

QCD Feynman Rules

The classical chromodynamics has a fairly simple Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{quarks}} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \sum_f \bar{\Psi}_{if}(i\not{D} + m_f)\Psi^{if} \quad (1)$$

where i denotes the color of a quark and f its flavor. In my notations, I follows the usual summation convention for the Lorentz or color indices, — and the Dirac indices are implicit altogether; but the sum over the quark flavors is explicit since the mass m_f depends on the flavor. OOH, the covariant derivatives D_μ are flavor-blind, $D_\mu\Psi^{if} = \partial_\mu\Psi^{if} + igA_\mu^a(t^a)^i_j\Psi^{jf}$ where t^a are matrices representing the gauge group generators in the quark representation; in QCD the quarks belong to the fundamental $\mathbf{3}$ representation of the $SU(3)_C$ so t^a are $\frac{1}{2} \times$ Gell-Mann matrices λ^a .

The Quantum ChromoDynamics is more complicated, even at the Lagrangian level: Besides the physical terms (1), there are gauge-fixing and ghost terms, and then there the counterterms. Altogether, bare QCD Lagrangian is

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{c.t.}}, \quad (2)$$

$$\begin{aligned} \mathcal{L}_{\text{ren}} &= \mathcal{L}_{\text{phys}} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{gh}} \\ &= -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \sum_f \bar{\Psi}_{if}(i\not{D} + m_f)\Psi^{if} - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \partial_\mu \bar{c}^a D^\mu c^a \end{aligned} \quad (3)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu \bar{c}^a)(\partial^\mu c^a) + \sum_f \bar{\Psi}_{if}(i\not{\partial} - m_f)\Psi^{if} \quad (4)$$

$$+ gf^{abc}A_\mu^b A_\nu^c \partial_\mu A^{a\nu} - \frac{g^2}{4}(f^{abc}A_\mu^b A_\nu^c)^2 - gf^{abc}(\partial_\mu \bar{c}^a)A^{b\mu}c^c \quad (5)$$

$$- gA_\mu^a \sum_f \bar{\Psi}_{if}\gamma^\mu(t^a)^i_j\Psi^{jf}, \quad (5)$$

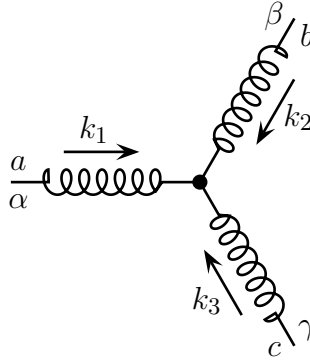
and

$$\mathcal{L}_{\text{c.t.}} = -\frac{\delta_3}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + g\delta_1^{(3g)}f^{abc}A_\mu^b A_\nu^c \partial_\mu A^{a\nu} - \frac{g^2\delta_1^{(4g)}}{4}(f^{abc}A_\mu^b A_\nu^c)^2 \quad (6)$$

$$+ \delta_2^{(\text{gh})}\partial_\mu \bar{c}^a \partial^\mu c^a - g\delta_1^{(\text{gh})}f^{abc}\partial_\mu \bar{c}^a A^{b\mu}c^c \quad (6)$$

$$+ \sum_f \bar{\Psi}_{if}\left(i\delta_2^{(q_f)}\not{\partial} + \delta_m^{(q_f)} - g\delta_1^{(q_f)}A^a t^a\right)\Psi^{if}. \quad (6)$$

- The first term on the top line (5) gives rise to the **three-gluon vertex**



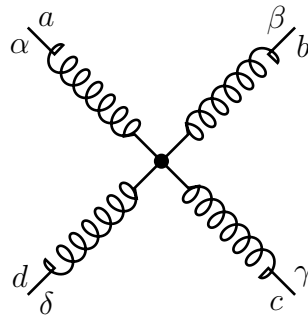
$$= -gf^{abc} \left[g^{\alpha\beta} (k_1 - k_2)^\gamma + g^{\beta\gamma} (k_2 - k_3)^\alpha + g^{\gamma\alpha} (k_3 - k_1)^\beta \right]. \quad (10)$$

