NON LINEAR SIGMA MODELS

Ordinary scalar, vector, spinor, *etc.*, fields are maps from the Minkowski spacetime to some *linear* spaces such as real or complex numbers, vectors, spinors, *etc.*. But a generic non-linear sigma model is a map from the Minkowski spacetime to a non-linear target space, namely a curved Riemannian manifold. In terms of the target space's coordinates ϕ^a ($a = 1, \ldots, N$) and its metric

$$ds^2 \;=\; g^{ab}(\phi)\, d\phi^a\, d\phi^b, \tag{1}$$

the non-linear sigma model comprises N scalar fields $\phi^a(x)$ with the purely-kinetic Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2} g^{ab}(\phi) \times \partial_\mu \phi^a\, \partial^\mu \phi^b. \tag{2}$$

Note that the specific fields ϕ^a and the specific metric $g^{ab}(\phi)$ depends on a particular coordinate system for the target space. For a different coordinate system, the NLΣM would have different fields ϕ'^a related to the ϕ^b in some non-linear fashion and a correspondingly different metric such that

$$g'^{ab}(\phi') \times d\phi'^a\, d\phi'^b \;=\; g^{ab}(\phi) \times d\phi^a\, d\phi^b. \tag{3}$$

The non-linear sigma models are renormalizable in $1+1$ dimensions but not in higher dimensions such as $3+1$. Consequently, loop corrections generate all kinds extra terms in the Lagrangian, especially the higher-derivative terms such as

$$\Delta\mathcal{L} \;=\; \tfrac{1}{8} h^{abcd}(\phi) \times \partial_\mu \phi^a\, \partial^\mu \phi^b\, \partial_\nu \phi^x\, \partial^\nu \phi^d \;+\; \cdots \tag{4}$$

Because of such higher-derivative terms, an NLΣM becomes useless at high energies. On the other hand, the higher-derivative terms become irrelevantly small at low energies, so an NLΣM can be a good low-energy effective theory despite its non-renormalizability.

Besides the higher-derivative terms, the loop corrections may also generate the no-derivative terms which together would comprise a scalar potential $V(\phi)$. If the target space of the NLΣM happens to be a homogeneous space of some symmetry — *i.e.*, if all points of the target space are related by symmetries, — then that symmetry would lead to $V(\phi) = \text{const}$, so the potential can be ignored. Otherwise, we would get a non-constant $V(\phi)$ which at low energies would produce stronger interactions than the curvature of the metric $g^{ab}(\phi)$. Consequently, the low-energy effective theory would be dominated by the potential rather than by the NLΣM.

For this reason, the non-supersymmetric non-linear sigma models usually involve homogeneous target spaces.[⋆] Of particular importance are target spaces of the type G/H where G is the manifold of some continuous symmetry group and H is a subgroup of G. Such target spaces often appear in the context of spontaneous symmetry breaking from G — the symmetry group of the Lagrangian — to H, the unbroken symmetry group of the vacuum.

In these notes I would like to focus on the spontaneous breakdown of the chiral symmetry, $SU(N)_L \times SU(N)_R \to SU(N)_V$, especially for $N = 2$ or $N = 3$, hence the NLΣM whose target space is the $SU(N)$ group manifold. Instead of using an explicit coordinate systems for such manifolds, let me use the non-linear, $SU(N)$–matrix valued field $W(x)$:

- For each x, the $W(x)$ is a $N \times N$ unitary matrix of determinant $= 1$.

In terms of this matrix-value field, the NLΣM Lagrangian is simply

$$\mathcal{L} \;=\; F^2 \,\mathrm{tr}\big(\partial_\mu W^\dagger \,\partial^\mu W\big) \tag{5}$$

for some constant F^2; in 4D, F^2 has dimension mass^2. The Lagrangian (5) has a global $SU(N)_L \times SU(N)_R$ symmetry which acts as

$$W'(x) \;=\; U_L\, W(x)\, U_R^\dagger, \quad W'^\dagger(x) \;=\; U_R\, W^\dagger(x)\, U_L^\dagger \tag{6}$$

for any two *independent* $SU(N)$ matrices U_L and U_R. However, any vacuum expectation value $\langle W\rangle \in SU(N)$ spontaneously breaks the symmetry to a single $SU(N)$ by imposing a

