1. In class we have focused on QCD and QCD-like theories of non-abelian gauge fields coupled to Dirac fermions in some multiplet(s) of the gauge group G, *cf.* my notes on QCD Feynman rules and Ward identities. This problem is about the scalar QCD, or more generally a non-abelian gauge theory with some gauge group G and complex scalar fields $\Phi^i(x)$ in some multiplet (r) of G.

   (a) Write down the physical Lagrangian of this theory, the complete bare Lagrangian of the quantum theory in the Feynman gauge, and the Feynman rules.

   Now consider the annihilation process $\Phi + \Phi^* \to 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

   (b) Draw the diagrams and write down the tree-level annihilation amplitude.

   As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: *On-shell Amplitudes involving* **a** *longitudinally polarized gauge bosons vanish, provided all the other gauge bosons are transversely polarized.* In other words,

$$\mathcal{M} \equiv e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n} \mathcal{M}_{\mu_1\mu_2\cdots\mu_n}(\text{momenta}) = 0$$
$$\text{when} \quad e_1^\mu \propto k_1^\mu \quad \text{but} \quad e_2^\nu k_{2\nu} = \cdots = e_n^\nu k_{n\nu} = 0.$$

   (c) Verify this identity for the scalar annihilation amplitude: Show that IF $e_2^\nu k_{2\nu} = 0$ THEN $k_{1\mu}\mathcal{M}^{\mu\nu}e_{2\nu} = 0$.

   Similar to what we had in class for the quark-antiquark annihilation, there are non-zero amplitudes for the scalar 'quark' and 'antiquark' annihilating into a pair of longitudinal gluons or a ghost-antighost pair, but the cross-sections for these two unphysical processes cancel each other.

   (d) Take both final-state gluons to be longitudinally polarized; specifically, in the center-of-mass frame let $e_1^\mu = (1, +\mathbf{n}_1)/\sqrt{2}$ for the first gluon and $e_2^\nu = (1, -\mathbf{n}_2)/\sqrt{2}$ for the second gluon.

   Calculate the tree-level annihilation amplitude $\Phi + \Phi^* \to g_L + g_L$ for these polarizations.

(e) Next, calculate the tree amplitude for the $\Phi + \Phi^* \to \mathrm{gh} + \overline{\mathrm{gh}}$. There is only one tree graph for this process, so evaluating it should not be hard.

(f) Compare the two un-physical amplitudes and show that the corresponding partial cross-sections cancel each other, thus

$$\frac{d\sigma_{\mathrm{net}}}{d\Omega} = \frac{d\sigma_{\mathrm{physical}}}{d\Omega}. \tag{1}$$

2. Continuing the previous problem, consider the tree-level annihilation of a scalar 'quark' Φ^i and an 'antiquark' Φ^*_j into a pair of gauge bosons with adjoint colors a and b. But this time, we focus on the group theory and on the physical cross-sections rather than the Ward identities.

(a) Take the annihilation amplitude from part (b) of problem 1, focus on its color dependence, and rewrite it in the form

$$\mathcal{M}(j + i \to a + b) = F \times \{T^a, T^b\}^i{}_j + iG \times [T^a, T^b]^i{}_j \tag{2}$$

where F and G are some functions of all the momenta and of the two vectors' polarizations. Give explicit formulae for F and G.

(b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalars' colors. Show that

$$\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{C(r)}{\dim(r)} \times \Big(\big(4C(r) - C(\mathrm{adj})\big) \times |F|^2 + C(\mathrm{adj}) \times |G|^2 \Big). \tag{3}$$

In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

$$\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{N^2 - 1}{2N^2} \left(\frac{N^2 - 2}{N} \times |F|^2 + N \times |G|^2 \right). \tag{4}$$

(c) Evaluate F and G in the center of mass frame, where the vector particles' polarizations $e^{\mu}_{1,2} = (0, \mathbf{e}_{1,2})$ are purely spatial and transverse to the vectors' momenta $\pm\mathbf{k}$. For simplicity, use planar rather than circular polarizations.

(d) Assemble your results and calculate the (polarized, partial) cross-section for the annihilation process.

3. Finally, let's evaluate a few one-loop diagrams. In class, I have calculated the (infinite parts of the) δ_2 and δ_1 counterterms for the quarks, *cf.* my notes on QCD beta-function. Your task is to calculate the analogous $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$ counterterms for the *ghosts fields.*

   (a) Draw one-loop diagrams whose divergences are canceled by the respective counterterms $\delta_2^{(\mathrm{gh})}$ and $\delta_1^{(\mathrm{gh})}$, and calculate the group factors for each diagrams.

   (b) Calculate the momentum integrals for the diagrams. Focus on the UV divergences and ignore the finite parts of the integrals.

   (c) Assemble your results and show that the *difference* $\delta_1^{(\mathrm{gh})} - \delta_2^{(\mathrm{gh})}$ for the ghosts is exactly the same as the $\delta_1 - \delta_2$ difference for the quarks.