omega}{256\pi^5}\times dE_1\,dE_2\,dE_3\,\delta(E_1+E_2+E_3-M_0), \tag{22}$$

where $d^3\Omega$ comprises three angular variables parametrizing the directions of the three final-state particles relative to some external frame, but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0$) and one more angle to fix the direction of *e.g.*, $\mathbf{p}_1$ in that plane. Altogether, $\int d^3\Omega=4\pi\times2\pi=8\pi^2$.

(e) Prove this lemma.

The electron and the neutrinos are so much lighter then the muon that in most decay events all three final-state particles are ultra-relativistic. This allows us to approximate $m_e\approx m_\nu\approx m_{\bar\nu}\approx 0$, which gives us rather simple limits for the final particles' energies.

(f) Show that when $m_1=m_2=m_3=0$, the kinematically allowed range of the final particles' energies is given by

$$0\;\le\;E_1,E_2,E_3\;\le\tfrac12 M_0\qquad\text{while}\quad E_1+E_2+E_3\;=\;M_0. \tag{23}$$

Note however that for non-zero masses $m_{1,2,3}$, the allowed energy range becomes much more complicated.

Experimentally, the neutrinos and the antineutrinos are hard to detect. But it is easy to measure the muon's net decay rate $\Gamma=1/\tau_\mu$ and the energy distribution $d\Gamma/dE_e$ of the electrons produced by decaying muons.

(g) Integrate the muon's partial decay rate over the final particle energies and derive first the $d\Gamma/dE_e$ and then the total decay rate.