This homework set has 4 problems. The first 3 problems are about spontaneous symmetry breaking and the Higgs mechanism. The last problem is about the $e^- + e^+ \to \mu^- + \mu^+$ process in the Standard Model, where the virtual particle in the s channel can be a photon, a Z^0, or a Higgs.

1. Let's start with the $SU(2)$ gauge symmetry coupled to a doublet of Higgs fields $\Phi^{1,2}(x)$,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; D_\mu\Phi^\dagger D^\mu\Phi \;-\; \frac{\lambda}{2}\left(\Phi^\dagger\Phi \;-\; \frac{v^2}{2}\right)^2. \tag{1}$$

As discussed in class (*cf.* my notes on the Higgs mechanism, pages 8–10), the Higgs doublet develops a vacuum expectation value (VEV)

$$\langle\Phi\rangle \;=\; \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \tag{2}$$

which completely breaks the $SU(2)$ gauge symmetry and gives the vector bosons equal masses $M_V = \frac{1}{2}gv$.

The reason all 3 vector fields get the same mass is a non-obvious global $SU(2)$ symmetry of the scalar fields. Indeed, the scalar potential is invariant under the $SO(4)$ symmetry which mixes the real and the imaginary parts of the scalar fields, and the $SO(4)$ or rather Spin(4) group happens to be isomorphic to $SU(2)\times SU(2)$. The gauge symmetry acts as one of these $SU(2)$ factors while the other $SU(2)$ factor of the $SO(4)$ remains a global symmetry.

To make both $SU(2)_{\text{local}}$ and $SU(2)_{\text{global}}$ symmetries manifest, consider a 2×2 matrix

$$W \;=\; \begin{pmatrix} +\Phi_2^* & +\Phi^1 \\ -\Phi_1^* & +\Phi^2 \end{pmatrix} \tag{3}$$

(a) Show that $\sigma_2 W^* \sigma_2 = W$, and that any matrix obeying this skewed reality condition has form (3) for come complex Φ^1 and Φ^2.

(b) Show that under a gauge transform of the $\Phi(x)$ doublet, the $W(x)$ matrix-valued field transforms in the similar manner,

$$\Phi'(x) \;=\; U(x)\Phi(x) \quad\Longrightarrow\quad W'(x) \;=\; U(x)W(x) \quad \forall U(x)\in SU(2). \tag{4}$$

(c) Show that in terms of the matrix-valued field $W(x)$ the Lagrangian (1) becomes

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \;+\; \tfrac{1}{2}\operatorname{tr}\bigl(D_\mu W^\dagger D^\mu W\bigr) \;-\; \frac{\lambda}{8}\bigl(\operatorname{tr}(W^\dagger W) \;-\; v^2\bigr)^2, \tag{5}$$

and that it is invariant under $SU(2)_{\rm local}\times SU(2)_{\rm global}$ symmetry which acts as

$$\begin{aligned} W'(x) \;&=\; U_L(x)W(x)U_G^\dagger,\\ A'_\mu(x) \;&=\; U_L(x)A_\mu(x)U_L^{-1}(x) \;+\; \frac{i}{g}(\partial_\mu U_L(x))U_L^{-1}(x),\\ &\text{local } U_L(x)\in SU(2),\text{ global } U_G\in SU(2). \end{aligned} \tag{6}$$

The vacuum expectation value (2) — which translates to

$$\langle W\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{7}$$

— is not invariant under the $SU(2)_{\rm global}$ symmetry factor of (6). Nevertheless, there are no Goldstone bosons because $\langle W\rangle$ is invariant under a modified global $SU(2)$ symmetry:

- Any $U_G\in SU(2)$ accompanied by $U_L(x)=U_G\ \forall x$.

(d) Argue that the generators of this modified $SU(2)'_{\rm global}$ symmetry are

$$T^a[SU(2)'_{\rm global}] \;=\; T^a[SU(2)_{\rm global}] \;+\; T^a[SU(2)_{\rm local}] \tag{8}$$

and check that the VEV $\langle W\rangle$ is indeed invariant under the $SU(2)'_{\rm global}$.