⋆ Supersymmetry severely restricts quantum correction to the scalar potential, and usually leads to $V(\phi) = 0$ over the entire target space of an NLΣM. Consequently, supersymmetric NLΣM's with non-homogeneous target spaces are OK.

relation between U_L and U_R. Indeed, to keep

$$\langle W\rangle' \;=\; U_L\,\langle W\rangle\,U_R^\dagger \;=\; \langle W\rangle \tag{7}$$

we would need

$$U_R \;=\; \langle W\rangle^\dagger\,U_L\,\langle W\rangle\,. \tag{8}$$

In particular, the simplest VEV $\langle W\rangle = \mathbf{1}_{N\times N}$ remains invariant only for $U_R = U_L$, thus spontaneous symmetry breaking $SU(N)_L \times SU(N)_R \to SU(N)_V$.

Now consider the symmetry currents. In general, for any continuous global internal symmetry with infinitesimal action $\delta\phi^a(x) \;= \epsilon Q\phi^a(x)$ for some operator Q, the conserved current obtains from the Noether theorem as follows: For x-independent ϵ, $\delta\mathcal{L} = 0$ by the symmetry, but if we make ϵ x-dependent, we generally get

$$\delta\mathcal{L} \;=\; \partial_\mu\epsilon \times J^\mu \tag{9}$$

for some current $J^\mu(\phi, \partial\phi)$. By Noether symmetry, J^μ is precisely the conserved current of the symmetry.

Now let's apply this rule to the chiral symmetry of the NLΣM. The infinitesimal $SU(N)_L \times SU(N)_R$ symmetries act on the W and $W^\dagger$ fields as

$$\delta W(x) \;=\; \tfrac{i}{2}\,\epsilon_L^a\lambda^a \times W(x) + W(x) \times \tfrac{-i}{2}\,\epsilon_R^a\lambda^a, \quad \delta W^\dagger(x) \;=\; \tfrac{i}{2}\,\epsilon_R^a\lambda^a \times W^\dagger(x) + W^\dagger(x) \times \tfrac{-i}{2}\,\epsilon_L^a\lambda^a, \tag{10}$$

where λ^a are the Gell-Mann matrices of the $SU(N)$. (Or the Pauli matrices for $N = 2$.) Consequently, for x-dependent ϵ_L^a and ϵ_R^a,

$$\begin{aligned}
\delta\partial_\mu W \;&=\; \tfrac{i}{2}\,(\partial_\mu\epsilon_L^a)\lambda^a W \;+\; \tfrac{i}{2}\,\epsilon_L^a\lambda^a\,\partial_\mu W \;-\; \tfrac{i}{2}\,\epsilon_R^a\,(\partial_\mu W)\lambda^a \;-\; \tfrac{i}{2}\,(\partial_\mu\epsilon_R^a)W\lambda^a,\\
\delta\partial_\mu W^\dagger \;&=\; \tfrac{i}{2}\,(\partial_\mu\epsilon_R^a)\lambda^a W^\dagger \;+\; \tfrac{i}{2}\,\epsilon_R^a\lambda^a\,\partial_\mu W^\dagger \;-\; \tfrac{i}{2}\,\epsilon_L^a\,(\partial_\mu W^\dagger)\lambda^a \;-\; \tfrac{i}{2}\,(\partial_\mu\epsilon_L^a)W^\dagger\lambda^a,
\end{aligned} \tag{11}$$

and therefore

$$\begin{aligned}
\delta\mathcal{L} \;=\; &\frac{iF^2}{2}\,(\partial_\mu\epsilon_L^a) \times \mathrm{tr}\Big(-(W^\dagger\lambda^a) \times \partial^\mu W \;+\; (\partial^\mu W^\dagger) \times (\lambda^a W)\Big)\\
+\; &\frac{iF^2}{2}\,(\partial_\mu\epsilon_R^a) \times \mathrm{tr}\Big(+(\lambda^a W^\dagger) \times \partial^\mu W \;-\; (\partial^\mu W^\dagger) \times (W\lambda^a)\Big).
\end{aligned} \tag{12}$$

