⋆ This whole homework follows on the two reading assignments from the previous homework set#22. Specifically, problems 1 and 2 about axial anomalies in various spacetime dimensions follow up on *Weinberg* §22.3–3, while problems 3 and 4 about pion decays follow up on *Peskin & Schroeder* §19.3 and *Weinberg* chapter 19.

1. Following up on *Weinberg*'s analysis of the axial anomaly of the fermionic functional integral's measure (§22.2-3) in $d = 4$ dimensions, let's generalize it to other *even* spacetime dimensions $d = 2n$. In any such dimension there a matrix Γ which acts as the γ^5 in 4D — $\Gamma\gamma^\mu = -\gamma^\mu\Gamma$ for all $\mu = 1, 2, \ldots d$. Consequently, a massless Dirac fermion in $d = 2n$ dimensions has a classical axial symmetry

$$\Psi(x) \;\to\; \exp(i\theta\Gamma)\Psi(x), \qquad \overline{\Psi}(x) \;\to \Psi(x)\exp(i\theta\Gamma), \tag{1}$$

which leads to a classically conserved current

$$J^\mu_A \;=\; \overline{\Psi}\gamma^\mu\Gamma\Psi, \qquad \partial_\mu J^\mu_A \;= \text{classically} =\; 0. \tag{2}$$

But when the fermion Ψ is coupled to a gauge field — or a multiplet of such fermions is coupled to a non-abelian gauge field — the axial symmetry is broken by the anomaly, thus

$$\partial_\mu J^\mu_A \;=\; +\frac{2}{n!}\left(\frac{-g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\,\operatorname{tr}\Big(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}\Big). \tag{3}$$

Generalize Weinberg's calculation of the anomaly via the Jacobian of the fermionic path integral to any even spacetime dimension $d = 2n$.

For your information, in $2n$ Euclidean dimensions $\{\gamma^\mu, \gamma^\nu\} = +2\delta^{\mu\nu}$, $\Gamma = i^{n-2}\gamma^1\gamma^2\cdots\gamma^{2n}$, $\{\Gamma, \gamma^\mu\} = 0$, $\Gamma^2 = +1$, and for any $2n = d$ matrices $\gamma^\alpha, \ldots, \gamma^\omega$, $\operatorname{tr}(\Gamma\gamma^\alpha\gamma^\beta\cdots\gamma^\omega) = 2^n i^{2-n}\epsilon^{\alpha\beta\cdots\omega}$.

2. In any even dimension $d = 2n$, the right hand side of the anomaly equation (3) is always a total derivative,

$$\epsilon^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(\mathcal{F}_{\alpha_1\beta_1}\cdots\mathcal{F}_{\alpha_n\beta_n}\Big) \;=\; \partial_\mu\Omega^\mu_{(2n-1)} \tag{4}$$

where $\Omega^\mu_{(2n-1)}$ is some polynomial in gauge fields $\mathcal{A}^\nu = gA^\nu$ and $\mathcal{F}^{\rho\sigma} = gF^{\rho\sigma}$, for example

$$\begin{aligned}
&\text{in } d=2,\ \Omega^\mu_{(1)} &&= 2\epsilon^{\mu\nu}\,\mathrm{tr}(\mathcal{A}_\nu) \quad [\text{abelian } A_\nu \text{ only}],\\
&\text{in } d=4,\ \Omega^\mu_{(3)} &&= 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma} - \tfrac{2i}{3}\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\right),\\
&\text{in } d=6,\ \Omega^\mu_{(5)} &&= 2\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\left(\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}\mathcal{F}_{\alpha\beta} - iA_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{F}_{\alpha\beta} - \tfrac{2}{5}\,\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\mathcal{A}_\alpha A_\beta\right),
\end{aligned} \tag{5}$$

etc., *etc.* The $\Omega^\mu_{(2n-1)}$ vectors are equivalent to $(2n-1)$–index totally antisymmetric tensors called the *Chern–Simons forms*, and those forms play many important roles in gauge theory and string theory. In particular, we may use the $\Omega^\mu_{(2n-1)}$ to define a conserved axial current

$$J^\mu_A \;\to\; J^\mu_{AC} \;=\; \overline{\Psi}\gamma^\mu\Gamma\Psi \;+\; \frac{1}{n!}\left(\frac{g}{4\pi}\right)^n \times\,\Omega^\mu_{(2n-1)}\,. \tag{6}$$

(Its conservation follows from eqs. (3) and (4).) However, the price of this current conservation is the loss of gauge invariance: the original axial current J^μ_A is gauge invariant, but the J^μ_{AC} is not.