Since the global $SU(2)$ symmetry is modified (by mixing with the gauge symmetry) rather than spontaneously broken, there are no Goldstone bosons for this symmetry. Moreover, the modification (8) make the symmetry act non-trivially on the massive vector fields.

(e) Show that the massive vector fields form a triplet of the $SU(2)'_{\rm global}$, and that's why they all have the same mass.

2. Next, a more complicated problem on symmetry breaking. Consider an $N \times N$ matrix $\Phi(x)$ of complex scalar fields $\Phi^i{}_j(x)$, $i, j = 1, \ldots, N$. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; \operatorname{tr}\left(\partial^\mu \Phi^\dagger\, \partial_\mu \Phi\right) \;-\; V(\Phi^\dagger \Phi) \tag{9}$$

where the potential is

$$V \;=\; \frac{\alpha}{2}\operatorname{tr}\left(\Phi^\dagger\Phi\Phi^\dagger\Phi\right) \;+\; \frac{\beta}{2}\left(\operatorname{tr}\left(\Phi^\dagger\Phi\right)\right)^2 \;+\; m^2 \operatorname{tr}\left(\Phi^\dagger\Phi\right). \tag{10}$$

(a) Show that this theory has global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ acting as

$$\Phi(x) \;\to\; e^{i\theta} U_L \Phi(x) U_R^\dagger, \qquad U_L, U_R \in SU(N). \tag{11}$$

For $\alpha = 0$ the theory would have a much larger continuous symmetry group $SO(2N^2)$ which mixes up the real and the imaginary parts of all the $\Phi^i{}_j$ regardless of their complex structure or the $N \times N$ matrix structure of Φ. But for $\alpha \neq 0$ this huge symmetry group is reduced to $G = SU(N)_L \times SU(N)_R \times U(1)$. Proving that the $\operatorname{tr}\left(\Phi^\dagger\Phi\Phi^\dagger\Phi\right)$ term has no other continuous symmetries besides G is a hard exercise in group theory, which fortunately is beyond the scope of this homework.

From now on, we take $\alpha, \beta > 0$ but $m^2 < 0$. In this regime, V is minimized for non-zero vacuum expectation values $\langle \Phi \rangle \neq 0$ of the scalar fields.

(b) Let $(\kappa_1, \ldots, \kappa_N)$ be eigenvalues of the hermitian matrix $\Phi^\dagger\Phi$. Express the potential (10) in terms of these eigenvalues and show that the minimum lies at

$$\kappa_1 \;=\; \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} \;>\; 0. \tag{12}$$

In terms of the matrix Φ, eq. (12) means $\Phi = C \times$ a unitary matrix. All such minima are related by symmetries (11) to $\Phi = C \times$ the unit matrix, so without loss of generality we may assume that the vacuum lies at

$$\langle\Phi\rangle = C\times\mathbf{1}_{N\times N} \quad i.e. \quad \left\langle\Phi^i{}_j\right\rangle = C\times\delta^i_j\,. \tag{13}$$

(c) Show that the symmetries (11) preserving these VEVs are limited to the $U_L = U_R \in SU(N)$ and $\theta = 0$. In other words, the $SU(N)_L \times SU(N)_R \times U(1)$ symmetry of the theory is spontaneously broken down to the $SU(N)_V$ subgroup.

Now that we know the vacuum state of the theory and its symmetry, let's find the particle spectrum of the theory.

(d) Expand the scalar potential in powers of the $\delta\Phi(x) = \Phi(x) - \langle\Phi\rangle$ and the $\delta\Phi^\dagger(x)$,

$$V(\delta\Phi^\dagger,\delta\Phi) = \text{const} + V_1 + V_2 + V_3 + V_4\,. \tag{14}$$

Show that $V_1 = 0$ while

$$V_2 = \frac{\alpha C^2}{2}\,\mathrm{tr}\big((\delta\Phi^\dagger + \delta\Phi)^2\big) + \frac{\beta C^2}{2}\,\mathrm{tr}^2(\delta\Phi^\dagger + \delta\Phi). \tag{15}$$

(e) Altogether, the N^2 complex scalar fields give rise to $2N^2$ particle species. Find the masses of all those particles from eq. (15).
Hint: Split the complex matrix Φ into its hermitian and antihermitian parts, and also into trace and traceless parts,

$$\delta\Phi(x) = \frac{\chi_1(x) + i\chi_2(x)}{\sqrt{2N}}\times\mathbf{1}_{N\times N} + \frac{\varphi_1(x) + i\varphi_2(x)}{\sqrt{2}} \tag{16}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are traceless hermitian matrices (or rather matrix-valued fields) while $\chi_1(x)$ and $\chi_2(x)$ are ordinary real fields.

