PHY–396 K. Optional problem about the magnetic monopoles.
Due by the end of the Fall 22 semester.

Some theories of fundamental interaction predict the existence of *dyons* — magnetic monopoles that also have electric charges. Dyons are usually very heavy compared to ordinary particles, so when an ordinary charged particle orbits a dyon, the latter can be thought as a static source of the electric and the magnetic fields: In Gauss units,

$$\mathbf{E}(\mathbf{x}) \;=\; \frac{Q}{r^2}\mathbf{n}, \qquad \mathbf{B}(\mathbf{x}) \;=\; \frac{M}{r^2}\mathbf{n}. \tag{1}$$

So let's consider the motion of a spinless non-relativistic particle of mass m and electric charge q in these static fields.

Let's start with the classical motion of the particle in question. It's net angular momentum is

$$\mathbf{J} \;=\; \mathbf{L}_{\text{mech}} \;+\; \mathbf{J}_{\text{EM}} \;=\; \mathbf{x}\times\vec{\pi} \;-\; \frac{qM}{c}\mathbf{n} \tag{2}$$

where $\vec{\pi} = m\mathbf{v}$ is the kinematic momentum of the particle rather that its canonical momentum.

(a) Verify that it is this net angular momentum that is conserved by the classical motion of the particle, $d\mathbf{J}/dt = 0$.

In quantum mechanics, we have a similar formula for the net angular momentum,

$$\hat{\mathbf{J}} \;=\; \hat{\mathbf{x}}\times\hat{\vec{\pi}} \;-\; \frac{qM}{c}\frac{\hat{\mathbf{x}}}{\hat{r}} \tag{3}$$

where

$$\hat{\vec{\pi}} \;=\; \hat{\mathbf{p}} \;-\; \frac{q}{c}\mathbf{A}(\hat{\mathbf{x}}). \tag{4}$$

In light of eq. (4), the (equal time) commutation relations for the position and *kinematic* momentum operators are

$$[\hat{x}_i, \hat{x}_j] \;=\; 0, \qquad [\hat{x}_i, \hat{\pi}_j] \;=\; i\hbar\delta_{ij}\,, \tag{5}$$

but

$$[\hat{\pi}_i, \hat{\pi}_j] \;=\; \frac{iq\hbar}{c}\epsilon_{ijk}B_k(\hat{\mathbf{x}}) \;\xrightarrow[\text{in the dyon field}]{}\; \frac{iqM\hbar}{c}\,\epsilon_{ijk}\frac{\hat{x}_k}{\hat{r}^3}\,. \tag{6}$$

(b) Use these commutation relation to show that the components of the angular momentum operator (3) indeed commute with each other — and with the other vectors — as legitimate angular momentum operators. Specifically,

$$[\hat{x}_i, \hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\,\hat{x}_k\,, \tag{7}$$

$$[\hat{\pi}_i, \hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\,\hat{\pi}_k\,, \tag{8}$$

$$[\hat{J}_i, \hat{J}_j] \;=\; i\hbar\epsilon_{ijk}\,\hat{J}_k\,. \tag{9}$$

(c) Show that the operators $\hat{J}_i$ are conserved, *i.e.*, that they commute with the particle's Hamiltonian

$$\hat{H} \;=\; \frac{\vec{\hat{\pi}}^2}{2m} \;+\; \frac{Qq}{\hat{r}}\,. \tag{10}$$

The vector potential due to the magnetic charge of the dyon can be written in spherical coordinates as

$$\mathbf{A}_{N,S}(r,\theta,\phi) \;=\; M\,\frac{\pm 1 - \cos\theta}{r\sin\theta}\cdot\mathbf{e}_\phi\,, \tag{11}$$

where $\mathbf{e}_\phi$ is the unit vector in the ϕ direction while the two signs correspond to the two different gauge choices for the Dirac monopole: '+' for the $\mathbf{A}_N$ potential on the Northern side of the dyon ($0 \le \theta < \pi - \epsilon$), and '−' for the $\mathbf{A}_S$ potential on the Southern side ($\epsilon < \theta \le \pi$).

(d) Show that for these gauge choices, the $\hat{J}_z$ operator acts in the spherical coordinate basis as

$$\hat{J}_z = -i\hbar\frac{\partial}{\partial\phi} \mp \frac{qM}{c}\psi. \tag{12}$$

Note that thanks to the Dirac's charge quantization rule, the $\mp(qM/c)$ factor in the second term here is always an integer or half-integer multiple of $\hbar$.

(e) Likewise, show that the other two components of the angular momentum have form

$$\begin{aligned}
\hat{J}_+ &= \hat{J}_x + i\hat{J}_y = \hbar e^{+i\phi}\left[+\frac{\partial}{\partial\theta} + i\cot\theta\times\frac{\partial}{\partial\phi} - \frac{qM}{\hbar c}\frac{1\mp\cos\theta}{\sin\theta}\right],\\
\hat{J}_- &= \hat{J}_x - i\hat{J}_y = \hbar e^{-i\phi}\left[-\frac{\partial}{\partial\theta} + i\cot\theta\times\frac{\partial}{\partial\phi} - \frac{qM}{\hbar c}\frac{1\mp\cos\theta}{\sin\theta}\right],
\end{aligned} \tag{13}$$

Now let's look for the simultaneous eigenstates $|n,j,m\rangle$ of the $\hat{\mathbf{J}}^2$ and $\hat{J}_z$ operators. By the usual rules of the angular momenta, for each given n and j, m runs from $-j$ to $+j$ by 1. However, in presence of the dyon, the spectrum of j is different from the spectrum of ℓ for the ordinary orbital angular momentum: Instead of $\ell = 0,1,2,3,\ldots$, we now have

$$j = j_{\min}, j_{\min}+1, j_{\min}+2, \ldots \quad \text{where} \quad j_{\min} = \frac{|qM|}{\hbar c}. \tag{14}$$

In particular, for a half-integral $qM/\hbar c$, we have j running over half-integral rather than integral values.

(f) Use eqs. (12) and (13) to obtain this spectrum of allowed values of j.

Now let's diagonalize the Hamiltonian (10). As a first step, let's separate the radial and the angular directions of the operator $\hat{\vec{\pi}}^2$.

(g) Use the commutation relations (5) through (9) to show that

$$\hat{\vec{\pi}}^2 = \hat{\pi}_r^2 + \frac{1}{\hat{r}^2}\left(\hat{\vec{J}}^2 - \left(\frac{qM}{c}\right)^2\right) \tag{15}$$

where

$$\hat{\pi}_r = \frac{1}{2}\{\hat{n}_i, \hat{\pi}_i\} \xrightarrow[\text{coordinate basis}]{} -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right). \tag{16}$$

(h) Finally, write down the radial Schrödinger equation for a given j and show that for $qQ < 0$ the bound state energies are

$$E(n_r, j) \;=\; -\frac{m(qQ)^2}{2\hbar^2} \times \frac{1}{(n_r + \lambda)^2} \tag{17}$$

where n_r is a positive integer $1, 2, 3, \ldots$ while ν is the positive root of

$$\lambda(\lambda + 1) \;=\; j(j + 1) \;-\; (qM/\hbar c)^2. \tag{18}$$

By comparison, in the absence of the magnetic charge j is $\ell = 0, 1, 2, 3, \ldots$, hence $\lambda = \ell$, and $n_r + \lambda = n_r + \ell$ is the principle quantum number N.