1. Consider the three gauge couplings of the $SU(3) \times SU(2) \times U(1)$ Standard Model and their one-loop beta-functions

$$\beta_1^{1\,\text{loop}} = \frac{b_1 g_1^3}{16\pi^2}, \quad \beta_2^{1\,\text{loop}} = \frac{b_2 g_2^3}{16\pi^2}, \quad \beta_3^{1\,\text{loop}} = \frac{b_3 g_3^3}{16\pi^2}. \tag{1}$$

In this exercise, you do not need to calculate these beta-function from scratch by evaluating the UV divergences of a bunch of loop diagrams. Instead, use eqs. (122) and (124–5) from my notes on QCD beta-function (pages 25–26).

(a) Calculate the $b_1, b_2, b_3$ coefficients for the minimal version of the Standard Model: the $SU(3) \times SU(2) \times U(1)$ gauge fields, one Higgs doublet, three families of quarks and leptons, and nothing else.

⋆ FYI, each family comprises 8 left-handed Weyl fields in the $(\mathbf{3}, \mathbf{2}, y = +\frac{1}{6})$ and $(\mathbf{1}, \mathbf{2}, y = -\frac{1}{2})$ multiplets of the gauge symmetry and 7 right-handed Weyl fermions in the $(\mathbf{3}, \mathbf{1}, y = +\frac{2}{3})$, $(\mathbf{3}, \mathbf{1}, y = -\frac{1}{3})$, and $(\mathbf{1}, \mathbf{1}, y = -1)$ multiplets.

(b) Re-calculate the $b_1, b_2, b_3$ for the MSSM — the Minimal Supersymmetric Standard Model. FYI, here is complete list of the MSSM fields:

- ∘ The $SU(3) \times SU(2) \times U(1)$ gauge fields, same as the non-SUSY SM.
- • For each vector field there is a Majorana fermion (a gaugino) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, there is an adjoint multiplet of gauginos for each factor of the gauge symmetry.
- ∘ 3 families of quarks and leptons, same as the non-SUSY SM.
- • For each Weyl fermion — left-handed or right-handed — in these three families, the MSSM also have a complex scalar field (a squark or a slepton) with similar $SU(3) \times SU(2) \times U(1)$ quantum numbers. Altogether, this makes 45 complex scalar fields in similar multiplets to the quarks and leptons.
- • The Higgs sector of the MSSM comprises *two* $SU(2)$ doublets of complex scalars accompanied by one $SU(2)$ doublet of Dirac fermions (the higgsinos); all these doublets have $y = \frac{1}{2}$.

— There are all kinds of Yukawa and $\phi^4$ interactions between the MSSM fields, but you do not need them for the one-loop calculation of the gauge couplings' beta-functions.

In Grand Unified Theories

$$\alpha_3 \;=\; \alpha_2 \;=\; \tfrac{5}{3}\alpha_1 \;=\; \alpha_{\rm GUT} \quad \text{at the GUT scale.} \tag{2}$$

At lower energy scales $E \ll M_{\rm GUT}$ the SM couplings are given (lo the leading one-loop order) by the Georgi–Quinn–Weinberg equations

$$\begin{aligned} \frac{1}{\alpha_3(E)} &= \frac{1}{\alpha_{\rm GUT}} + b_3 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,, \\ \frac{1}{\alpha_2(E)} &= \frac{1}{\alpha_{\rm GUT}} + b_2 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,, \\ \frac{1}{\alpha_1(E)} &= \frac{5/3}{\alpha_{\rm GUT}} + b_1 \times \frac{1}{2\pi}\log\frac{M_{\rm GUT}}{E}\,. \end{aligned} \tag{3}$$

(c) Derive these equations from eqs. (1).

The experimental data are usually interpreted in terms of the $\overline{\rm MS}$ gauge couplings at the $Z^0$ mass $M_Z \approx 91$ GeV; according to the latest particle data group publication

$$\frac{1}{\alpha_3(M_Z)} \approx 8.45 \pm 0.12, \quad \frac{1}{\alpha_2(M_Z)} \approx 29.585 \pm 0.005, \quad \frac{1}{\alpha_1(M_Z)} \approx 98.369 \pm 0.009. \tag{4}$$

Since the top quark and the Higgs boson are heavier than $M_Z$, let me translate these data to the $\overline{\rm MS}$ couplings at $E = M_{\rm top} \approx 173$ GeV:

$$\frac{1}{\alpha_3(M_t)} \approx 9.18 \pm 0.12, \quad \frac{1}{\alpha_2(M_t)} \approx 30.028 \pm 0.005, \quad \frac{1}{\alpha_1(M_t)} \approx 97.84 \pm 0.01. \tag{5}$$

(d) Check that these data are *not* consistent with eq. (3) for the minimal Standard Model.