In terms of the symmetry currents, this means

$$\begin{aligned}
J_L^{\mu,a} &= \frac{iF^2}{2}\,\mathrm{tr}\Big(-(W^\dagger\lambda^a)\times\partial^\mu W \,+\, (\partial^\mu W^\dagger)\times(\lambda^a W)\Big)\\
&= -\frac{iF^2}{2}\,\mathrm{tr}\Big(W^\dagger\lambda^a\times\partial^\mu W \,+\, \big(-\partial^\mu W^\dagger = +W^\dagger(\partial^\mu W)W^\dagger\big)\times\lambda^a W\Big)\\
&= -\frac{iF^2}{2}\,\mathrm{tr}\Big(W^\dagger\lambda^a\times\partial^\mu W \,+\, (\partial^\mu W)\times W^\dagger\lambda^a\Big)\\
&= -\frac{iF^2}{2}\times 2\,\mathrm{tr}\Big(\lambda^a\times(\partial^\mu W)W^\dagger\Big),
\end{aligned}\tag{13}$$

and likewise

$$J_R^{\mu,a} \;=\; +\frac{iF^2}{2}\times 2\,\mathrm{tr}\Big(\lambda^a\times W^\dagger(\partial^\mu W)\Big).\tag{14}$$

Or in terms of the vector and the axial currents,

$$J_V^{\mu,a} \;=\; \tfrac{1}{2}J_R^{\mu,a} \,+\, \tfrac{1}{2}J_L^{\mu,a} \;=\; \frac{iF^2}{2}\,\mathrm{tr}\Big(\lambda^a\times\big[W^\dagger,\partial^\mu W\big]\Big),\tag{15}$$

$$J_A^{\mu,a} \;=\; \tfrac{1}{2}J_R^{\mu,a} \,-\, \tfrac{1}{2}J_L^{\mu,a} \;=\; \frac{iF^2}{2}\,\mathrm{tr}\Big(\lambda^a\times\big\{W^\dagger,\partial^\mu W\big\}\Big).\tag{16}$$

In particular, in the vicinity of the $\langle W\rangle = 1$ vacuum state where

$$W(x) \;=\; 1 \,+\, \frac{i}{2F}\,\pi^a(x)\lambda^a \,+\, O(\pi^2/F^2),\tag{17}$$

the currents become

$$\text{(vector)}\quad J_\mu^a \;=\; -f^{abc}(\partial_\mu\pi^b)\pi^c \,+\, O(\pi^3/F),\tag{18}$$

$$\text{(axial)}\quad J_{\mu 5}^a \;=\; -F\partial_\mu\pi^a \,+\, O(\pi^2).\tag{19}$$

Note the linear (in the Goldstone fields π^a) term in the axial current. Thanks to this terms the current operator can create the Goldstone bosons from the vacuum or annihilate the Goldstone bosons:

$$\hat{J}_{\mu 5}^a\,|\mathrm{vac}\rangle \;=\; -iFp_\mu\,|\pi^a\rangle\,,\quad \hat{J}_{\mu 5}^a\,|\pi^b\rangle \;=\; iFp_\mu\delta^{ab}\,|\mathrm{vac}\rangle \,+\, \text{other states}.\tag{20}$$

When the NLΣM is used to model the $SU(2)\times SU(2)\to SU(2)$ chiral symmetry breaking in QCD, the Goldstone bosons π^a are identified as pions, and the F constant as $f_\pi\approx 93$ MeV, the pion decay constant. This name follows from the f_π governing the amplitude of the weak decay of a charged pion into a muon and an (anti)neutrino, $\pi^+ \to \mu^+ + \nu_\mu$ or $\pi^- \to \mu^- + \bar{\nu}_\mu$. You should calculate this decay amplitude — and hence the decay rate — as a part of your next homework set#23.

NLΣM in QCD Context

Let's see how the NLΣM of chiral symmetry breaking emerges from QCD. For simplicity, let's start with QCD with N_f exactly massless quark flavors,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} \;+\; \overline{\Psi}_{i\alpha}(i\not{D})^i{}_j\Psi^{j\alpha} \tag{21}$$

where indices $i, j, \ldots$ denote quark colors while $\alpha, \beta, \ldots$ denote their flavors. The theory has exact $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry which acts as

$$\Psi^{i\alpha}_L \;=\; \frac{1-\gamma^5}{2}\Psi^{i\alpha} \;\mapsto\; (U_L)^\alpha_\beta\Psi^{i\beta}_L, \quad \Psi^{i\alpha}_R \;=\; \frac{1+\gamma^5}{2}\Psi^{i\alpha} \;\mapsto\; (U_R)^\alpha_\beta\Psi^{i\beta}_R, \tag{22}$$

for independent $SU(N_f)$ matrices U_L and U_R. However, the non-perturbative dynamics of low-energy QCD *spontaneously* breaks this chiral symmetry down to the $SU(N_f)_V$ spanned by $U_L = U_R$.

