Phase Space Factors

Consider a quantum transition from some initial state to a continuum of unbound states. For example, an excited atom emitting a photon, or an unstable particle decaying into two or more lighter particles. Another example would be scattering, in which the initial unbound state of two particles about to collide transitions into another unbound state of particles moving in different directions. In all such cases, the final states form a continuum, the transition not to a specific final state but to a continuous family of similar final states. Fermi's golden rule gives the rate of such transitions:

$$\Gamma \stackrel{\text{def}}{=} \frac{d\,\text{probability}}{d\,\text{time}} = \frac{2\pi\rho}{\hbar} \times \left|\langle\text{final}|\,\hat{T}\,|\text{initial}\rangle\right|^2 \tag{1}$$

where $\hat{T} = \hat{V} +$ higher order corrections, and ρ is the *final states' density*

$$\rho = \frac{dN_{\text{final states}}}{dE_{\text{final}}}\,. \tag{2}$$

Equivalently,

$$\Gamma = \int dN_{\text{final}} \left|\langle\text{final}|\,\hat{T}\,|\text{initial}\rangle\right|^2 \times \frac{2\pi}{\hbar}\delta(E_{\text{final}} - E_{\text{initial}}). \tag{3}$$

For an example, consider an atom in an excited state emitting a photon while the atom itself drops to a lower energy state. Thus, the initial and the final states are eigenstates of the free Hamiltonian

$$\hat{H}_0 = \hat{H}(\text{atom}) + \hat{H}(\text{free photons}) \tag{4}$$

while the interactions between the quantum EM fields and the atom's electrons comprise the perturbation $\hat{V}$. For a moment, let's fix the specific initial and final states of the atom as well as the photon's polarization λ. However, the final states still form a continuous family parametrized by the photon's momentum $\mathbf{p}_\gamma$. In the large-box normalization, the number of such final states is

$$dN_{\text{final}} = \left(\frac{L}{2\pi}\right)^3 d^3\mathbf{k}_\gamma = \frac{L^3}{(2\pi)^3} \times k_\gamma^2\,dk_\gamma\,d^2\Omega_\gamma \tag{5}$$

where $d^2\Omega_\gamma$ is the infinitesimal solid angle into which the photon is emitted. At the same time,

$$E^{\rm net}_{\rm final}\ -\ E^{\rm net}_{\rm initial}\ =\ \hbar ck_\gamma\ +\ E^{\rm atom}_{\rm final}\ -\ E^{\rm atom}_{\rm initial}\ =\ \hbar ck_\gamma\ -\ \Delta E^{\rm atom}, \tag{6}$$

hence (to the first order of the perturbation $\hat{V}$)

$$\Gamma\ =\ \frac{1}{(2\pi)^2\hbar}\int\!d^2\Omega_\gamma\int\!dk_\gamma\,k_\gamma^2\times L^3\left|\left\langle {\rm atom}_f+\gamma\right|\hat{V}\left|{\rm atom}_i\right\rangle\right|^2\times\delta(\hbar ck_\gamma-\Delta E^{\rm atom}). \tag{7}$$

In this formula, the L^3 factor in the density of states factor cancels against the (square of the) $L^{-3/2}$ factor in the matrix element due to the photon's wave function in the large-box normalization. Specifically, in the electric dipole approximation to the interaction between the EM fields and the atom

$$\hat{V}\ \approx\ -\hat{\mathbf{E}}(\mathbf{x}_{\rm atom})\cdot\hat{\mathbf{d}}\qquad(\hat{\mathbf{d}}\text{ being the atom's electric dipole moment}), \tag{8}$$

we have

$$\begin{aligned}\left\langle {\rm atom}_f+\gamma\right|\hat{V}\left|{\rm atom}_i\right\rangle\approx\ &-\left\langle\gamma(\mathbf{k},\lambda)\right|\hat{\mathbf{E}}\left|{\rm vac}\right\rangle\cdot\left\langle f\right|\hat{\mathbf{d}}\left|i\right\rangle_{\rm atom}\\ =\ &-iL^{-3/2}\sqrt{2\pi\hbar\omega_{\mathbf{k}}}\,e^{-i\mathbf{k}\cdot\mathbf{x}_{\rm atom}}\,\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\left\langle f\right|\hat{\mathbf{d}}\left|i\right\rangle_{\rm atom}\,.\end{aligned} \tag{9}$$

hence

$$\Gamma\ \approx\int\!d^2\Omega_\gamma\int\!dk_\gamma\,k_\gamma^2\times\frac{\omega=ck_\gamma}{2\pi}\left|\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\left\langle f\right|\hat{\mathbf{d}}\left|i\right\rangle_{\rm atom}\right|^2\times\delta(\hbar ck_\gamma-\Delta E^{\rm atom}). \tag{10}$$