The specific indexology of this vertex follows from totally symmetrizing the 3 gluon fields A_α^a , A_β^b , and A_γ^c in the interaction term $gf^{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu}$. Since the f^{abc} factor is totally antisymmetric in the adjoint colors of the 3 gluons, this means totally antisymmetrizing the Lorentz and derivative indices, thus

$$\mathcal{L} \supset gf^{abc}(\partial^\gamma A_\alpha^a)A_\beta^b A_\gamma^c g^{\alpha\beta} = \frac{gf^{abc}}{6} \times \begin{pmatrix} g^{\alpha\beta}(\partial^\gamma A_\alpha^a)A_\beta^b A_\gamma^c - g^{\alpha\beta}A_\alpha^a(\partial^\gamma A_\beta^b)A_\gamma^c \\ + g^{\beta\gamma}(\partial^\alpha A_\beta^b)A_\gamma^c A_\alpha^a - g^{\beta\gamma}A_\beta^b(\partial^\alpha A_\gamma^c)A_\alpha^a \\ + g^{\gamma\alpha}(\partial^\beta A_\gamma^c)A_\alpha^a A_\beta^b - g^{\gamma\alpha}A_\gamma^c(\partial^\beta A_\alpha^a)A_\beta^b \end{pmatrix}. \quad (11)$$

After that, in the momentum space each ∂ becomes the appropriate ik and then there is an overall factor of i , hence the 3-gluon vertex (10).

- The second term on the top line (5) is quartic in the gauge field, so it gives rise to the **four-gluon vertex**



$$= -ig^2 \begin{bmatrix} f^{abe} f^{cde} (g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}) \\ + f^{ace} f^{bde} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\gamma\beta}) \\ + f^{ade} f^{bce} (g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\delta\beta}) \end{bmatrix}. \quad (12)$$

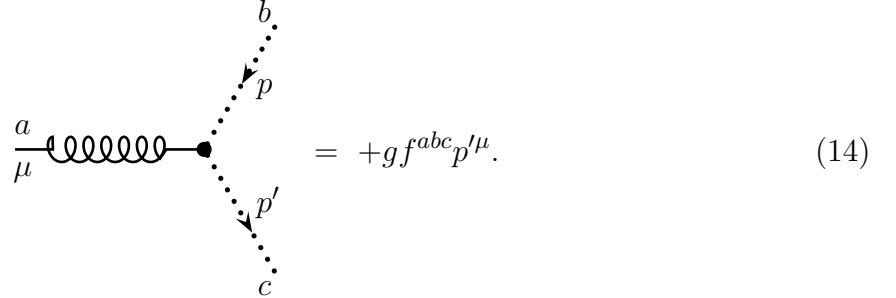
Again, the messy indexology of this vertex stems from totally symmetrizing the quartic

interaction terms WRT to the 4 gauge fields A_α^a , A_β^b , A_γ^c and A_δ^d :

$$\begin{aligned} \mathcal{L} \supset -\frac{g^2}{4}(f^{abe}A_\alpha^aA_\beta^b)^2 &= -\frac{g^2}{4}(f^{abe}A_\alpha^aA_\beta^b)g^{\alpha\gamma}g^{\beta\delta}(f^{cde}A_\gamma^cA_\delta^d) \\ &= -\frac{g^2}{24}A_\alpha^aA_\beta^bA_\gamma^cA_\delta^d \times \begin{bmatrix} f^{abe}f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace}f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade}f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{bmatrix} \end{aligned} \quad (13)$$

where $f^{abe}f^{cde}$ is antisymmetric WRT $a \leftrightarrow b$ or $c \leftrightarrow d$ and likewise for the other color-index structures, hence the 4-gluon vertex (12).

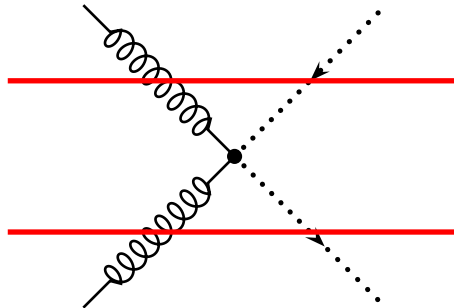
- The third term $-gf^{abc}(\partial^\mu\bar{c}^a)A_\mu^b c^c$ on the top line (5) gives rise to the **ghost-antighost-gluon vertex**