The *order parameter* of this spontaneous symmetry breaking is non-zero vacuum expectation value of the scalar quark-antiquark bilinear $\langle\overline{\Psi}\Psi\rangle$. In StatMech terms, we may think of this VEV as the Bose–Einstein condensate of scalar mesons made from quark-antiquark pairs. This condensate breaks the chiral symmetry because the scalars are made from a quark and an antiquark of opposite chiralities; in terms of the Weyl fermions Ψ_L and Ψ_R,

$$\overline{\Psi}\Psi \;=\; \overline{\Psi}_R\Psi_L \;+\; \overline{\Psi}_L\Psi_R\,, \tag{23}$$

so the $\langle\overline{\Psi}\Psi\rangle$ condensate connects the left-handed and the right-handed fermionic fields to each other.

Let's take a closer look at the color and the flavor indices of the quark-antiquark condensate. Since the $SU(N_c)$ gauge symmetry of QCD is NOT spontaneously broken, the condensate must be color singlet, so the quark and the antiquark in the condensate must have matching colors. On the other hand, their flavor indices do not have to match, so in general we have

$$\left\langle \overline{\Psi}_{R,i\alpha} \Psi_L^{j\beta} \right\rangle \;=\; \frac{\delta_i^j}{N_c} \times T^\beta_{\ \alpha} \tag{24}$$

for some complex $N_f \times N_f$ matrix $T^\beta_{\ \alpha}$, and by Hermitian conjugation of the fields and their VEVs,

$$\left\langle \overline{\Psi}_{L,j\beta} \Psi_R^{i\alpha} \right\rangle \;=\; \frac{\delta_j^i}{N_c} \times (T^\dagger)^\alpha_{\ \beta}\,. \tag{25}$$

Thus, instead of a single condensate, we actually have a whole $N_f \times N_f$ complex matrix of condensates.

In excited states of QCD, the quark-antiquark condensate may fluctuate rather than being stuck to its vacuum value. The simplest way to describe such fluctuations — some of which may be rather long-ranged — is to promote the $T^\beta_{\ \alpha}$ matrix to a matrix-valued complex field $T^\beta_{\ \alpha}(x)$ and to write down an effective Lagrangian for this field,

$$\mathcal{L}_{\text{eff}} \;=\; -V(T, T^\dagger) \;+\; \mathcal{L}_{\text{kinetic}}(\partial_\mu T, \partial_\mu T^\dagger) \;+\; \mathcal{L}^{\text{higher}}_{\text{derivatives}}\,. \tag{26}$$

All parameters of this effective Lagrangian ultimately follow from QCD, but there is no perturbation theory for them, so until we have a better non-perturbative understanding of QCD we may only guess at those parameters or fit them to the experimental data. However, we know that the effective Lagrangian (26) must be invariant under all exact symmetries of QCD, even if they happen to be spontaneously broken.

In particular, consider the chiral $SU(N)_L \times SU(N)_R$ symmetries of QCD under which the condensate matrix T transforms as

$$\begin{aligned} T^\beta_{\ \alpha} \;&=\; \left\langle \overline{\Psi}_{R,i\alpha} \Psi_L^{i\beta} \right\rangle \\ &\downarrow \\ T'^\beta_{\ \alpha} \;&=\; \left\langle \overline{\Psi}'_{R,i\alpha} \Psi_L'^{i\beta} \right\rangle \;=\; \left\langle \overline{\Psi}_{R,i\gamma} (U_R^\dagger)^\gamma_{\ \alpha} \times (U_L)^\beta_{\ \delta} \Psi_L^{i\delta} \right\rangle \\ &=\; (U_L)^\beta_{\ \delta} \left\langle \overline{\Psi}_{R,i\gamma} \Psi_L^{i\delta} \right\rangle (U_R^\dagger)^\gamma_{\ \alpha} \;=\; (U_L)^\beta_{\ \delta}\, T^\delta_{\ \gamma}\, (U_R^\dagger)^\gamma_{\ \alpha}\,, \end{aligned} \tag{27}$$