Integrating over the k_γ removes the delta-function for the energy, and we are left with

$$\Gamma\ =\ \frac{(\Delta E^{\rm atom})^3}{2\pi\hbar^4c^3}\int\!d^2\Omega_\gamma\left|\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\left\langle f\right|\hat{\mathbf{d}}\left|i\right\rangle_{\rm atom}\right|^2. \tag{11}$$

Moreover, we may drop the $\int d\Omega$ integral and get the partial rate of photon emission in a particular direction,

$$\frac{d\Gamma}{d\Omega_\gamma}\ =\ \frac{(\Delta E^{\rm atom})^3}{2\pi\hbar^4c^3}\times\left|\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\left\langle f\right|\hat{\mathbf{d}}\left|i\right\rangle_{\rm atom}\right|^2. \tag{12}$$

Alternatively, we may not only integrate over the photon's direction but also sum over its polarization as well as some quantum numbers of the atom's final state — such as m_j —

that we are not bothering to measure. This gives us a more inclusive transition rate

$$\begin{aligned}\Gamma &= \frac{(\Delta E^{\rm atom})^3}{2\pi\hbar^4 c^3}\times\int\!d^2\Omega_\gamma\sum_\lambda\sum_{m_j(f)}\left|\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\langle f|\,\hat{\mathbf{d}}\,|i\rangle_{\rm atom}\right|^2\\ &= \frac{4(\Delta E^{\rm atom})^3}{3\hbar^4 c^3}\times\sum_{m_j(f)}\left|\langle f|\,\hat{\mathbf{d}}\,|i\rangle_{\rm atom}\right|^2.\end{aligned} \tag{13}$$

For another example, consider the decay of an unstable particle into n daughter particles. Due to momentum conservation, only $n-1$ of the daughter particle momenta $\mathbf{p}'_i$ are independent, but formally we may integrate over all n of the $\mathbf{p}'_i$ but include a delta-function to reimpose the momentum conservation. Thus,

$$\begin{aligned}\Gamma = \int\!\frac{L^3d^3\mathbf{p}'_1}{(2\pi\hbar)^3}\cdots\int\!\frac{L^3d^3\mathbf{p}'_n}{(2\pi\hbar)^3}\,\left|\left\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n\right|\hat{T}\left|\mathbf{p}_{\rm in}\right\rangle\right|^2\times\\ \times\left(\frac{2\pi\hbar}{L}\right)^3\delta^{(3)}(\mathbf{p}'_1+\cdots+\mathbf{p}'_n-\mathbf{p}_{\rm in})\times\frac{2\pi}{\hbar}\delta(E'_1+\cdots+E'_n-E_{\rm in}).\end{aligned} \tag{14}$$

This formula assumes non-relativistic big-box normalization of quantum states and matrix elements. Changing to the continuum normalization — which removes the powers of L^3 in both phase-space factors and in the matrix element — and also using the $\hbar=c=1$ units, turns eq. (14) to

$$\begin{aligned}\Gamma = \int\!\frac{d^3\mathbf{p}'_1}{(2\pi)^3}\cdots\int\!\frac{d^3\mathbf{p}'_n}{(2\pi)^3}\,\left|\left\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n\right|\hat{T}\left|\mathbf{p}_{\rm in}\right\rangle\right|^2\times\\ \times(2\pi)^3\delta^{(3)}(\mathbf{p}'_1+\cdots+\mathbf{p}'_n-\mathbf{p}_{\rm in})\times(2\pi)\delta(E'_1+\cdots+E'_n-E_{\rm in}).\end{aligned} \tag{15}$$

In a relativistic theory, we may combine the δ-functions for the momentum conservation and the energy conservation into a single 4D δ-function

$$(2\pi)^3\delta^{(3)}(\mathbf{p}'_1+\cdots+\mathbf{p}'_n-\mathbf{p}_{\rm in})\times(2\pi)\delta(E'_1+\cdots+E'_n-E_{\rm in}) = (2\pi)^4\delta^{(4)}(p'_1+\cdots+p'_n-p_{\rm in}). \tag{16}$$

Also, we should use the relativistic normalization of the particle states, which changes the transition matrix element to

$$\left\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n\right|\hat{\mathcal{M}}\left|\mathbf{p}_{\rm in}\right\rangle \equiv \left\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n\right|\hat{T}\left|\mathbf{p}_{\rm in}\right\rangle_{\rm rel} = \sqrt{2E_{\rm in}}\prod_{i=1}^n\sqrt{2E'_i}\times\left\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n\right|\hat{T}\left|\mathbf{p}_{\rm in}\right\rangle_{\rm nonrel}. \tag{17}$$