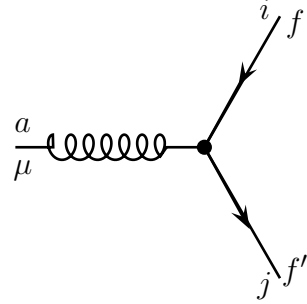
$$= +gf^{abc}p'^\mu. \quad (14)$$

Note that although the ghosts and antighosts look like massless scalar fields, the momentum structure of this vertex is different from a similar vertex for a physical colored scalar field we might add to the theory, or for that matter, from the scalar-photon vertex in scalar QED. For a physical scalar field, the single-gluon vertex involves the sum of two scalar lines' momenta $(p + p')^\mu$, but the ghost vertex (14) involves only the antighost's momentum p'^μ but not the ghost's momentum p^μ .

- Also, unlike a physical scalar, the ghost fields do not have the $A^\mu A_\mu \bar{c}c$ term in their Lagrangian (color indices suppresses), hence there is no seagull vertex for the ghosts



- Finally, the term on the second line (5) gives rise to the **quark-antiquark-gluon vertex** similar to the electron-positron-photon vertex in QED,

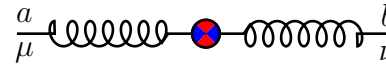


$$= -ig\gamma^\mu \times \delta_f^{f'} \times (t^a)^j_i. \quad (15)$$

The $\delta_f^{f'}$ factor for the quark flavors is similar to not mixing the electron, muon, and tau lepton species in QED, but the color matrix $(t^a)^j_i$ is new to QCD. I shall return to this matrix later in these notes.

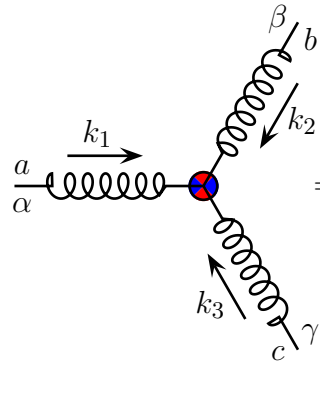
In addition, the renormalized theory has a whole bunch of the counterterm vertices:

- * The two-gluon counterterm vertex



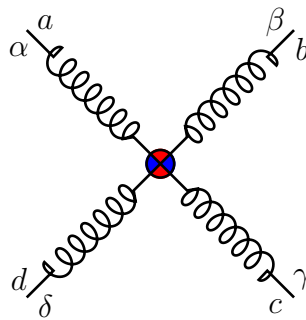
$$= -i\delta_3\delta^{ab} (k^2g^{\mu\nu} - k^\mu k^\nu). \quad (16)$$

- * The three-gluon counterterm vertex



$$= -g\delta_1^{(3g)} \times f^{abc} \left[g^{\alpha\beta}(k_1 - k_2)^\gamma + g^{\beta\gamma}(k_2 - k_3)^\alpha + g^{\gamma\alpha}(k_3 - k_1)^\beta \right]. \quad (17)$$

- * The four-gluon counterterm vertex



$$= -ig^2\delta_1^{(4g)} \times \left[\begin{array}{l} fabe fcd e (g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + face fbde (g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + fade fbce (g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{array} \right]. \quad (18)$$

* The two-quark counterterm vertex

$$\frac{f}{i} \leftarrow \text{---} \bullet \text{---} \leftarrow \frac{f'}{j} = \delta_f^{f'} \delta_i^j \times \left(i\delta_m^{(q_f)} - i\delta_2^{(q_f)} \times \not{p} \right). \quad (19)$$

* The quark-gluon counterterm vertex

$$\frac{a}{\mu} \text{---} \text{---} \bullet \begin{cases} \nearrow \frac{i}{f} \\ \searrow \frac{j}{f'} \end{cases} = -ig\delta_1^{(q_f)} \delta_f^{f'} \times \gamma^\mu \times (t^a)_i^j. \quad (20)$$

* The two-ghost counterterm vertex

$$\dots \leftarrow \dots \bullet \leftarrow \dots = \delta^{ab} \times i\delta_2^{(\text{gh})} \times k^2. \quad (21)$$

