1. First, read about the *optical theorem* in §7.3 of *Peskin and Schroeder*, and also in §3.6 of *Weinberg*. In *Peskin and Schroeder*, pay particular attention to the optical theorem for the Feynman diagrams and the Cutkosky's cutting rules, but don't skip the other subjects such as the optical theorem for particle decays. In *Weinberg*, note the relation between the S-matrix unitarity $SS^\dagger = 1$ and the Boltzmann's *H theorem*.

2. Next, consider the Yukawa theory of a Dirac fermion field Ψ coupled to a real scalar field Φ according to

$$\mathcal{L} \;=\; \overline{\Psi}(i\partial\!\!\!/ - m_f)\Psi \;+\; \tfrac{1}{2}(\partial_\mu\Phi)^2 \;-\; \tfrac{1}{2}m_s^2\Phi^2 \;+\; g\Phi\overline{\Psi}\Psi. \tag{1}$$

For $M_s > 2M_f$, the scalar particle becomes unstable: it decays into a fermion and an antifermion, $S \to f + \bar{f}$.

(a) Calculate the tree-level decay rate $\Gamma(S \to f + \bar{f})$.

(b) In class, we have calculated

$$\Sigma_\Phi^{1\,\text{loop}}(p^2) \;=\; \frac{12g^2}{16\pi^2}\int_0^1 d\xi\,\Delta(\xi)\times\left[\frac{1}{\epsilon} - \gamma_E + \frac{1}{3} + \log\frac{4\pi\mu^2}{\Delta(\xi)}\right] \tag{2}$$

$$\text{for } \Delta(\xi) \;=\; m_f^2 \;-\; \xi(1-\xi)p^2. \tag{3}$$

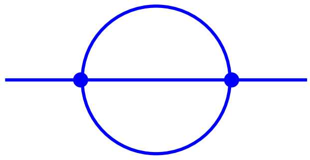
Show that for $p^2 > 4m_f^2$, this $\Sigma_\Phi(p^2)$ has an imaginary part and calculate it for $p^2 = M_s^2 + i\epsilon$.
Note: at this level, you may neglect the difference between m_f^{bare} and M_f^{physical}.

(c) Verify that

$$\operatorname{Im}\Sigma_\Phi^{1\,\text{loop}}(p^2 = M_s^2 + i\epsilon) \;=\; -M_s\Gamma^{\text{tree}}(S \to f + \bar{f}) \tag{4}$$

and explain this relation in terms of the optical theorem.

3. And now, a harder exercise about the scalar $\lambda\phi^4$ theory. As discussed in class, in this theory the field strength renormalization begins at the two-loop level. Specifically, the leading contribution to the $d\Sigma(p^2)/dp^2$ — and hence to the $Z - 1$ — comes from the two-loop 1PI diagram


(5)

Your task is to evaluate this contribution.

(a) First, write the $\Sigma(p^2)$ from the diagram (5) as an integral over two independent loop momenta, say q_1^μ and q_2^μ, then use the Feynman's parameter trick — *cf.* eq. (F.d) of the homework set#13 — to write the product of three propagators as

$$\iiint d\xi\, d\eta\, d\zeta\, \delta(\xi + \eta + \zeta - 1)\, \frac{2}{(\mathcal{D})^3} \tag{6}$$

where $\mathcal{D}$ is a quadratic polynomial of the momenta q_1, q_2, p, and mass m with Feynman-parameter dependent coefficients.

Warning: Do not set $p^2 = m^2$ but keep p an independent variable.

(b) Next, change the independent loop momentum variables from q_1 and q_2 to $k_1 = q_1 +$ something $\times\, q_2 +$ something $\times\, p$ and $k_2 = q_2 +$ something $\times\, p$ to give $\mathcal{D}$ a simpler form

$$\mathcal{D} \;=\; \alpha \times k_1^2 \;+\; \beta \times k_2^2 \;+\; \gamma \times p^2 \;-\; m^2 \;+\; i0 \tag{7}$$

for some (ξ, η, ζ)–dependent coefficients α, β, γ, for example

$$\alpha \;=\; (\xi + \zeta), \qquad \beta \;=\; \frac{\xi\eta + \xi\zeta + \eta\zeta}{\xi + \zeta}, \qquad \gamma \;=\; \frac{\xi\eta\zeta}{\xi\eta + \xi\zeta + \eta\zeta}. \tag{8}$$

Make sure the momentum shift has unit Jacobian $\partial(q_1, q_2)/\partial(k_1, k_2) = 1$.