Consequently, in eq. (15) the mod-square of the relativistic decay amplitude $\left\langle\mathbf{p}'_1,\ldots,\mathbf{p}'_n\right|\hat{\mathcal{M}}\left|\mathbf{p}_{\rm in}\right\rangle$

should be divided by a $2E$ factor for each initial or final particle, thus

$$\Gamma = \frac{1}{2E_{\rm in}} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \cdots \int \frac{d^3\mathbf{p}'_n}{(2\pi)^3\, 2E'_n} \left|\langle p'_1, \ldots, p'_n \right| \mathcal{M} \left| p_{\rm in} \rangle\right|^2 \times (2\pi^4) \delta^{(4)}(p'_1 + \cdots + p'_n - p_{\rm in}). \tag{18}$$

In other words, an unstable particle (0) decays into n final-state particles $(1') + \cdots + (n')$ at the rate

$$\Gamma = \int d\mathcal{P}_{\rm decay} \times \left|\left\langle 1', 2', \ldots, n' \right| \mathcal{M} \left| 0 \right\rangle\right|^2 \tag{19}$$

where $\mathcal{M}(0 \to 1' + \cdots + n') \equiv \langle 1', \ldots, n' | \hat{\mathcal{M}} | 0 \rangle$ is the relativistic decay amplitude calculates according to the Feynman rules, and $d\mathcal{P}$ is the *phase space factor*

$$d\mathcal{P}_{\rm decay} = \frac{1}{2E_0} \times \prod_{i=1}^{n} \frac{d^3\mathbf{p}'_i}{(2\pi)^3 2E'_i} \times (2\pi)^4 \delta^{(4)}(E'_1 + \cdots + E'_n - E_0). \tag{20}$$

Likewise, the transition rate for a generic 2-particle to n-particle scattering process is given by

$$\Gamma = \frac{1}{2E_1 \times 2E_2} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \cdots \int \frac{d^3\mathbf{p}'_n}{(2\pi)^3\, 2E'_n} \left|\langle p'_1, \ldots, p'_n \right| \mathcal{M} \left| p_1, p_2 \rangle\right|^2 \times \\ \times (2\pi^4) \delta^{(4)}(p'_1 + \cdots + p'_n - p_1 - p_2). \tag{21}$$

In terms of the scattering cross-section σ, the rate (21) is $\Gamma = \sigma \times$ flux of initial particles. In the large-box normalization the flux is $L^{-3}|\mathbf{v}_1 - \mathbf{v}_2|$, so in the continuum normalization it's simply the relative speed $|\mathbf{v}_1 - \mathbf{v}_2|$. Consequently, the total scattering cross-section is given by

$$\sigma_{\rm tot} = \frac{1}{4E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2|} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3\, 2E'_1} \cdots \int \frac{d^3\mathbf{p}'_n}{(2\pi)^3\, 2E'_n} \left|\langle p'_1, \ldots, p'_n \right| \mathcal{M} \left| p_1, p_2 \rangle\right|^2 \times \\ \times (2\pi^4) \delta^{(4)}(p'_1 + \cdots + p'_n - p_1 - p_2), \tag{22}$$

or in other words

$$\sigma_{\rm tot} = \int d\mathcal{P}_{\rm scattering} \times \left|\left\langle 1', 2', \ldots, n' \right| \mathcal{M} \left| 1, 2 \right\rangle\right|^2 \tag{23}$$

$$\text{for } d\mathcal{P}_{\text{scattering}} = \frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|} \times \prod_{i=1}^{n} \frac{d^3\mathbf{p}'_i}{(2\pi)^3 2E'_i} \times (2\pi)^4\delta^{(4)}(E'_1+\cdots+E'_n-E_0). \tag{24}$$

A note on Lorentz invariance of decay rates or cross-sections. The matrix elements $\langle\text{final}|\,\mathcal{M}\,|\text{initial}\rangle$ are Lorentz invariant, and so are all the integrals over the final-particles' momenta and the δ-functions. The only non-invariant factor in the decay-rate formula (18) is the pre-integral $1/E_0$, hence the decay rate of a moving particle is

$$\Gamma(\text{moving}) = \Gamma(\text{rest frame}) \times \frac{M}{E} \tag{25}$$

where M/E is precisely the time dilation factor in the moving frame.

