QED: Feynman Rules, Divergences, and Renormalizability

QED Feynman Rules in the Counterterm Perturbation Theory

The simplest version of Quantum Electrodynamics (QED) has only 2 field types — the electromagnetic field A^μ and the electron field Ψ — and its physical Lagrangian is

$$\mathcal{L}_{\text{phys}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\gamma^\mu D_\mu - m_e)\Psi = -\tfrac{1}{4}F_{\mu\nu}^2 + \overline{\Psi}(i\partial\!\!\!/ - m)\Psi + eA_\mu\overline{\Psi}\gamma^\mu\Psi. \quad (1)$$

The bare Lagrangian of the perturbation theory has a similar form, except for the bare coupling e_{bare} instead of the physical coupling e, the bare electron mass m_{bare} instead of the physical mass m, and the bare fields $A^\mu_{\text{bare}}(x)$ and $\Psi_{\text{bare}}(x)$ instead of the renormalized fields $A^\mu(x)$ and $\Psi(x)$. By convention, the fields strength^2 factors Z for the EM and the electron fields are called respectively the Z_3 and the Z_2, while the Z_1 is the electric charge renormalization factor. Thus,

$$A^\mu_{\text{bare}}(x) = \sqrt{Z_3} \times A^\mu(x), \qquad \Psi_{\text{bare}}(x) = \sqrt{Z_2} \times \Psi(x), \quad (2)$$

and plugging these bare fields into the bare Lagrangian we obtain

$$\mathcal{L}_{\text{bare}} = -\frac{Z_3}{4} F_{\mu\nu}F^{\mu\nu} + Z_2\overline{\Psi}(i\partial\!\!\!/ - m_{\text{bare}})\Psi + Z_1 e \times A_\mu\overline{\Psi}\gamma^\mu\Psi \quad (3)$$

where

$$Z_1 \times e = Z_2\sqrt{Z_3} \times e_{\text{bare}} \quad (4)$$

by definition of the Z_1.

As usual in the counterterm perturbation theory, we split

$$\mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{phys}} + \mathcal{L}^{\text{counter}}_{\text{terms}} \quad (5)$$

where the physical Lagrangian $\mathcal{L}_{\text{phys}}$ is exactly as in eq. (1) while the counterterms comprise the difference. Specifically,

$$\mathcal{L}^{\rm counter}_{\rm terms} \;=\; -\frac{\delta_3}{4}\times F_{\mu\nu}F^{\mu\nu} \;+\; \delta_2\times\overline{\Psi}i\partial\!\!\!/\Psi \;-\; \delta_m\times\overline{\Psi}\Psi \;+\; e\delta_1\times A_\mu\overline{\Psi}\gamma^\mu\Psi \tag{6}$$

for

$$\delta_3 \;=\; Z_3-1,\quad \delta_2 \;=\; Z_2-1,\quad \delta_1 \;=\; Z_1-1,\quad \delta_m \;=\; Z_2 m_{\rm bare}-m_{\rm phys}\,. \tag{7}$$

Actually, the bare Lagrangian (5) is not the whole story, since in the quantum theory the EM field $A^\mu(x)$ needs to be gauge-fixed. The most commonly used Feynman gauge (or similar Lorentz-invariant gauges) stem from the gauge-averaging of the functional integral. We shall learn how this works in late March, although the impatient students are welcome to read my notes on the subject. For the moment, all we need to know is that the net effect of the gauge-averaging procedure amounting to adding an extra gauge-symmetry breaking term to the Lagrangian:

$$\mathcal{L}_{\rm bare} \;=\; \mathcal{L}_{\rm phys} \;+\; \mathcal{L}^{\rm gauge}_{\rm fixing} \;+\; \mathcal{L}^{\rm counter}_{\rm terms} \tag{8}$$

for

$$\mathcal{L}^{\rm gauge}_{\rm fixing} \;=\; -\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 \tag{9}$$

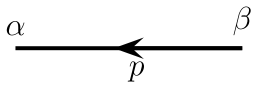
where ξ is a constant parametrizing a specific gauge. In the Feynman gauge $\xi=1$.