(c) Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1\, d^4k_2\, \frac{1}{\mathcal{D}^4}. \tag{9}$$

Note that although this momentum integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

(d) To evaluate the momentum integral (9), Wick-rotate the momenta k_1 and k_2 to the Euclidean space, and then use the dimensional regularization. Here are some useful formulæ for this calculation:

$$\frac{6}{A^4} = \int_0^\infty dt\, t^3\, e^{-At}, \tag{10}$$

$$\int \frac{d^D k}{(2\pi)^D}\, e^{-ctk^2} = \big(4\pi ct\big)^{-D/2}, \tag{11}$$

$$\Gamma(2\epsilon)X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \tfrac{1}{2}\log X + O(\epsilon). \tag{12}$$

(e) Assemble your results as

$$\frac{d\Sigma(p^2)}{dp^2} = -\frac{\lambda^2}{12(4\pi)^4} \iiint_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \times \left(\frac{1}{\epsilon} - 2\gamma_E + 2\log\frac{4\pi\mu^2}{m^2} + \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2}\right). \tag{13}$$

(f) Before you evaluate the Feynman parameter integral (13) — which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually calculating the integrals

$$\iiint_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}, \tag{14}$$

$$\iiint_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \times \frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \times \log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta(p^2/m^2))^2},$$

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where $\xi,\eta \to 0$ while $\zeta \to 1$ (or $\xi,\zeta \to 0$, or $\eta,\zeta \to 0$).

- This calculation shows that

$$\frac{d\Sigma}{dp^2} = \frac{\text{constant}}{\epsilon} + \mathit{a\_finite\_function}(p^2) \tag{15}$$

and hence

$$\Sigma(p^2) \;=\; (\text{a divergent constant}) \;+\; (\text{another divergent constant})\times p^2 \\ +\; \textit{a\_finite\_function}(p^2) \tag{16}$$

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of $\Sigma(p^2)$ may be canceled with just two counterterms, δ^m and $\delta^Z \times p^2$.

For the purposes of calculating the field strength renormalization factor

$$Z \;=\; \left[1 \;-\; \frac{d\Sigma}{dp^2}\right]^{-1} \tag{17}$$

we need to evaluate the derivative $d\Sigma/dp^2$ at $p^2 = M_{\rm ph}^2$ — the physical mass² of the scalar particle. However, to the leading non-trivial order in λ we may approximate $M_{\rm ph}^2 \approx m_{\rm bare}^2$ and set $p^2 = m^2$ in the Feynman-parameter integral (13). Consequently, the second integral (14) becomes a little simpler, although it is still a frightful mess.

⋆ Optional exercise: Evaluate the integrals (14) for $p^2 = m^2$ and show that

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1)\times\frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3} \;=\; \frac{1}{2}\,, \tag{18}$$

$$\iiint\limits_{\xi,\eta,\zeta\geq 0} d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1)\times\frac{\xi\eta\zeta}{(\xi\eta+\xi\zeta+\eta\zeta)^3}\times\log\frac{(\xi\eta+\xi\zeta+\eta\zeta)^3}{(\xi\eta+\xi\zeta+\eta\zeta-\xi\eta\zeta)^2} \;=\; -\frac{3}{4}\,.$$

Do not try to do this calculation by hand — it would take way too much time. Instead, use *Mathematica* or equivalent software. To help it along, replace the (ξ,η,ζ) variables with (x,w) according to

$$\xi \;=\; w\times x, \quad \eta \;=\; w\times(1-x), \quad \zeta \;=\; 1-w, \\ \iiint d\xi d\eta d\zeta\, \delta(\xi+\eta+\zeta-1) \;=\; \int_0^1 dx \int_0^1 dw\, w, \tag{19}$$

then integrate over w first and over x second.

Alternatively, you may evaluate the integrals like this numerically. In this case, don't bother changing variables, just use a simple 2D grid spanning a triangle defined by $\xi + \eta + \zeta = 1$, $\xi, \eta, \zeta \geq 0$; modern computers can sum up a billion grid points in less than a minute. But watch out for singularities at the corners of the triangle.

(g) Finally, assemble your results and calculate the field strength renormalization factor Z to the two-loop order.