or in matrix language

$$T' \;=\; U_L \times T \times U_R^\dagger. \tag{28}$$

Using this symmetry, any complex T matrix can be brought to the form

$$T' \;=\; U_L \times T \times U_R^\dagger \;=\; e^{i\theta} \times \mathcal{D} \tag{29}$$

where $\theta = (1/N_f) \arg\det(T)$ and $\mathcal{D}$ is a real diagonal matrix made from the eigenvalues of $T^\dagger T$, or rather from the square roots of those eigenvalues. Consequently, the effective potential $V(T, T^\dagger)$ for the quark-antiquark condensate must have form

$$V(T, T^\dagger) \;=\; \textit{function\_of}\big(\text{eigenvalues of } T^\dagger T \text{ and } \arg\det(T)\big). \tag{30}$$

We do not know the specific form of this function. However, judging by the phenomenology of the spontaneous chiral symmetry breaking in real life, *we presume that the minimum of this function obtains when*

$$\begin{aligned} &\text{all eigenvalues of } T^\dagger T \text{ are equal to some positive constant } \mathcal{C}^2 \;=\; O(\Lambda_{\mathrm{QCD}}^6) \\ &\qquad\qquad\text{and } \arg\det(T) \;=\; 0. \end{aligned} \tag{31}$$

Thus, the effective potential has minima for T such that

$$\text{for some } U_L, U_R \in SU(N_f), \quad U_L T U_R^\dagger \;=\; \mathcal{C} \times \mathbf{1}_{N_f \times N_f}\,, \tag{32}$$

and hence

$$T_\alpha^\beta \;=\; \mathcal{C} \times W_\alpha^\beta \quad \text{for an } SU(N_f) \text{ matrix } W_\alpha^\beta\,. \tag{33}$$

Thanks to the chiral symmetry, the minima for all $SU(N_f)$ matrices W are exactly degenerate, and as we saw earlier, any $\langle T\rangle = \mathcal{C} \times W$ *spontaneously* breaks the chiral symmetry down to the vector $SU(N_f)$.

The fluctuations $T(x) - \langle T\rangle$ of the quark-antiquark condensate give rise to $2N_f^2$ species of scalar particles, or rather scalar and pseudoscalar particles since the chiral symmetry does not commute with parity. Specifically, there are $N_f^2 - 1$ massless Goldstone bosons π^a corresponding to fluctuations of the form $T(t) = \mathcal{C} \times W(x)$, while the remaining $N^2 + 1$ particles are massive. In terms of their $SU(N_f)_V$ and parity quantum numbers:

- The Goldstone bosons are pseudoscalar and form the adjoint multiplet of the $SU(N_f)_V$. For $N_f = 2$ these Goldstone bosons approximate the pions.

* One massive particle is also pseudoscalar, but it's a singlet of $SU(N_f)_V$. For $N_f = 2$ this particle may be identified as the η meson with real-life mass $M_\eta \approx 550$ MeV.

* The rest of the massive particles are scalars (positive parity); they form an adjoint + singlet multiplet of the $SU(N_f)$. For $N_f = 2$, the scalar isosinglet can be identified with the broad resonance σ centered at 500 MeV, while the scalar isotriplet is harder to identify with real-life particles. Perhaps it's the lightest isotriplet of scalar mesons $f_0(980)$ with mass of 980 MeV.

At low energies / long distances, only the massless particles — the Goldstone bosons — become excited, so the effective field theory for those particles can be described in terms of the condensate fields $T^\beta_\alpha(x)$ limited to their Goldstone modes $T^\beta_\alpha(x) = \mathcal{C} \times W^\beta_\alpha(x)$ for $W(x) \in SU(N_f)$. For such modes, the potential $V(T, T^\dagger)$ is constant, the kinetic term in the effective Lagrangian is restricted by chiral symmetries to

$$\mathcal{L}_{\text{kin}} \;=\; \text{const} \times \text{tr}\bigl(\partial_\mu T\, \partial^\mu T^\dagger\bigr) \;=\; F^2\, \text{tr}\bigl(\partial_\mu W\, \partial^\mu W^\dagger\bigr), \tag{34}$$

so altogether

$$\mathcal{L}_{\text{eff}} \;=\; F^2\, \text{tr}\bigl(\partial_\mu W\, \partial^\mu W^\dagger\bigr) \;+\; \text{higher derivative terms.} \tag{35}$$

But the higher derivative terms decouple in the low-energy limit, so we are left with just the kinetic term of NLΣM.