(a) You task is to verify eqs. (4) for $d = 2, 4, 6$.

The Chern–Simons vectors (5) are not gauge invariant, but their variations under the infinitesimal gauge transforms are total derivatives of antisymmetric tensors,

$$\delta\Omega^\mu_{(2n-1)} \;=\; -2\partial_\nu H^{\mu\nu}_{(2n-2)}, \qquad H^{\mu\nu}_{(2n-2)} \;=\; -H^{\nu\mu}_{(2n-2)}. \tag{7}$$

Specifically, for $d = 2n = 2, 4, 6$:

$$\begin{aligned}
&\text{in } d=2,\ H^{\mu\nu}_{(0)} &&= \epsilon^{\mu\nu}\,\mathrm{tr}(\Lambda) \quad [\text{abelian } \mathcal{A}_\nu \text{ only}],\\
&\text{in } d=4,\ H^{\mu\nu}_{(2)} &&= 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\big(\Lambda\times\partial_\rho\mathcal{A}_\sigma\big),\\
&\text{in } d=6,\ H^{\mu\nu}_{(4)} &&= 4\epsilon^{\mu\nu\rho\sigma\alpha\beta}\,\mathrm{tr}\left(\Lambda\times\partial_\rho\left(\mathcal{A}_\sigma\partial_\alpha\mathcal{A}_\beta + \tfrac{i}{2}A_\sigma\mathcal{A}_\alpha\,A_\beta\right)\right).
\end{aligned} \tag{8}$$

(b) Verify eqs. (7) for these H tensors.

Note: for $d = 2$ eq. (7) is trivial, while for $d = 4$ it's very similar to problem 2(b) of the Fall˜2022 midterm exam. But for $d = 6$ you have to work it out from scratch.

3. The pions are pseudo-Goldstone bosons of the spontaneously broken chiral symmetry of QCD, so they can be created or annihilated by the axial isospin currents

$$J^a_{\mu 5}(x) \;=\; \overline{\Psi}(\bar{u},\bar{d})\gamma^\mu\gamma^5\left(\frac{\tau^a}{2}\right)_{\text{isospin}}\Psi(u,d) = \; -f_\pi\partial_\mu\pi^a(x) \;+\; \text{multi-pion terms.} \tag{9}$$

The f_π in this formula is the *pion decay constant* because it controls the decay rate of the charged pions, mostly into muons and neutrinos, $\pi^+ \to \mu^+\nu_\mu$ and $\pi^- \to \mu^-\bar{\nu}_\mu$. In this exercise, we shall see how this works. Experimentally, $f_\pi \approx 93$ MeV.

The weak interactions at energies $O(M_\pi) \ll M_W$ are governed by the Fermi's current-current effective Lagrangian

$$\mathcal{L}_{\text{Fermi}} \;=\; -2\sqrt{2}G_F J^{+\alpha}_L J^-_{L\alpha} \tag{10}$$

where $L^{\pm\alpha}_L = \frac{1}{2}(J^{\pm\alpha}_V - J^{\pm\alpha}_A)$ are the left-handed charged currents. In terms of the quark and lepton fields,

$$\begin{aligned} J^{+\alpha}_L \;&=\; \tfrac{1}{2}\overline{\Psi}(\nu_\mu)(1-\gamma^5)\gamma^\alpha\Psi(\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(u)(1-\gamma^5)\gamma^\alpha\Psi(d) \;+\; \cdots, \\ J^{-\alpha}_L \;&=\; \tfrac{1}{2}\overline{\Psi}(\mu)(1-\gamma^5)\gamma^\alpha\Psi(\nu_\mu) \;+\; \cos\theta_c \times \tfrac{1}{2}\overline{\Psi}(d)(1-\gamma^5)\gamma^\alpha\Psi(u) \;+\; \cdots, \end{aligned} \tag{11}$$

where the $\cdots$ stand for other fermions of the Standard Model, and $\theta_c \approx 13°$ is the Cabibbo angle.