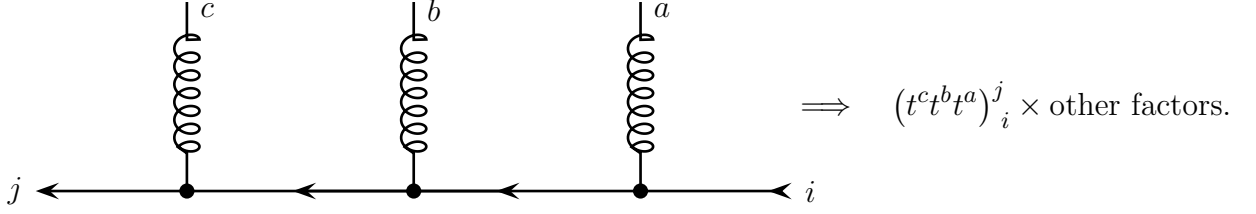
* The ghost-gluon counterterm vertex

$$\frac{a}{\mu} \text{---} \text{---} \bullet \begin{cases} \nearrow \dots \frac{b}{p} \\ \searrow \dots \frac{p'}{c} \end{cases} = +g\delta_1^{(\text{gh})} \times f^{abc} p'^\mu. \quad (22)$$

In addition to all these propagators and vertices, there are a few general rules:

- * Remember that the ghost fields are fermionic, so each closed loop of ghost propagators carries a minus sign.
- * The flavor f remains constant along any quark line, open or closed. For an open line, f matches both the incoming and the outgoing quarks (or antiquarks); for closed quark loops, we sum over all the flavors.

- ★ The color of a quark changes from propagator to propagator since the quark-quark-gluon vertices carry the $(t^a)^j_i$ factors. In matrix notations, the t^a generators should be multiplied right-to-left in the order of arrows on the quark line, for example

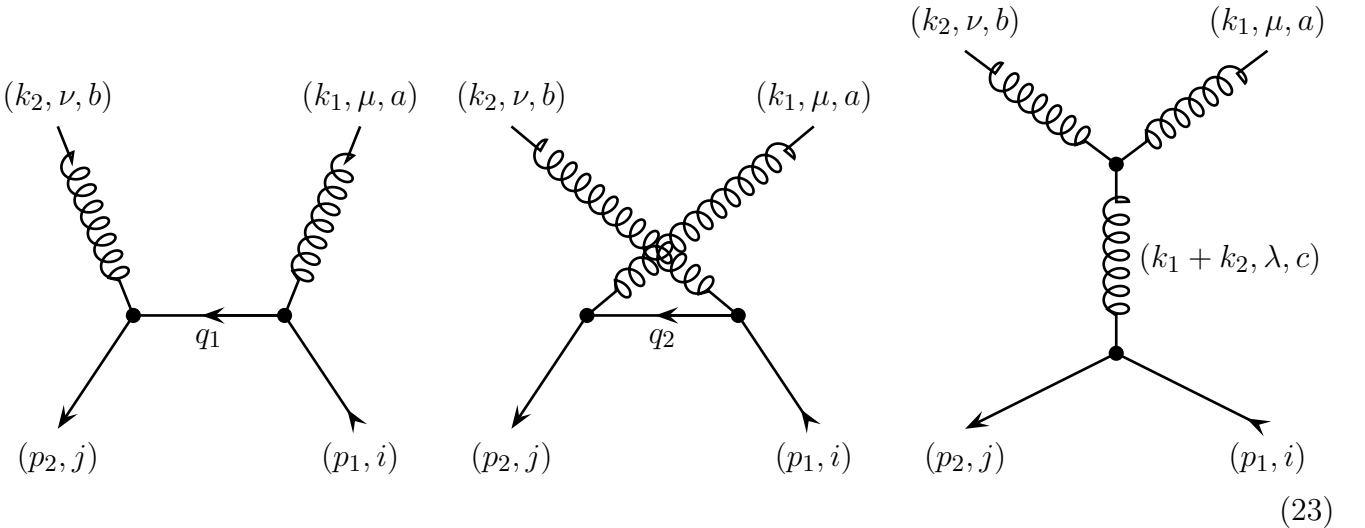


For the closed quark lines, one starts at an arbitrary vertex, multiplies all the generators right-to-left in the order of the arrows, then takes the trace over the color indices, $\text{tr}(\dots t^c t^b t^a)$.