(f) Finally, organize the $2N^2$ particles into multiplets of the unbroken $SU(N)_V$ symmetry and make sure that all members of each multiple have the same mass.

Also, check the Nambu–Goldstone theorem for this model — verify that for each *spontaneously broken* generator of the symmetry (11) there is a massless particle with similar quantum numbers WRT the unbroken $SU(N)_V$ subgroup.

3. In the previous problem we had the continuous global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ spontaneously broken down to its $H = SU(N)_V$ subgroup. Now let's gauge the entire $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry and work out the Higgs mechanism.

Thus, consider a theory of N^2 complex scalar fields $\Phi^i{}_j(x)$ organized into an $N \times N$ matrix Φ, and $2N^2 - 1$ real vector fields $B_\mu(x)$, $L^a_\mu(x)$, and $R^a_\mu(x)$, the latter organized into traceless hermitian matrices $L_\mu(x) = \sum_a L^a_\mu(x) \times \frac{1}{2}\lambda^a$ and $R_\mu(x) = \sum_a R^a_\mu(x) \times \frac{1}{2}\lambda^a$, where $a = 1, \ldots, (N^2 - 1)$ and λ^a are the Gell-Mann matrices of $SU(N)$. The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} - \tfrac{1}{2} \operatorname{tr}\left(L_{\mu\nu} L^{\mu\nu}\right) - \tfrac{1}{2} \operatorname{tr}\left(R_{\mu\nu} R^{\mu\nu}\right) + \operatorname{tr}\left(D^\mu \Phi^\dagger \, D_\mu \Phi\right) - V(\Phi^\dagger \Phi), \tag{17}$$

where

$$\begin{aligned}
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,, \\
L_{\mu\nu} &= \partial_\mu L_\nu - \partial_\nu L_\mu + ig[L_\mu, L_\nu], \\
R_{\mu\nu} &= \partial_\mu R_\nu - \partial_\nu R_\mu + ig[R_\mu, R_\nu], \\
D_\mu \Phi &= \partial_\mu \Phi + ig' B_\mu \Phi + ig L_\mu \Phi - ig \Phi R_\mu\,, \\
D_\mu \Phi^\dagger &= (D_\mu \Phi)^\dagger = \partial_\mu \Phi^\dagger - ig' B_\mu \Phi^\dagger + ig R_\mu \Phi^\dagger - ig \Phi^\dagger L_\mu\,.
\end{aligned} \tag{18}$$

For simplicity, I assume equal gauge couplings $g_L = g_R = g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

Finally, the scalar potential V is precisely as in the previous problem,

$$V = \frac{\alpha}{2} \operatorname{tr}\left(\Phi^\dagger \Phi \Phi^\dagger \Phi\right) + \frac{\beta}{2} \operatorname{tr}^2\left(\Phi^\dagger \Phi\right) + m^2 \operatorname{tr}\left(\Phi^\dagger \Phi\right), \qquad \alpha, \beta > 0, \quad m^2 < 0, \tag{19}$$

hence similar VEVs of the scalar fields: up to a gauge symmetry,

$$\langle \Phi \rangle = C \times \mathbf{1}_{N\times N} \quad \text{where} \quad C = \sqrt{\frac{-m^2}{\alpha + N\beta}}\,, \tag{20}$$

which breaks the $G = SU(N)_L \times SU(N)_R \times U(1)$ symmetry down to the $H = SU(N)_V$ subgroup.

(a) The Higgs mechanism makes N^2 out of $2N^2 - 1$ vector fields massive. Calculate their masses by plugging $\langle \Phi \rangle$ for the $\Phi(x)$ into the $\operatorname{tr}(D_\mu \Phi^\dagger \, D^\mu \Phi)$ term of the Lagrangian.