In the counterterm perturbation theory, we take the free Lagrangian to be the quadratic part of the physical Lagrangian plus the gauge fixing term, thus

$$\mathcal{L}_{\rm free} \;=\; \overline{\Psi}(i\partial\!\!\!/-m)\Psi \;-\; \tfrac14 F^2_{\mu\nu} \;-\; \frac{1}{2\xi}\,(\partial_\mu A^\mu)^2 \tag{10}$$

(where m is the physical mass of the electron), while all the other terms in the bare Lagrangian — the physical coupling $eA_\mu\overline{\Psi}\gamma^\mu\Psi$ and all the counterterms (6) — are treated as perturbations. Consequently, the QED Feynman rules have the following propagators and vertices:

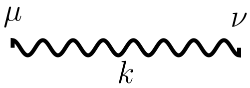
- The electron propagator



$$= \left[\frac{i}{\not p - m + i0}\right]_{\alpha\beta} = \frac{i(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i0} \qquad (11)$$

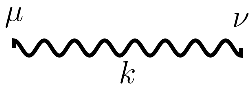
where α and β are the Dirac indices, usually not written down.

- The photon propagator



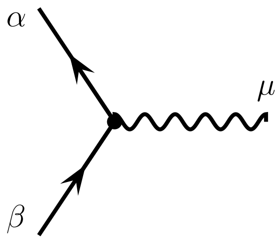
$$= \frac{-i}{k^2 + i0} \times \left(g^{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2 + i0}\right). \qquad (12)$$

In the Feynman gauge $\xi = 1$ this propagator simplifies to



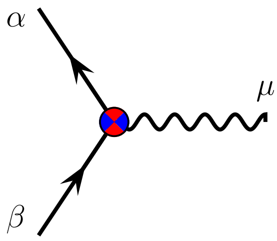
$$= \frac{-ig^{\mu\nu}}{k^2 + i0}. \qquad (13)$$

- The physical vertex

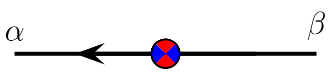


$$= \left(+ie\gamma^\mu\right)_{\alpha\beta}. \qquad (14)$$

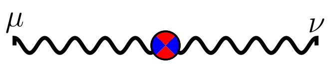
⋆ And then there are three kinds of counterterm vertices:



$$= +ie\delta_1 \times (\gamma^\mu)_{\alpha\beta}, \qquad (15)$$



$$= +i\big(\delta_2 \times \not p - \delta_m\big)_{\alpha\beta}, \qquad (16)$$



$$= -i\delta_3 \times \big(g^{\mu\nu}k^2 - k^\mu k^\nu\big). \qquad (17)$$

Renormalizability (I):
Power Counting, Divergences, and Counterterms

The only physical coupling of QED is e, which is dimensionless in 4D; its numeric value is $e \approx 0.302\,822\,120\,872$. Consequently, by dimensional analysis — also called the *power-counting analysis* — QED should be a renormalizable theory. That is, it has infinite number of UV-divergent Feynman diagrams but only a finite number of UV-divergent *amplitudes*, and all such divergences can be canceled *in situ* by a finite set of counterterms. (Although the counterterm couplings δ_{whatever} should be adjusted order-by-order at all orders of the perturbation theory.)

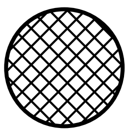
However, by the power-counting analysis, canceling all the divergences seems to require a few more counterterms than QED actually has. Such missing counterterms raise a question of whether QED — with the Feynman rules exactly as in the previous section — is actually a renormalizable theory. And in the next section we shall learn that QED is indeed renormalizable thanks to the Ward–Takahashi identities which make the missing counterterms unneeded. But before we go there, we need to work out the power-counting analysis of QED divergences, and that's what this section is about.

In QED, like in any 4D theory with dimensionless coupling(s), the superficial degree of divergence of any 1PI Feynman graph follows from the numbers of its external legs as

$$\mathcal{D} \;=\; 4 \;-\; \tfrac{3}{2}E_f \;-\; E_b \;\xrightarrow[\text{QED}]{}\; 4 \;-\; \tfrac{3}{2}E_e \;-\; E_\gamma\,. \tag{18}$$

(See my notes on dimensional analysis for explanation.) Consequently, all the *superficially divergent* Feynman graphs — that is, the graphs suffering from the overall UV divergence when *all* $q_j^\mu \to \infty$ rather than from a subgraph divergence — must have $\frac{3}{2}E_e + E_\gamma \le 4$, which means either ($E_e = 0$ and $E_\gamma \le 4$) or ($E_e = 2$ and $E_\gamma \le 1$). So let's take a closer look at such superficially divergent graphs and at the amplitudes to which they contribute:

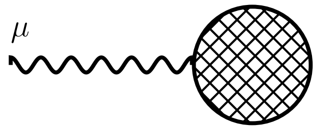
- $E_e = 0$ and $E_\gamma = 0$,



is a vacuum bubble. (19)

  Since such bubbles affect nothing but the vacuum energy of the theory, we shall ignore them from now on.