As to the scattering cross-section, it should be invariant under Lorentz boosts along the initial axis of scattering, thus the same cross-section in any frame where $\mathbf{p}_1 \parallel \mathbf{p}_2$. This includes the *lab frame* where one of the two particles is initially at rest, the *center-of-mass frame* where $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and any other frame where the two particles collide head-on. And indeed, in any frame where both $\mathbf{p}_1$ and $\mathbf{p}_2$ are parallel to the z axis, the pre-integral factor in eq. (22) for the cross-section becomes

$$\frac{1}{4E_1E_2|\mathbf{v}_1-\mathbf{v}_2|} = \frac{1}{4|E_1\mathbf{p}_2 - E_2\mathbf{p}_1|} = \frac{1}{4|\epsilon_{\mu\nu xy}p_1^\mu p_2^\nu|}\,, \tag{26}$$

which is manifestly invariant under the $SO^+(1,1)$ group of Lorentz boosts along the z axis.

Let's simplify eq. (22) for a 2 particle $\to$ 2 particle scattering process in the center-of-mass frame where $\mathbf{p}_1 + \mathbf{p}_2 = 0$. In this frame,

$$E_1E_2\,|\mathbf{v}_1-\mathbf{v}_2| = |E_2\mathbf{p}_1 - E_1\mathbf{p}_2| = |\mathbf{p}| \times (E_1+E_2 = E_{\text{tot}}), \tag{27}$$

hence the phase-space factor

$$\mathcal{P}_{\text{scattering}} = \frac{1}{4|\mathbf{p}|E_{\text{tot}}} \times \mathcal{P}_{\text{int}} \tag{28}$$

for

$$\begin{aligned}
\mathcal{P}_{\rm int} &= \int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\,2E'_1}\int\frac{d^3\mathbf{p}'_2}{(2\pi)^3\,2E'_2}\,(2\pi)^4\delta^{(3)}(\mathbf{p}'_1+\mathbf{p}'_2)\delta(E'_1+E'_2-E_{\rm net}) \\
&= \int\frac{d^3\mathbf{p}'_1}{(2\pi)^3\times 2E'_1\times 2E'_2}\,(2\pi)\delta(E'_1(\mathbf{p}'_1)+E'_2(-\mathbf{p}'_1)-E_{\rm net}) \\
&= \int d^2\Omega_{\mathbf{p}'}\times\int_0^\infty dp'\,\frac{p'^2}{16\pi^2E'_1E'_2}\times\delta(E'_1+E'_2-E_{\rm tot}) \\
&= \int d^2\Omega_{\mathbf{p}'}\left[\frac{p'^2}{16\pi^2E'_1E'_2}\Big/\frac{d(E'_1+E'_2)}{dp'}\right]^{\rm when}_{E'_1+E'_2=E_{\rm tot}}.
\end{aligned} \tag{29}$$

On the last 3 lines here $E'_1=E'_1(\mathbf{p}'_1)=\sqrt{p'^2+m'^2_1}$ while $E'_2=E'_2(\mathbf{p}'_2=-\mathbf{p}'_1)=\sqrt{p'^2+m'^2_2}$. Consequently,

$$\frac{dE'_1}{dp'} = \frac{p'}{E'_1}\,,\quad \frac{dE'_2}{dp'} = \frac{p'}{E'_2}\,, \tag{30}$$

$$\frac{d(E'_1+E'_2)}{dp'} = \frac{p'}{E'_1}+\frac{p'}{E'_2} = \frac{p'}{E'_1E'_2}\times(E'_2+E'_1 = E_{\rm tot}), \tag{31}$$

$$\left[\frac{p'^2}{16\pi^2E'_1E'_2}\Big/\frac{d(E'_1+E'_2)}{dp'}\right]^{\rm when}_{E'_1+E'_2=E_{\rm tot}} = \frac{p'}{16\pi^2E_{\rm tot}}\,, \tag{32}$$

and therefore

$$\begin{aligned}
\mathcal{P}_{\rm scattering} &= \frac{1}{4|\mathbf{p}|E_{\rm tot}}\times\frac{p'}{16\pi^2E_{\rm tot}}\times\int d^2\Omega_{\mathbf{p}'} \\
&= \frac{p'}{p}\times\frac{1}{64\pi^2E^2_{\rm tot}}\times\int d^2\Omega_{\mathbf{p}'}\,.
\end{aligned} \tag{33}$$