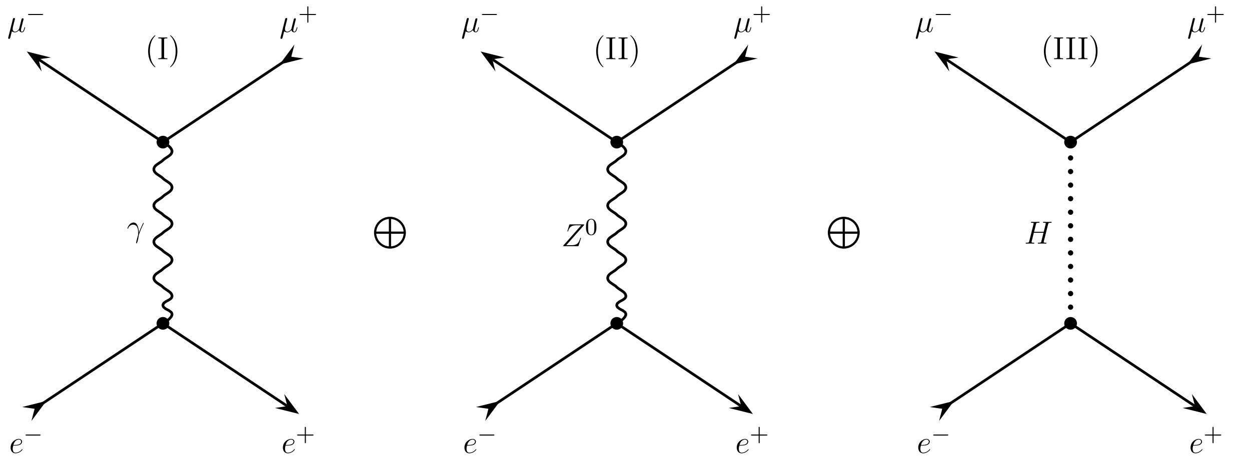
In particular, show that the abelian gauge field B_μ and the $X^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu - R^a_\mu)$ combinations of the $SU(N)$ gauge fields become massive, while the $V^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu + R^a_\mu)$ combinations remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V^a_\mu(x)$ by freezing all the other fields, *i.e.* setting $B_\mu(x) \equiv 0$, $X^a_\mu(x) \equiv 0$, and $\Phi(x) \equiv \langle\Phi\rangle$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

⋆ For extra challenge, allow for un-equal gauge couplings $g_L \neq g_R$. Find which combinations of the $L^a_\mu(x)$ and $R^a_\mu(x)$ fields remain massless in this case, then derive the effective Lagrangian for these massless fields by freezing everything else. As in part (b), you should get an $SU(N)$ Yang–Mills theory, but this time the gauge coupling is

$$g_v = \frac{g_L g_R}{\sqrt{g_L^2 + g_R^2}}. \tag{21}$$

4. Finally, consider the $e^+e^- \to \mu^+\mu^-$ pair production in the Standard Model of electroweak interactions rather than in just QED. Unlike QED, the Standard Model has three diagrams contributing to this process: one with the virtual photon in the s channel, one with the virtual Z^0 gauge boson, and one with the virtual Higgs scalar,

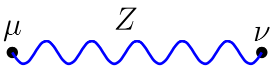


$$\mathcal{M}(e^-, e^+ \to \mu^-, \mu^+) = \mathcal{M}_\gamma + \mathcal{M}_Z + \mathcal{M}_H. \tag{22}$$

The first diagram was studied in class and also in the homework set#9. In this problem, we shall focus on the other two diagrams — especially the diagram (II) with a virtual Z^0 — and on their interference with the first diagram.

(a) Write down the amplitude $\mathcal{M}_H$ due to virtual Higgs scalar (diagram III). Also, relate the Yukawa couplings of the Higgs to the electrons and the muons to the fermion's masses, then argue that these couplings are so much smaller than the gauge couplings e or g' that the $\mathcal{M}_H$ is negligibly small compared to the $\mathcal{M}_Z$ or $\mathcal{M}_\gamma$ amplitudes.

