⋆ The problems **1**(b) and **3**(b) below use the Poisson resummation formula: If a function $F(n)$ of integer n can be analytically continued to a function $F(\nu)$ of arbitrary real ν, then

$$\sum_{n=-\infty}^{+\infty} F(n) = \int d\nu\, F(\nu) \times \sum_{n=-\infty}^{+\infty} \delta(\nu - n) = \sum_{\ell=-\infty}^{+\infty} \int d\nu\, F(\nu) \times e^{2\pi i\ell\nu}. \tag{1}$$

As an *optional* exercise, derive this formula by first calculating

$$S(\nu, \epsilon) = \sum_{\ell=-\infty}^{+\infty} e^{2\pi\ell\nu} \times e^{-\epsilon|\ell|} \tag{2}$$

and then showing that

$$\lim_{\epsilon\to+0} S(\nu, \epsilon) = \sum_{n=-\infty}^{+\infty} \delta(\nu - n). \tag{3}$$

1. First, a simple exercise on using path integrals. Consider a 1D particle living on a circle of radius R, or equivalently a 1D particle in a box of length $L = 2\pi R$ with periodic boundary conditions where moving past the $x = L$ point brings you back to $x = 0$. In other words, the particle's position $x(t)$ is defined modulo L.

   The particle has no potential energy, only the non-relativistic kinetic energy $p^2/2M$.

   (a) As a particle moves from some point x_1 (mod L) at time t_1 to some other point x_2 (mod L) at time t_2, it may travel directly from x_1 to x_2, or it may take a few turns around the circle before ending at the x_2. Show that the space of all such paths on a circle is isomorphic to the space of all paths on an infinite line which begin at fixed x_1 at time t_1 and end at time t_2 at any one of the points $x_2' = x_2 + nL$ where $n = 0, \pm1, \pm2, \ldots$ is any whole number.

   Then use path integrals to relate the evolution kernels for the circle and for the infinite line (over the same time interval $t_2 - t_1$) as

$$U_{\text{circle}}(x_2, t_2; x_1, t_1) = \sum_{n=-\infty}^{+\infty} U_{\text{line}}(x_2 + nL, t_2; x_1, t_1). \tag{4}$$

(b) A free particle living on an infinite 1D line has evolution kernel

$$U_{\text{line}}(x_2, t_2; x_1, t_1) = \sqrt{\frac{M}{2\pi i\hbar(t_2 - t_1)}} \times \exp\left(+\frac{iM(x_2 - x_1)^2}{2\hbar(t_1 - t_1)}\right). \tag{5}$$

Plug this free kernel into eq. (4) and use Poisson's formula (1) to sum over n.

(c) Verify that the resulting evolution kernel for the particle on the circle agrees with the usual QM formula

$$U_{\text{box}}(x_2, t_2; x_1, t_1) = \sum_p \psi_p(x_2) \times \exp\bigl(-i(p^2/2M)(t_2 - t_1)/\hbar\bigr) \times \psi_p^*(x_1) \tag{6}$$

where the momentum p takes circle-quantized values

$$p = \frac{2\pi\hbar}{L} \times \text{integer} \tag{7}$$

and

$$\psi_p(x) = L^{-1/2} \exp(ipx/\hbar) \tag{8}$$

is the normalized wavefunction of the momentum eigenstate $|p\rangle$.

2. As a more serious exercise in in path integrals — or rather in the functional quantization of fields — solve problem **11.1** from the *Peskin&Schroeder* textbook. In this exercise you should learn why spontaneous breakdown of continuous symmetries does not happen in spacetimes of dimensions $d \leq 2$.

Hint: for a massless free scalar field, the coordinate-space formula for the propagator becomes fairly simple. In d Euclidean dimensions,

$$G_0(x - y) \equiv \int \frac{d^d p_E}{(2\pi)^d} \frac{e^{ip(x-y)}}{p_E^2} = \frac{\Gamma\left(\frac{d}{2} - 1\right)}{4\pi^{d/2}} \times |x - y|^{2-d}, \tag{9}$$

except for $d = 2$ where $G_0(x - y) = \text{const} - \frac{1}{2\pi} \log|x - y|$.

3. Next, a modified textbook problem 9.2(c) about the Euclidean functional integrals of free quantum fields at finite temperatures. (Parts (a) and (b) of the textbook problem were done in class, while parts (d–f) are postponed to the next homework set).