And now we see how the NLΣM of the spontaneous chiral symmetry breaking emerges from QCD.

Mass Perturbation

Thus far, we have focused on QCD with exactly massless light flavors. Now let's consider the real life in which the light quarks have non-zero albeit small masses, thus

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_{\mathrm{massless}} + \mathcal{L}_{\mathrm{mass}} \tag{36}$$

for

$$\mathcal{L}_{\mathrm{mass}} = \sum_{\alpha}^{\mathrm{flavors}} m_\alpha \overline{\Psi}_{i\alpha} \Psi^{i\alpha}. \tag{37}$$

Let's promote the array of quark masses to a diagonal mass matrix

$$m^\alpha_{\ \beta} = m_\alpha \delta^\alpha_\beta\,, \tag{38}$$

so we may rewrite the mass terms in QCD Lagrangian as

$$\mathcal{L}_{\mathrm{mass}} = m^\alpha_{\ \beta} \overline{\Psi}_{R,i\alpha} \Psi^{i\beta}_L + (m^\dagger)^\beta_{\ \alpha} \overline{\Psi}_{L,i\beta} \Psi^{i\alpha}_R\,. \tag{39}$$

The mass term is NOT invariant under the chiral symmetry $SU(N_f)_L \times SU(N_f)_R$, but as long as the quark masses are much smaller then the energy scale of the QCD non-perturbative effects, we my still use the $SU(N_f)_L \times SU(N_f)_R$, as an *approximate* symmetry of the theory. Specifically, let's use the mass term (39) as a small perturbation to the rest of QCD.* Thus, the unperturbed theory has exact $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry, and only the small perturbation by the quark masses breaks this symmetry.

By the usual rules of perturbation theory, the first-order effect of a small perturbation $\Delta\hat{H}$ is limited to its diagonal matrix elements or off-diagonal elements between degenerate

$\star$ This perturbation has nothing to do with the Feynman perturbation theory in α_s. We are already in the deeply non-perturbative regime WRT that. Instead our 'unperturbed' theory is massless QCD in all its glory, with all the non-perturbative effects such as confinement and the spontaneous chiral symmetry breakdown already included. And then we add small quark masses as small perturbations on top of *that*.

or nearly-degenerate states of the un-perturbed theory. In QFT terms, this means the first order effect of small quark masses on the effective low-energy Lagrangian is

$$\Delta\mathcal{L}_{\text{eff}} \;=\; \langle\mathcal{L}_{\text{mass}}\rangle \tag{40}$$

where the VEVs are taken between the states of the un-perturbed (*i.e.*, massless) QCD. In particular, for the low-energy theory of the quark-antiquark condensates and their fluctuations,

$$\begin{aligned}\Delta\mathcal{L}_{\text{eff}} \;=\; \langle\mathcal{L}_{\text{mass}}\rangle \;&=\; m^{\alpha}_{\ \beta}\times\left\langle\overline{\Psi}_{R,i\alpha}\Psi^{i\beta}_{L}\right\rangle \;+\; (m^\dagger)^{\beta}_{\ \alpha}\left\langle\overline{\Psi}_{L,i\beta}\Psi^{i\alpha}_{R}\right\rangle\\ &=\; m^{\alpha}_{\ \beta}\times T^{\beta}_{\ \alpha} \;+\; (m^\dagger)^{\beta}_{\ \alpha}\times(T^\dagger)^{\alpha}_{\ \beta}\\ &=\; \operatorname{tr}(mT) \;+\; \operatorname{tr}(m^\dagger T^\dagger).\end{aligned} \tag{41}$$

And when we further restrict our analysis to the otherwise massless Goldstone modes of the chiral symmetry breaking,

$$T(x) \;=\; \mathcal{C}\times W(x) \quad\text{for}\quad \mathcal{C} \;=\; \left\langle\overline{\psi}\psi\right\rangle \;=\; O(\Lambda^3_{\text{QCD}}) \quad\text{and}\quad W(x)\in SU(N_f), \tag{42}$$

the first-order mass perturbation (41) becomes simply

$$\Delta\mathcal{L}_{\text{eff}} \;=\; \left\langle\overline{\psi}\psi\right\rangle\times\operatorname{tr}(mW+m^\dagger W^\dagger). \tag{43}$$