Next, consider the amplitude due to a virtual Z^0 in the s channel (diagram II). For simplicity, let's work in the unitary gauge where the 'eaten-up' scalars do not have any vertices or propagators, while the massive gauge bosons like Z_0 have propagators



$$= \frac{i}{q^2 - M_Z^2 + iM_Z\Gamma_Z} \times \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2}\right). \tag{23}$$

(b) Derive the electron-Z and muon-Z vertices in diagram (II) from the neutral week current,

$$\mathcal{L} \supset -g' Z_\lambda \times \sum_{\substack{\text{quarks and}\\ \text{leptons}}} \overline{\Psi}\gamma^\lambda \left(T^3 \frac{1-\gamma^5}{2} - Q\sin^2\theta\right)\Psi, \tag{24}$$

cf. my notes on the electroweak interactions of quarks and leptons, and write down the amplitude $\mathcal{M}_Z$. For simplicity, approximate $\sin^2\theta \approx \frac{1}{4}$ (experimentally, $\sin^2\theta \approx 0.233$) so that for the charged leptons like the electron or the muon

$$T^3 \frac{1-\gamma^5}{2} - Q\sin^2\theta = \frac{-1+\gamma^5}{4} + \sin^2\theta \approx \frac{\gamma^5}{4}, \tag{25}$$

(c) Assume both the electrons and the muons to be ultra-relativistic ($E_{\text{c.m.}} = O(M_Z) \gg m_\mu, m_e$) and evaluate the amplitude $\mathcal{M}_Z$ for all possible particle helicities. (Use the center-of-mass frame.) I suggest you proceed exactly as in homework set#9 (problem 2) for the $\mathcal{M}_\gamma$ amplitude, but mind the γ^5 factors in the vertices and the massive vector propagator for the Z_0. Your answer should have form

$$\mathcal{M}_Z = \pm F(s) \times \mathcal{M}_\gamma \tag{26}$$

where the $\pm$ sign depends on the helicities while

$$F(s) \;=\; \left(\frac{e}{4g'}\right)^2 \frac{s}{s - M_Z^2 + i\Gamma_Z M_Z}\,. \tag{27}$$

(d) Combine the amplitudes due to virtual Z and virtual photon and calculate the polarized partial cross-sections $d\sigma(e^-e^+ \to \mu^-\mu^+)/d\Omega$ as functions of EM energy$^2 = s$, scattering angle θ, and helicities of all 4 fermions involved. Specifically, show that

$$\begin{aligned}
\frac{d\sigma(e_L^- + e_L^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^- + e_R^+ \to \mu_{\rm any}^- + \mu_{\rm any}^+)}{d\Omega_{\rm c.m.}} = 0,\\
\frac{d\sigma(e_{\rm any}^- + e_{\rm any}^+ \to \mu_L^- + \mu_L^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_{\rm any}^- + e_{\rm any}^+ \to \mu_R^- + \mu_R^+)}{d\Omega_{\rm c.m.}} = 0.
\end{aligned} \tag{28}$$

while

$$\begin{aligned}
\frac{d\sigma(e_L^- + e_R^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\rm c.m.}}\\
&= \frac{\alpha^2}{4s} \times |1+F(s)|^2 \times (1+\cos\theta)^2,\\
\frac{d\sigma(e_L^- + e_R^+ \to \mu_R^- + \mu_L^+)}{d\Omega_{\rm c.m.}} &= \frac{d\sigma(e_R^- + e_L^+ \to \mu_L^- + \mu_R^+)}{d\Omega_{\rm c.m.}}\\
&= \frac{\alpha^2}{4s} \times |1-F(s)|^2 \times (1-\cos\theta)^2,
\end{aligned} \tag{29}$$

where the $|1 \pm F(s)|^2$ factor stem from the interference between the virtual-photon and virtual-Z diagrams.

(e) Finally, assume un-polarized electron and positron beams and a spin-blind muon detector. Calculate the total cross section $\sigma(e^+e^- \to \mu^+\mu^-)$ and the forward-backward asymmetry

$$A \;=\; \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \tag{30}$$

as functions of the total energy $E_{\rm c.m.}$.

Note: In QED, the tree-level pair production is symmetric with respect to $\theta \to \pi - \theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual-Z diagrams.