Note: Although I talked about temperature / coupling correspondence in class, in this exercise there are no couplings — all the fields are free — while the temperature is meant in the usual thermodynamical sense. That is, the fields being in thermal equilibrium with some heat reservoir of temperature $\mathcal{T}$, and we are looking for the StatMech partition function

$$Z(\mathcal{T}) \;=\; \mathrm{Tr}\big(\exp(-\beta\hat{H}_{\mathrm{QFT}})\big) \quad \text{for} \quad \beta \;=\; \frac{1}{\mathcal{T}}\,. \tag{10}$$

In class we saw that for an ordinary quantum system like a harmonic oscillator, the partition function obtains from the Euclidean path integral where all paths are periodic in Euclidean time with period β,

$$Z(\mathcal{T}) \;=\; \iiint\limits^{x(t_e=\beta)=x(t_e=0)} \mathcal{D}[x(t_e)]\exp\left(-\int_0^\beta dt_e\, L_E(x,\dot{x})\right). \tag{11}$$

Similarly, for a QFT in thermal equilibrium at a finite temperature $\mathcal{T}$, the partition function is

$$Z(\mathcal{T}) \;=\; \mathrm{Tr}\big(\exp(-\beta\hat{H})\big) \;=\; \iiint \mathcal{D}[\text{periodic }\Phi(x_e)]\,\exp\left(-\int_0^\beta dx^4\int d^3\mathbf{x}\,\mathcal{L}_E\right). \tag{12}$$

where ‘periodic’ means periodic in the x_4 direction with period $\beta = 1/\mathcal{T}$, $\Phi(\mathbf{x}, x_4+\beta) = \Phi(\mathbf{x}, x_4)$, and likewise for the non-scalar fields.

(a) Consider a free scalar field in 3 + 1 dimensions at finite temperature $\mathcal{T}$. Use the Euclidean functional integral (12) to calculate the partition function and hence the Helmholtz free energy $F(\mathcal{T}) = -T\log Z$. Show that formally

$$F(\mathcal{T} \;=\; \frac{\mathcal{T}}{2}\times\mathrm{Tr}\big[\log(-\partial_E^2+m^2)\big] \tag{13}$$

where the trace is over the Hilbert space of functions $\psi(x_1,x_2,x_3,x_4)$ that are periodic in x_4 with period $\beta = 1/\mathcal{T}$.

(b) Write down the trace in eq. (13) as a momentum space sum/integral. Then use the Poisson resummation formula (1) to show that the free energy *density* of the scalar field is

$$\mathcal{F}(\mathcal{T}) \;=\; \frac{\mathcal{T}}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{p_4}\log(p_E^2+m^2) \tag{14}$$

where the sum is over $p_4 \;=\; 2\pi\mathcal{T}\times\bigl(n=0,\pm1,\pm2,\pm3,\ldots\bigr)$

$$=\;\text{const}\;+\;\frac{1}{2}\sum_{\ell=-\infty}^{+\infty}\int\frac{d^4p_E}{(2\pi)^4}\,\exp(i\ell\beta p_4)\times\log(p_E^2+m^2) \tag{15}$$

$$=\;\mathcal{F}(0)\;+\;\sum_{\ell=1}^{\infty}\int\frac{d^4p_E}{(2\pi)^4}\,\exp(i\ell\beta p_4)\times\log(p_E^2+m^2). \tag{16}$$

(c) To evaluate the $\int dp_4$ integral in eq. (16), move the integration contour from the real axis to the two 'banks' of a branch cut. Show that

$$\int\limits_{-\infty}^{+\infty}\frac{dp_4}{2\pi}\,\exp(i\ell\beta p_4)\times\log(p_4^2+E^2)\;=\;-\frac{\exp(-\ell\beta E)}{\ell\beta}\,. \tag{17}$$

(d) Finally, use eqs. (16) and (17) to show that the free energy density of a free scalar field above the zero-point energy is

$$\mathcal{F}(\mathcal{T})\,-\,\mathcal{F}(0)\;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathcal{T}\,\log\left(1-e^{-\beta E_p}\right)\;=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\left(F^{\text{harmonic}}_{\text{oscillator}}(\mathcal{T},E_p)\,-\,\tfrac{1}{2}E_p\right). \tag{18}$$

4. Finally, a couple of optional reading assignment to the students who need them. These assignments are rather long, so I suggest you start reading over the Spring break and finish when you can — hopefully before the end of this semester, although there is no specific due date for these reading assignments.

   (a) Although I gave you a quick and dirty introduction to the path integrals this week, it would be very helpful — both for the QFT class and for your general education — to get more familiar with the subject. So please read *Quantum Mechanics and*

Path Integrals by Richard Feynman and Albert Hibbs about care and use of the Path Integrals. You can find this book in electronic format at the UT library at this link or read it online at scribd.com at this link.

(b) Basic group theory is rather important to the non-abelian gauge theories like QCD. I shall try to explain the relevant issues in class, but due to time constraints I would have to be brief, and it would help if you already know the basics. So, if you are unfamiliar with the group theory — especially the continuous group theory, — read *Lie Algebras in Particle Physics: from Isospin to Unified Theories* by Howard Georgi (1999, Westview press, ISBN 9780813346113). This book is available in electronic format at the UT library at (this link). Since you cannot finish the whole book by the time you would need the knowledge — which should be about a week after the break, – start by *carefully* reading the first 3 chapters, and then browse through chapters on the $SU(2)$, the $SU(3)$, and the color.