Thus, the effective low-energy theory becomes the NLΣM with a small potential term for the W field, namely

$$\mathcal{L}_{\text{eff}} \;=\; F^2\,\operatorname{tr}(\partial_\mu W^\dagger\,\partial^\mu W) \;-\; V \quad\text{for}\quad V \;=\; -\left\langle\overline{\psi}\psi\right\rangle\times\operatorname{tr}(mW+m^\dagger W^\dagger). \tag{44}$$

The potential (44) spoils the degeneracy between the vacuum states with different $W\in SU(N_f)$. In the basis for the quark fields where the mass matrix m is diagonal, real, and positive, the potential (44) has a unique minimum, namely $\langle W\rangle = 1$. Consequently, there is a unique vacuum state, and all fluctuations around this vacuum cost positive energy, so all the particles are massive. And that's why in real life, the π mesons are not exactly massless but merely light compared to the other mesons.

To calculate the pion mass — and also the masses of other light pseudoscalar mesons for $N_f = 3$ — we expand the non-linear W field around $\langle W\rangle = 1$,

$$W(x) \;=\; \exp\left(\frac{i\pi^a(x)\lambda^a}{2F}\right) \;=\; 1 \,+\, \frac{i}{2F}\,\pi^a(x)\lambda^a \;-\; \frac{1}{16F^2}\,\pi^a(x)\pi^b(x)\,\{\lambda^a,\lambda^b\} \;+\; O(\pi^3/F^3). \tag{45}$$

Consequently,

$$W \;+\; W^\dagger \;=\; 2 \;-\; \frac{1}{8F^2}\,\pi^a\pi^b \times \{\lambda^a,\lambda^b\} \;+\; O(\pi^4/F^4), \tag{46}$$

and therefore the potential V for the π^a fields

$$V(\pi) \;=\; -\left\langle\overline{\psi}\psi\right\rangle \times \mathrm{tr}\big(m(W^+W^\dagger)\big) \;=\; \mathrm{const} + \frac{\left\langle\overline{\psi}\psi\right\rangle}{8F^2}\times\mathrm{tr}(m\{\lambda^a,\lambda^b\})\times\pi^a\pi^b \,+\, O(\pi^4). \tag{47}$$

In particular, we get the mass matrix for the pions (and similar pseudoscalars for $N_f = 3$), namely

$$V \;=\; \tfrac{1}{2}(M^2)^{ab}\times\pi^a\pi^b \tag{48}$$

for

$$(M^2)^{ab} \;=\; \frac{\left\langle\overline{\psi}\psi\right\rangle}{4F^2}\times\mathrm{tr}\big(m\{\lambda^a,\lambda^b\}\big). \tag{49}$$

In particular, for the 2-flavor NLΣM, we get

$$\{\lambda^a,\lambda^b\} \;\to\; \{\tau^a,\tau^b\} \;=\; 2\delta^{ab}\times 1_{2\times 2}\,, \tag{50}$$

$$\mathrm{tr}\big(m\{\lambda^a,\lambda^b\}\big) \;=\; 2\delta^{ab}\times\mathrm{tr}(m) \;=\; 2\delta^{ab}\times(m_u+m_d), \tag{51}$$

and therefore equal masses for all 3 species of pions, namely

$$M_\pi^2 \;=\; \frac{\left\langle\overline{\psi}\psi\right\rangle}{2F_\pi^2}\times(m_u+m_d) \tag{52}$$

In real life, the charged pions $\pi^\pm$ are slightly heavier then the neutral pion π^0, — $M(\pi^\pm)\approx$ 139 MeV while $M(\pi^0)\approx 134$ MeV — but the difference stems from the electromagnetic effects rather than the quark masses. Indeed, the electromagnetism breaks the isospin symmetry $SU(2)$ down to $U(1)$, and likewise breaks the chiral isospin $SU(2) \times SU(2)$ down to $U(1) \times U(1)$, and that produces an extra chiral symmetry breaking besides the effect of the quark masses. Consequently, even without the quark masses, the charged pions $\pi^{\pm}$ would have small but non-zero mass and only the neutral pion would be massless. On the other hand, even without the electromagnetism, the quark mass difference $m_d - m_u$ would also produce a small $M(\pi^{\pm}) - M(\pi^0)$ pion mass splitting in the second order of the mass perturbation theory, but this effect is much smaller than the electromagnetic mass splitting.