Note: since the scattering amplitude $\mathcal{M}$ may depend on the directions of the scattered particles, we should multiply the phase space factor by the $|\mathcal{M}|^2$ *before* integrating over those directions. This means that we should not evaluate the angular integral in eq. (33) but rather re-interpret that formula as

$$d\mathcal{P}_{\rm scattering} = \frac{p'}{p}\times\frac{1}{64\pi^2E^2_{\rm cm}}\times d\Omega_{\rm cm} \tag{34}$$

where

$$d\Omega_{\rm cm} = d^2\Omega_{\mathbf{p}'_1} = d^2\Omega_{\mathbf{p}'_2}\quad\text{in the center-of-mass frame}$$

and also

$$E_{\rm cm}^2 \;=\; E_{\rm tot}^2(\text{in the center-of-mass frame}); \tag{35}$$

in frame-independent terms,

$$E_{\rm cm}^2 \;=\; (E_1+E_2)^2 \;-\; (\mathbf{p}_1+\mathbf{p}_2)^2 \;=\; (p_1+p_2)^2 \;=\; \text{Mandelstam's } s. \tag{36}$$

In light of eq. (34), in the center-of-mass frame

$$\begin{aligned} d\sigma(1+2\to 1'+2') \;&=\; \left|\langle p_1'+p_2'\right| \mathcal{M}\left|p_1+p_2\rangle\right|^2 \times d\mathcal{P}_{\rm scattering} \\ &=\; \frac{p'}{p}\,\frac{1}{64\pi^2 E_{\rm tot}^2} \times \left|\langle p_1'+p_2'\right| \mathcal{M}\left|p_1+p_2\rangle\right|^2 \times d\Omega_{\rm cm} \end{aligned} \tag{37}$$

and hence the *partial cross-section* for scattering in a particular direction is

$$\frac{d\sigma(1+2\to 1'+2')}{d\Omega_{\rm cm}} \;=\; \frac{p'}{p}\,\frac{1}{64\pi^2 E_{\rm tot}^2} \times \left|\langle p_1'+p_2'\right| \mathcal{M}\left|p_1+p_2\rangle\right|^2. \tag{38}$$

Finally, the net cross-section — into specific final particle species but emitted in any direction — obtains as an integral

$$\sigma_{\rm net}(1+2\to 1'+2') \;=\; \frac{p'}{p} \times \frac{1}{64\pi^2 E_{\rm cm}^2} \times \int d^2\Omega_{\rm cm}\; \left|\langle p_1'+p_2'\right| \mathcal{M}\left|p_1+p_2\rangle\right|^2. \tag{39}$$

Note: the net cross-sections have same values in all frames where the initial momenta are collinear, so you may use eq. (39) in any such frame, provided you evaluate the (p'/p) ration in the center-of-mass frame. But the infinitesimal solid angles $d\Omega$ are not invariant under Lorentz boosts along the scattering axis, so eq. (38) for the partial cross-section applies only in the center-of mass frame. In any other collinear frame, we would have

$$\frac{d\sigma}{d\Omega} \;=\; \frac{d\sigma}{d\Omega_{\rm cm}} \times \frac{d\Omega_{\rm cm}}{d\Omega} \tag{40}$$

with a non-trivial frame-dependent factor $d\Omega_{\rm cm}/d\Omega$.

Finally, let me write down the phase-space factor for a 2-body decay (1 particle $\to$ 2 particles) in the rest frame of the initial particle. The under-the-integral factors for such a decay are the same as in eq. (29) for a $2 \to 2$ scattering, but the pre-integral factor is $1/2M_{\rm in}$ instead of $1/4pE_{\rm cm}$, thus instead of eq. (34) we get

$$d\mathcal{P}_{\rm decay} \;=\; \frac{1}{2M_{\rm in}} \times \frac{p'}{16\pi^2(E_{\rm tot}=M_{\rm in})} \times d\Omega_{\rm cm} \;=\; \frac{p'}{32\pi^2 M_{\rm in}^2} \times d\Omega_{\rm cm}\,. \tag{41}$$

Consequently, the partial decay rate (for the final particles flying along a particular axis) is

$$\frac{d\Gamma(0\to 1'+2')}{d\Omega_{\rm cm}} \;=\; \frac{p'}{32\pi^2 M^2} \times \left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2, \tag{42}$$

and the net decay rate — into specific particle species but flying in any directions — is

$$\Gamma(0\to 1'+2') \;=\; \frac{p'}{32\pi^2 M^2} \times \int\! d^2\Omega_{\rm cm}\,\left|\langle p_1'+p_2'|\,\mathcal{M}\,|p_0\rangle\right|^2. \tag{43}$$

Postscript: In these notes, I have treated all particles as scalars and ignored their spin states. For scattering of decay processes involving particles with non-zero spins, we should distinguish between the polarized cross-sections or decay rates — in which we know the spin states of all the initial and final particles, — and the un-polarized cross-sections or decay rates — in which we sum over the final particles' spin states and average over the initial particles' spin states. I shall explain this issue later in class when we get to QED.