Ward Identities

QCD has weaker Ward identities than QED. In particular, consider the on-shell scattering amplitudes involving the longitudinally polarized gluons. *When one gluon is longitudinal and all other gluons are transverse, the amplitude vanishes. But when two or more gluons are longitudinal, the amplitude does not vanish; instead, it is related to the amplitudes involving the external ghosts instead of the longitudinal gluons.*

As an example, consider the tree level annihilation of a quark and an antiquark into a pair of gluons, $q\bar{q} \rightarrow gg$. In QED there are two tree diagrams for the $e^-e^+ \rightarrow \gamma\gamma$ annihilation, but in QCD there are three diagrams:



According to the QCD Feynman rules, these diagrams evaluate to

$$i\mathcal{M}_1 = \bar{v}(p_2) (-ig\gamma^\nu e_{2\nu}^*) \frac{i}{\not{q}_1 - m} (-ig\gamma^\mu e_{1\mu}^*) u(p_1) \times (t^b t^a)^j_i, \quad (24)$$

$$i\mathcal{M}_2 = \bar{v}(p_2) (-ig\gamma^\mu e_{1\mu}^*) \frac{i}{\not{q}_2 - m} (-ig\gamma^\nu e_{2\nu}^*) u(p_1) \times (t^a t^b)^j_i, \quad (25)$$

$$\begin{aligned} i\mathcal{M}_3 &= \bar{v}(p_2) (-ig\gamma_\lambda) u(p_1) \times (t^c)^j_i \times \frac{-i}{(k_1 + k_2)^2} \times \\ &\quad \times (-g) f^{abc} [g^{\mu\nu} (-k_1 + k_2)^\lambda + g^{\nu\lambda} (-k_2 - (k_1 + k_2))^\mu + g^{\lambda\mu} ((k_1 + k_2) + k_1)^\nu] \\ &\quad \langle\langle \text{the 3 gluon vertex; the unusual signs are due to directions of momenta} \rangle\rangle \\ &\quad \langle\langle \text{the } k_1 \text{ and } k_2 \text{ are outgoing while the } k_3 = k_1 + k_2 \text{ is incoming} \rangle\rangle \\ &\quad \times e_{1\mu}^* e_{2\nu}^*, \end{aligned} \quad (26)$$

$$\mathcal{M}_{\text{tree}}^{\text{net}} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3. \quad (27)$$

Clearly, each term in the net amplitude is $O(g^2)$ and each term includes the polarization vectors for the two gluons, thus

$$\mathcal{M} = e_{1\mu}^* e_{2\nu}^* \times \mathcal{M}^{\mu\nu}. \quad (28)$$

So let us check the Ward identity $k_{1\mu} \times \mathcal{M}^{\mu\nu} \stackrel{??}{=} 0$.

For the first diagram's amplitude we have

$$k_{1\mu} \times \mathcal{M}_1^{\mu\nu} = -g^2 (t^b t^a)^j_i \times \bar{v} \gamma^\nu \frac{1}{\not{q}_1 - m} \not{k}_1 u. \quad (29)$$

In the second factor here, $q_1 = p_1 - k_1$, hence

$$\frac{1}{\not{q}_1 - m} \not{k}_1 = \frac{1}{\not{q}_1 - m} (\not{p}_1 - \not{q}_1) = -1 + \frac{1}{\not{q}_1 - m} (\not{p}_1 - m), \quad (30)$$

which for the on-shell quark gives

$$\frac{1}{\not{q}_1 - m} \not{k}_1 u(p_1) = -u(p_1) + \frac{1}{\not{q}_1 - m} (\not{p}_1 - m) u(p_1) = -u(p_1) + 0 \quad (31)$$

because $(\not{p}_1 - m)u(p_1) = 0$. Thus,

$$k_{1\mu} \times \mathcal{M}_1^{\mu\nu} = +g^2 (t^b t^a)^j_i \times \bar{v}(p_2) \gamma^\nu u(p_1). \quad (32)$$