(e) Now consider the Minimal Supersymmetric Standard Model. For simplicity, assume that all the super-partners — or rather all particles of the MSSM not present in the non-supersymmetric minimal SM — have masses $M \approx M_{\rm top} \approx 173$ GeV; this has been ruled out experimentally, but it's a useful toy model.

Show that for this model — unlike for the minimal non-SUSY Standard Model, — the experimental gauge couplings (5) are consistent with the Georgi–Quinn–Weinberg eqs. (3). Also, calculate the GUT scale $M_{\rm GUT}$ for this model.

(f) Finally, consider a more realistic model, namely MSSM in which all the extra particles have the same mass $M_S = 2$ TeV, just out of LHS's reach.

To check the consistency of this model, first extrapolate the experimental gauge couplings (5) from the $M_{\rm top}$ scale to the $M_S$ scale using the beta-function coefficients $b_{1,2,3}$ of the non-SUSY Standard Model. And then check whether the resulting gauge couplings are consistent with eq. (3) for the MSSM.

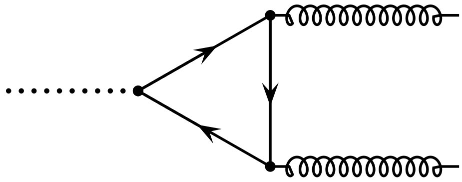
2. Now consider the axial anomaly in a non-abelian gauge theory, for example QCD with $N_f$ massless quark flavors,

$$J_A^\mu \;=\; \sum_{i,f} \overline{\Psi}_{if}\gamma^\mu\gamma^5\Psi^{if}, \qquad \partial_\mu J_A^\mu \;=\; -\frac{N_f g^2}{16\pi^2}\,\epsilon^{\alpha\beta\mu\nu}\,{\rm tr}\Big(F_{\alpha\beta}F_{\mu\nu}\Big) \tag{6}$$

where $F_{\mu\nu}$ is the non-abelian gauge field strength.

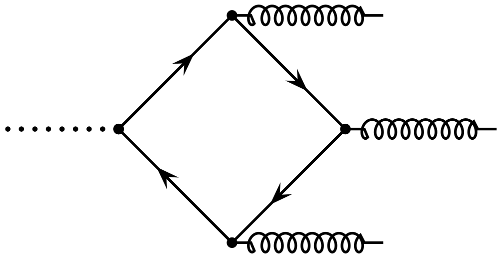
(a) Expand the right hand side of eq. (6) into 2–gluon, 3–gluon, and 4–gluon terms and show that the 4–gluon term vanishes identically.
Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams



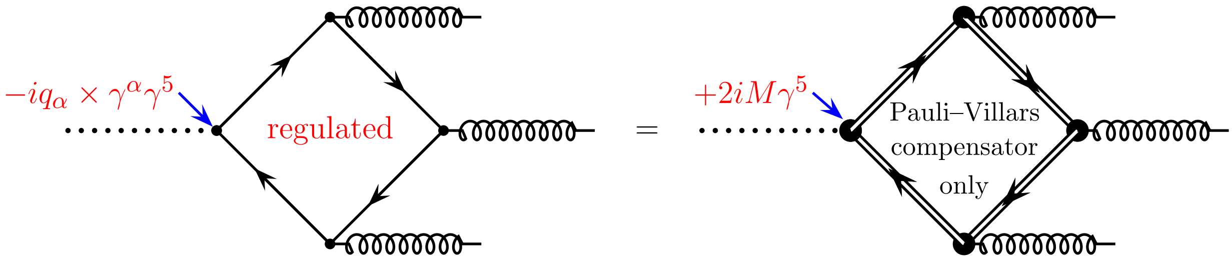
$+$ gluon permutation. (7)

This works exactly as discussed in class for the QED, except in QCD we should trace $F_{\alpha\beta}F_{\gamma\delta}$ over the quarks' colors and flavors. But in QCD there is also the three-gluon anomaly (*cf.* part (a)) which obtains from the quadrangle diagrams

 + gluon permutations. (8)

Since the quadrangle diagrams suffer from linear UV divergences, we need to regulate them, so let's use the Pauli–Villars regulator.

(b) First, show that




+ terms which cancel after summing over gluon permutations (9)

(c) Second, evaluate the the quadrangle diagrams for the Pauli–Villars compensators and derive the three-gluon anomaly in QCD.

3. Next, a reading assignment: §22.2–3 of *Weinberg* about the chiral anomaly. Pay particular attention to the Jacobian of the fermion path integral and to regularization of the functional trace.

4. Finally, one more reading assignment: §19.3 of *Peskin & Schroeder* about the chiral symmetry of QCD and the pions.

For a deeper discussion of pions (and Goldstone bosons in general), please also read chapter 19 of *Weinberg*.