1. As a warm-up exercise, consider the electric charge renormalization in QED.

   In general, a coupling g of an operator involving n fields $\hat{\phi}_a(x),\ldots,\hat{\phi}_n(x)$ has beta-function

$$\beta_g \;=\; (\gamma_1+\cdots+\gamma_n)\times\bigl(g+\delta^g\bigr)\;-\;\frac{d\delta^g}{d\log E}\,, \tag{1}$$

   cf. my notes on the renormalization group, eq. (131) on page 25. When applied to QED, this formula yields

$$\beta_e \;=\; (2\gamma_e+\gamma_\gamma)\times\bigl(e+e\delta_1\bigr)-\frac{d(e\delta_1)}{d\log E}\,. \tag{2}$$

   Use the Ward identity $\delta_1(E)=\delta_2(E)$ to reduce this formula to

$$\beta_e \;=\; \gamma_\gamma\times e. \tag{3}$$

2. Now consider the electron mass renormalization in QED. At high energies $E\gg m_e$, we may treat the electron mass m_e as a small coupling between the left-handed and the right-handed Weyl spinor components of the electron field $\hat{\Psi}(x)$. Consequently, we may write the renormalization group equation for the running electron mass $m_e(E)$ just as we would write it for any other kind of a small coupling,

$$\frac{dm_e(E)}{d\log E} \;=\; \beta_m(m_e,\alpha). \tag{4}$$

   Your task is to calculate the β-function here to the one-loop order, and then to solve the RGE (4).

   (a) Before you calculate anything, use the axial symmetry of a massless electron to argue that for a small but non-zero electron's mass, the counterterm δ_m should be proportional to m_e itself and therefore be logarithmically rather than quadratically divergent,

$$\delta_m \;\sim\; \alpha m_e\times\log\Lambda \quad\text{rather than}\quad \delta_m\;\sim\;\alpha\times\Lambda. \tag{5}$$

   (b) Now, a bit of hard work: Calculate to the one-loop order the UV-infinite parts of the δ_2 and δm counterterms as functions of the gauge-fixing parameter ξ. Assume

$E \gg m_e$ and use the off-shell renormalization condition for these counterterms:

$$\begin{aligned} \text{expand} \quad \Sigma_{\text{net}}(\not{p}) &= A(p^2)\times \not{p} + B(p^2)\times m \\ \text{and demand} \quad A = B &= 0 \quad \text{for} \quad p^2 = -E^2. \end{aligned} \tag{6}$$

The counterterms you obtain in part (a) should have form

$$\delta_2(E) = \frac{C_2(\xi)\alpha}{2\pi} \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + \text{const}\right), \tag{7}$$

$$\delta_m(E) = \frac{C_m(\xi)\alpha m}{2\pi} \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + \text{const}\right), \tag{8}$$

for some ξ–dependent coefficients $C_2(\xi)$ and $C_m(\xi)$.

(c) Check that the difference $C_m(\xi) - C_2(\xi)$ does not depend on the gauge parameter ξ. If it does, go back to part (b) and check for mistakes.

Applying the general formula (1) for the β-function of an n-field coupling to the running electron mass $m_e(E)$, we get

$$\beta_m = 2\gamma_2 \times \big(m(E) + \delta_m(E)\big) - \frac{d\delta_m(E)}{d\log(E)}. \tag{9}$$

(d) Use eqs. (7) and (8) to show that to the one-loop order this formula yields

$$\frac{dm_e(E)}{d\log E} = \beta_m = 2(C_m - C_2) \times \frac{\alpha(E)m(E)}{2\pi} + O(\alpha^2 m). \tag{10}$$

Note: the running mass should be gauge invariant, that's why I asked you to check the ξ–independence of $C_m - C_2$ in part (c).

(e) Solve the differential equation (10) and show that

$$m(E) = m_0 \times \left(\frac{\alpha(E)}{\alpha_0}\right)^r \tag{11}$$

for some power r, and calculate that power.
Hint: let $m(E) = F(\alpha(E))$ for some function $F(\alpha)$, then combine eq. (10) and

$$\frac{d\alpha(E)}{d\log E} = \frac{2\alpha^2}{3\pi} + O(\alpha^3) \tag{12}$$

into a differential equation for the $F(\alpha)$, and solve that equation.