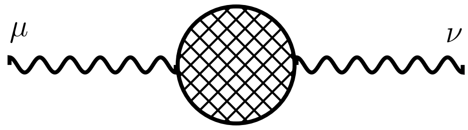
- $E_e = 0$ and $E_\gamma = 1$,



is a photon tadpole. (20)

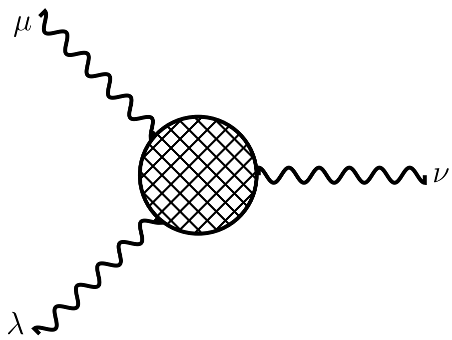
All such tadpole diagrams vanish by the Lorentz symmetry acting on the μ index of a zero-momentum photon.

- $E_e = 0$ and $E_\gamma = 2$,



contributes to $i\Sigma^{\gamma}_{\mu\nu}(k)$. (21)

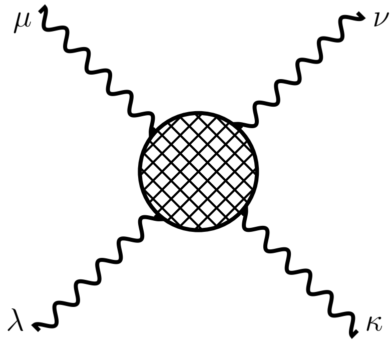
- $E_e = 0$ and $E_\gamma = 3$,



(22)

While individual 3–photon diagrams like this do not vanish, they form pairs which precisely cancel each other, so the net 3-photon amplitude vanishes. This is the *Furry's theorem*, which stems from the charge conjugation symmetry **C** of QED. Indeed, **C** acts on the EM field $A^\mu(x)$ by flipping its sign, $A^\mu(x) \to -A^\mu(x)$, hence any amplitude involving an odd number of photons — and no electrons — must vanish.

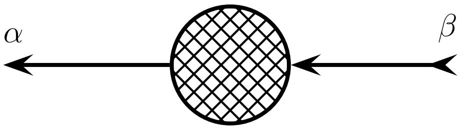
- $E_e = 0$ and $E_\gamma = 4$,



(23)

contributes to the 4-photon amplitude $i\mathcal{V}_{\kappa\lambda\mu\nu}(k_1, k_2, k_3, k_4)$.

- $E_e = 2$ and $E_\gamma = 0$,



contributes to $-i\Sigma^e_{\alpha\beta}(p)$ (24)

where α and β are the Dirac indices, usually not written down. By Lorentz+parity symmetry of this amplitude,

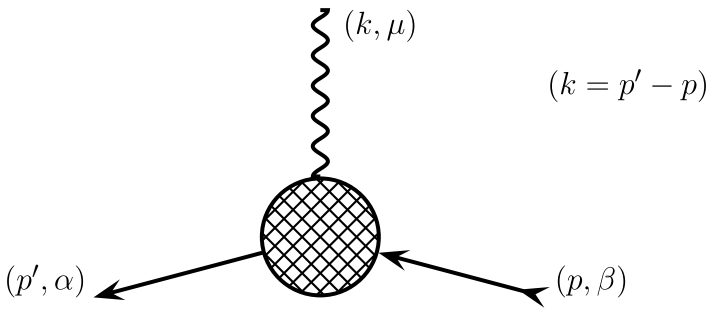
$$\Sigma^e_{\alpha\beta}(p^\mu) \;=\; A(p^2)\times\delta_{\alpha\beta} \;+\; B(p^2)\times p_\mu\gamma^\mu_{\alpha\beta} \tag{25}$$

for some scalar functions A and B of p^2, or suppressing the Dirac indices

$$\Sigma^e(p^\mu) \;=\; A(p^2) \;+\; B(p^2)\times\not{p}. \tag{26}$$

Moreover, expanding $A(p^2)$ and $B(p^2)$ into power series in p^2 and making use of $p^2 = \not{p}^2$, we can turn the RHS of this formula into a power series in $\not{p}$, where the even powers of $\not{p}$ come from $A(p^2)$ and the odd powers from $B(p^2)\times\not{p}$. Consequently, the 1PI 2-point function for the electron field is usually written down as $\Sigma^e(\not{p})$.