Likewise, for the second diagram

$$k_{1\mu} \times \mathcal{M}_2^{\mu\nu} = -g^2 (t^b t^a)^j_i \times \bar{v} k_1 \frac{1}{\not{q}_2 - m} \gamma^\nu u. \quad (33)$$

where in the second factor

$$q_2 = k_1 - p_2 \implies k_1 \frac{1}{\not{q}_2 - m} = 1 - (\not{p}_2 + m) \frac{1}{\not{q}_2 - m} \implies \bar{v}(p_2) k_1 \frac{1}{\not{q}_2 - m} = +\bar{v}(p_2) - 0, \quad (34)$$

thus

$$k_{1\mu} \times \mathcal{M}_2^{\mu\nu} = -g^2 (t^a t^b)^j_i \times \bar{v}(p_2) \gamma^\nu u(p_1). \quad (35)$$

In QED, $k_{1\mu} \times \mathcal{M}_1^{\mu\nu}$ and $k_{1\mu} \times \mathcal{M}_2^{\mu\nu}$ would have canceled each other, but in QCD eqs. (32) and (35) carry different color-dependent factors. So instead of cancellation, we have

$$k_{1\mu} \times \mathcal{M}_{1+2}^{\mu\nu} = g^2 \bar{v} \gamma^\nu u \times (t^b t^a - t^a t^b)^j_i = g^2 \bar{v} \gamma^\nu u \times -i f^{abc} (t^c)^j_i. \quad (36)$$

But the net color-dependent factor is similar to the third amplitude, so there is a hope that the Ward identity might work when all three diagrams are put together.

For the third diagram we have

$$k_{1\mu} \times \mathcal{M}_3^{\mu\nu} = -i g^2 f^{abc} (t^c)^j_i \times \bar{v} \gamma^\lambda u \times \frac{1}{(k_1 + k_2)^2} \times \\ \times k_{1\mu} \times [g^{\mu\nu} (k_2 - k_1)^\lambda + g^{\nu\lambda} (-k_1 - 2k_2)^\mu + g^{\lambda\mu} (2k_1 + k_2)^\nu], \quad (37)$$

where on the second line

$$k_{1\mu} \times [\dots] = k_1^\nu (k_2 - k_1)^\lambda + g^{\nu\lambda} (-k_1^2 - 2k_1 k_2) + k_1^\lambda (2k_1 + k_2)^\nu \\ = g^{\lambda\nu} (-(k_1 + k_2)^2 + k_2^2) + [(2-1)k_1^\lambda k_1^\nu + k_1^\lambda k_2^\nu + k_2^\lambda k_1^\nu] \\ \langle\langle \text{on shell} \rangle\rangle \\ = -g^{\lambda\nu} (k_1 + k_2)^2 + (k_1 + k_2)^\lambda (k_1 + k_2)^\nu - k_2^\lambda k_2^\nu. \quad (38)$$

Plugging the three terms here back into eq. (37), we obtain

$$k_{1\mu} \times \mathcal{M}_3^{\mu\nu} = k_{1\mu} \times \mathcal{M}_{3,a}^{\mu\nu} + k_{1\mu} \times \mathcal{M}_{3,b}^{\mu\nu} + k_{1\mu} \times \mathcal{M}_{3,c}^{\mu\nu} \quad (39)$$

where

$$k_{1\mu} \times \mathcal{M}_{3,a}^{\mu\nu} = +i g^2 f^{abc} (t^c)^j_i \times \bar{v}(p_2) \gamma^\nu u(p_1), \quad (40)$$

$$k_{1\mu} \times \mathcal{M}_{3,b}^{\mu\nu} = -ig^2 f^{abc} (t^c)^j_i \times \bar{v}(p_2) \not{k}_1 + \not{k}_2 u(p_1) \times \frac{(k_1 + k_2)^\nu}{(k_1 + k_2)^2}, \quad (41)$$

$$k_{1\mu} \times \mathcal{M}_{3,c}^{\mu\nu} = +ig^2 f^{abc} (t^c)^j_i \times \bar{v}(p_2) \not{k}_2 u(p_1) \times \frac{k_2^\nu}{(k_1 + k_2)^2}. \quad (42)$$

By inspection of eqs. (40) and (36), the first term in eq. (39) precisely cancels the contributions of the first two diagrams,

$$k_{1\mu} \mathcal{M}_{1+2}^\mu + k_{1\mu} \times \mathcal{M}_{3,a}^{\mu\nu} = 0. \quad (43)$$