For the pion decay process, the axial part of one of the charged currents annihilates the charged pion according to eq. (9) while the other charged current creates the lepton pair.

(a) Show that

$$\langle\text{vacuum}|\,\hat{J}^{-\alpha}_L\,|\pi^+\rangle \;=\; \frac{if_\pi\cos\theta_c}{\sqrt{2}} \times p^\alpha(\pi^+) \tag{12}$$

and therefore the tree-level pion decay amplitude is

$$\mathcal{M} \;=\; \langle\mu^+,\bar{\nu}_\mu|\,\hat{\hat{L}}_{\text{Fermi}}\,|\pi^+\rangle \;=\; iG_f f_\pi\cos\theta_c \times p^\alpha(\pi^+) \times \bar{u}(\nu_\mu)(1-\gamma^5)\gamma_\alpha v(\mu^+). \tag{13}$$

(b) Sum over the fermion spins and calculate the decay rate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$.

(c) Experimentally, $f_\pi \approx 93$ MeV, $M_\pi \approx 140$ MeV, $M_\mu \approx 106$ MeV, $M_\nu \approx 0$, $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$, and $\theta_c \approx 13°$. Use these data to calculate the charged pion's lifetime and compare to the experimental value $\tau(\pi^\pm) = 2.6 \times 10^{-8}$ s.

(d) The charged pions decay to muons much more often than they decay to electrons,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{M_e^2}{M_\mu^2} \frac{(1-(M_e/M_\pi)^2)^2}{(1-(M_\mu/M_\pi)^2)^2} \approx 1.2 \cdot 10^{-4}. \tag{14}$$

Derive this formula, then explain this preference for the heavier final-state lepton in terms of the mismatch between that lepton's chirality and helicity.

4. Finally, consider the neutral pion decay into two photons, $\pi^0 \to \gamma\gamma$. This decay is facilitated by the QED anomaly of the axial isospin current $J^3_{\mu 5} = -f_\pi \partial_\mu \pi^0 + \cdots$, *cf.* eq. (9). As explained in class,

$$\operatorname{tr}\left(\frac{\tau^3}{2} \times Q^2_{\text{el}}\right) = \frac{e^2}{2} \tag{15}$$

hence

$$\left(\partial^\mu J^3_{\mu 5}\right)_{\text{anomalous}} = -\frac{e^2}{32\pi^2}\,\epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}\,, \tag{16}$$

which may be explained by an effective Lagrangian for the neutral pion field

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}(\partial_\mu \pi^0)^2 + \frac{e^2}{32\pi^2 f_\pi}\,\pi^0 \times \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}\,. \tag{17}$$

In real life, there is an additional contribution to the axial current divergence $\partial^\mu J^3_{\mu 5}$ due to non-zero quark masses; in terms of the effective Lagrangian (17) this extra term can be accounted by the pions mass^2 term, thus

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \pi^0)^2 - \frac{M_\pi^2}{2}(\pi^0)^2 + \frac{e^2}{32\pi^2 f_\pi}\,\pi^0 \times \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}\,. \tag{18}$$

The interaction term here gives rise to the pion decay amplitude

$$\mathcal{M}(\pi^0 \to \gamma\gamma) = -\frac{\alpha}{\pi f_\pi} \times \epsilon^{\alpha\beta\mu\nu} (k_\alpha e^*_\beta)_1 (k_\mu e^*_\nu)_2\,. \tag{19}$$

(a) Derive this amplitude.

(b) Sum $|\mathcal{M}|^2$ over the two photon's polarizations and calculate the neutral pion's decay rate.

(c) Experimentally, $M_\pi \approx 135$ MeV (for the neutral pion), $f_\pi \approx 93$ MeV, and $\alpha \approx 1/137$. Calculate the numerical value of the neutral pion's lifetime for these data and compare to the experimental value of $\tau(\pi^0) \approx 8.5 \times 10^{-17}$ s.