For the 3-flavor NLΣM, we have 8 pseudo-Goldstone pseudoscalar mesons, namely the 3 pions, the 4 kaons (the isospin doublet (K^+, K^0) and their antiparticles $(\overline{K}^0, K^-)$), and one eta meson. Eq. (49) gives a diagonal mass matrix for these 8 mesons, with eigenvalues

$$
\begin{aligned}
M^2(\pi\pm) \;=\; M^2(\pi^0) \;&=\; \frac{\langle\overline{\psi}\psi\rangle}{2F_\pi^2} \times (m_u + m_d), &(53)\\
M^2(K^{\pm}) \;&=\; \frac{\langle\overline{\psi}\psi\rangle}{2F_\pi^2} \times (m_s + m_u), &(54)\\
M^2(K^0 \text{ or } \overline{K}^0) \;&=\; \frac{\langle\overline{\psi}\psi\rangle}{2F_\pi^2} \times (m_s + m_d), &(55)\\
M^2(\eta) \;&=\; \frac{\langle\overline{\psi}\psi\rangle}{2F_\pi^2} \times \left(\tfrac{4}{3}m_s + \tfrac{1}{3}(m_d + m_u)\right). &(56)
\end{aligned}
$$

Adding the EM corrections to the M^2 of the charged pions or kaons, we get a pretty good fit to the real-life meson masses

$$
M(\pi) = 135 \text{ MeV}, \quad M(K^{\pm}) = 494 \text{ MeV}, \quad M(K^0) = 498 \text{ MeV}, \quad M(\eta) = 547 \text{ MeV} \quad (57)
$$

for

$$
m_u \;\approx\; 2.16 \text{ MeV}, \quad m_d \;\approx\; 4.67 \text{ MeV}, \quad m_s \;\approx\; 93 \text{ MeV} \quad (58)
$$

(renormalized to $\mu = 2$ GeV), while $F \approx f_\pi = 93$ MeV and $\langle\overline{\psi}\psi\rangle \approx (360 \text{ MeV})^3$.

Note: without the axial anomaly, there would be 9 rather than 8 pseudo-Goldstone pseudoscalar mesons in the octet+singlet of the flavor $SU(3)$. And indeed in real life there is a ninth pseudoscalar meson η', but the anomaly gives it a larger mass $M(\eta') = 947$ MeV than the other 8 pseudoscalar mesons. Moreover, the η' is not exactly an $SU(3)$ singlet while η is not exactly a member of an octet; instead there is a roughly $15°$ mixing between the two mesons because the quark masses give mass^2 to one combination of two isosinlget mesons while the anomaly gives mass^2 to another combination. The easiest way to see this is in terms of the $U(3)$ NLΣM, where the matrix-valued field $W(x)$ is unitary but its determinant is not restricted, and the Lagrangian is

$$\mathcal{L} \;=\; F^2\,\mathrm{tr}(\partial^\mu W^\dagger\,\partial_\mu W) \;+\; \left\langle\overline{\psi}\psi\right\rangle \times \mathrm{tr}(mW + m^\dagger W^\dagger) \;+\; A\times\bigl(\det(W)+\det(W^\dagger)\bigr) \quad (59)$$

for some constant $A = O(\Lambda_{\mathrm{QCD}})$. The A term breaks the $U(1)_A$ subgroup of the chiral symmetry of the NLΣM, so it represents the effect of the axial anomaly on the meson masses.

Together, the A term and the quark mass terms yield a non-diagonal 2×2 mass^2 matrix for the two isosinglet mesons — namely the $SU(3)$ singlet and the isosinglet member of the octet. with the eigenstates being the physical η and η' mesons. For $A \approx \frac{2}{3}m_s\left\langle\overline{\psi}\psi\right\rangle \approx (230\ \mathrm{GeV})^4$, this mass^2 matrix is

$$M^2 \;\approx\; 0.16\ \mathrm{GeV}^2 \times \begin{pmatrix} 2.0 & -1.4 \\ -1.4 & 7.0 \end{pmatrix}, \quad (60)$$

which gives a good fit to the $15°$ mixing angle between the η meson and the octet state (or between the η' meson and the singlet state), although the masses $M_1 \approx 510$ MeV and $M_2 \approx 1080$ MeV come out a few percent off from the real-life values $M_\eta = 547$ MeV and $M_{\eta'} = 947$ MeV.