The second term's contribution (41) vanishes for the on-shell quarks. Indeed, by momentum conservation $k_1 + k_2 = p_1 + p_2$, hence

$$\bar{v}(p_2) \not{k}_1 + \not{k}_2 u(p_1) = \bar{v}(p_2) \not{p}_1 + \not{p}_2 u(p_2) = \bar{v}(p_2) (\not{p}_2 + m) u(p_1) + \bar{v}(p_2) (\not{p}_1 - m) u(p_1) = 0 + 0 \quad (44)$$

and therefore $k_{1\mu} \times \mathcal{M}_{3,b}^{\mu\nu} = 0$.

But the third term's contribution (41) does not vanish, and this breaks the Ward identity for the net QCD amplitude:

$$k_{1\mu} \times \mathcal{M}_{\text{net}}^{\mu\nu} = k_{1\mu} \times \mathcal{M}_{3,c}^{\mu\nu} = +ig^2 f^{abc} (t^c)^j_i \times \bar{v} \not{k}_2 u \times \frac{1}{(k_1 + k_2)^2} \times k_2^\nu \neq 0. \quad (45)$$

However, the net violation of the Ward identity is proportional to the k_2^ν factor. Therefore, when we contract the amplitude $\mathcal{M}_{\text{net}}^{\mu\nu}$ with the polarization vector of the second gluon, we obtain

$$k_{1\mu} \times \mathcal{M}_{\text{net}}^{\mu\nu} e_{2\nu}^* = [\dots] \times (k_2 e_2^*), \quad (46)$$

which vanishes when the second gluon is transversely polarized! This agrees with the weakened Ward identity of QCD: *Amplitudes involving one longitudinal gluon vanish if all the other gluons are transverse, but if two (or more) gluons are longitudinal, the amplitude does not have to vanish. Instead, such amplitudes are related to the amplitudes involving ghosts and antighosts.*

Indeed, consider the annihilation amplitude of two quarks into two longitudinal gluons, $\mathcal{M}(q\bar{q} \rightarrow g_L g_L)$. In Minkowski space, there are two distinct longitudinal polarizations for a

gluon moving in the direction \mathbf{n} , namely $e_{\pm}^{\mu} = (1, \pm \mathbf{n})/\sqrt{2}$. In light of eq. (46), the amplitude for producing two gluons with polarizations $L+$ (*i.e.*, $e^{\mu} \propto k^{\mu}$ for each gluon) vanishes, $\mathcal{M}(q\bar{q} \rightarrow g_{L+}g_{L+}) = 0$. The amplitude for producing two gluons with longitudinal polarizations $L-$ also vanishes, $\mathcal{M}(q\bar{q} \rightarrow g_{L-}g_{L-}) = 0$, although I am not going to prove it in these notes. Instead, let me focus on the non-zero amplitude for producing one gluon with the longitudinal $L+$ polarization and the other gluons with the longitudinal $L-$ polarization.

In light of eq. (46), we get

$$\mathcal{M}(q\bar{q} \rightarrow g_{L+}g_{L-}) = \left[e_{1\mu}^*(L+) = \frac{k_{1\mu}}{\omega_1\sqrt{2}} \right] \times \mathcal{M}_{\text{net}}^{\mu\nu} \times e_{2\nu}^*(L-) = \frac{1}{\omega_1\sqrt{2}} \times [\dots] \times (e_{2\nu}^*(L-)k_2^{\nu}), \quad (47)$$

where $[\dots]$ stands for the factors from eq. (45) which I did not write down explicitly in eq. (46), namely

$$[\dots] = +ig^2 f^{abc} (t^c)^j_i \times \bar{v}k_2u \times \frac{1}{(k_1 + k_2)^2}, \quad (48)$$

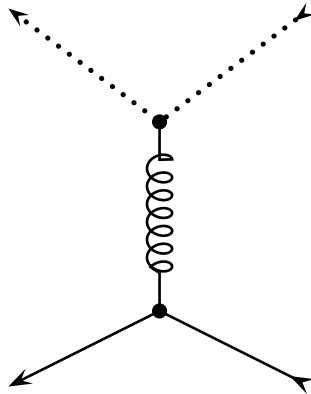
while

$$e_2^{*\mu}(L-)g_{\mu\nu}k_2^{\nu} = \frac{(1, -\mathbf{n})^{\mu}}{\sqrt{2}} \times g_{\mu\nu} \times \omega_2(1, \mathbf{n})^{\nu} = \frac{\omega_2}{\sqrt{2}} \times 2. \quad (49)$$

Thus, in the center of mass frame where $\omega_1 = \omega_2 = \frac{1}{2}E_{\text{cm}}$ while $(k_1 + k_2)^2 = s = E_{\text{cm}}^2$, we have

$$\mathcal{M}(q + \bar{q} \rightarrow g_{L+} + g_{L-}) = \frac{ig^2}{s} f^{abc} (t^c)^j_i \times \bar{v}(p_2)k_2u(p_1). \quad (50)$$

Let's compare this amplitude to the annihilation of the same quark and the same antiquark into a ghost and antighost. At the tree level, there is only one diagram for the later process,



which yields the amplitude

$$\begin{aligned}
i\mathcal{M}_{\text{tree}}(q + \bar{q} \rightarrow \text{gh} + \overline{\text{gh}}) &= \bar{v}(p_2)(-ig\gamma^\lambda)u(p_1) \times (t^c)^j_i \times \frac{-ig_{\lambda\nu}}{s} \times gf^{abc}k_2^\nu \\
&= -\frac{g^2}{s} f^{abc} (t^c)^j_i \times \bar{v}(p_2)k_2 u(p_1).
\end{aligned} \tag{51}$$

By inspection, this amplitude is equal to the amplitude (50) for $q + \bar{q}$ annihilating into two longitudinal gluons instead of a ghost and an antighost,

$$\mathcal{M}(q + \bar{q} \rightarrow \text{gh} + \overline{\text{gh}}) = \mathcal{M}(q + \bar{q} \rightarrow g_{L+} + g_{L-}). \tag{52}$$

In the next set of notes we shall learn that such relations stem from the BRST symmetry, but right now we may use eq. (52) to understand how the physical cross-sections work in QCD.

The ghosts violate the spin-statistics theorem, so we must give up one of its assumptions: relativity, positive particle energies, or the positive norm in the Hilbert space. The correct choice is to give up on the norm positivity in the extended Hilbert space including both the physical and the unphysical quanta: While the physical (anti)quarks and transverse gluons must have positive norm, the norm for the unphysical longitudinal gluons is ghost has mixed signature — positive for the longitudinal gluons but negative for the ghosts and antighosts. And because of the negative norm for the (anti)ghosts states, the cross-section for the annihilation-into-ghosts process comes out negative,

$$\frac{d\sigma}{d\Omega} = -\frac{|\mathcal{M}|^2}{64\pi^2 s}. \tag{53}$$

By themselves, the negative cross-sections are impossible, but they make sense in the context of net unpolarized cross-section where the final states could be either gluons or ghosts,

$$\frac{d\sigma(q + \bar{q} \rightarrow \dots)}{d\Omega} = \frac{d\sigma(q + \bar{q} \rightarrow g_T + g_T)}{d\Omega} + \frac{d\sigma(q + \bar{q} \rightarrow g_L + g_L)}{d\Omega} + \frac{d\sigma(q + \bar{q} \rightarrow \text{gh} + \overline{\text{gh}})}{d\Omega}. \tag{54}$$

Thanks to eq. (52), the negative cross-section for the annihilation into ghosts precisely cancels the positive cross-section for the annihilation into longitudinal gluons,

$$\frac{d\sigma(q + \bar{q} \rightarrow g_L + g_L)}{d\Omega} + \frac{d\sigma(q + \bar{q} \rightarrow \text{gh} + \overline{\text{gh}})}{d\Omega} = 0. \tag{55}$$

Thus, the un-physical processes cancel each other, and the net annihilation cross-section is just

the cross-section for producing the physical states only. At the $O(g^4)$ level, this means annihilation into a pair of transverse gluons only,

$$\frac{d\sigma(q + \bar{q} \rightarrow g + g \text{ or } gh + \overline{gh})}{d\Omega} = \frac{d\sigma(q + \bar{q} \rightarrow g_T + g_T \text{ only})}{d\Omega}. \quad (56)$$

Note: this relation is important for the unitarity of QCD.