- $E_e = 2$ and $E_\gamma = 1$,



(27)

contributes to the "dressed" electron-photon vertex $ie\Gamma^\mu_{\alpha\beta}(p',p)$. We shall address the Lorentz symmetry properties of this vertex later in class.

⋆ And this is it! QED has no other kinds of superficially-divergent amplitudes.

Now let's take a closer look at the non-trivial superficially-divergent amplitudes $\Gamma^\mu(p',p)$, $\Sigma^e(\not{p})$, $\mathcal{V}_{\kappa\lambda\mu\nu}(k_1,\ldots,k_4)$, and $\Sigma^\gamma_{\mu\nu}(k)$ and focus on their infinite parts.

1. Let's start with the dressed electron-photon vertex $ie\Gamma^\mu_{\alpha\beta}(p',p)$. Loop diagrams contributing to this dressed vertex have superficial degree of divergence

$$\mathcal{D}[\Gamma^\mu] \;=\; 4 \;-\; \tfrac{3}{2}(E_e=2) \;-\; (E_\gamma=1) \;=\; 0, \tag{28}$$

so they suffer from a logarithmic UV divergence. The derivatives of the dressed vertex with respect to the electron's or photon's momenta

$$\frac{\partial\Gamma^\mu}{\partial p_\nu},\; \frac{\partial\Gamma^\mu}{\partial p'_\nu},\; \frac{\partial\Gamma^\mu}{\partial k_\nu},\quad \text{have }\; \mathcal{D} \;=\; \mathcal{D}[\Gamma^\mu] \;-\; 1 \;=\; -1, \tag{29}$$

which makes them UV-finite. Consequently, the UV-infinite part of the dressed vertex must be a momentum-independent constant, or rather a constant array indexed by μ and Dirac indices α,β. By the Lorentz+parity symmetry, such constant array must be proportional to the Dirac's $\gamma^\mu_{\alpha\beta}$, hence

$$\Gamma^\mu_{\alpha\beta}(p',p) \;=\; [O(\log\Lambda)\ \text{constant}] \times \gamma^\mu_{\alpha\beta} \;+\; \textit{finite}^\mu_{\alpha\beta}(p',p). \tag{30}$$

Moreover, in the counterterm perturbation theory

$$\Gamma^\mu_{\text{net}}(p',p) \;=\; \left(\Gamma^\mu_{\text{tree}} \;=\; \gamma^\mu\right) \;+\; \Gamma^\mu_{\text{loops}}(p',p) \;+\; \delta_1\times\gamma^\mu, \tag{31}$$

so the UV divergences (30) of the diagrams contributing to the dressed vertex can be canceled by suitable value of the counterterm coupling δ_1.

2. Next, consider the electron's 2-point 1PI amplitude $-i\Sigma^e(\not{p})$. Loop diagrams contributing to this amplitude have superficial degree of divergence

$$\mathcal{D}(\Sigma^e) \;=\; 4 \;-\; \tfrac{3}{2}(E_e=2) \;-\; (E_\gamma=0) \;=\; +1, \tag{32}$$

so they diverge as $O(\Lambda)$. But taking the derivatives of Σ^e with respect to the electron's momentum $\not{p}$ reduces the superficial degree of divergence,

$$\mathcal{D}[d\Sigma^2/d\not{p}] \;=\; \mathcal{D}[\Sigma^e] \;-\; 1 \;=\; 0, \quad \mathcal{D}[d^2\Sigma^2/(d\not{p})^2] \;=\; \mathcal{D}[\Sigma^e] \;-\; 2 \;=\; -1, \;\ldots \tag{33}$$

so the first derivative diverges as $O(\log\Lambda)$ while the second and higher derivatives are UV-finite. Consequently, the infinite part of $\Sigma^e(\not{p})$ must be a linear function of $\not{p}$, thus

$$\Sigma^e \;=\; [O(\Lambda)\text{ constant}] \;+\; [O(\log\Lambda)\text{ constant}]\times\not{p} \;+\; \mathit{finite}(\not{p}). \tag{34}$$

Moreover, in the counterterm perturbation theory

$$\Sigma^e_{\text{net}}(\not{p}) \;=\; \Sigma^e_{\text{loops}}(\not{p}) \;+\; \delta_m \;-\; \delta_2\times\not{p}, \tag{35}$$

so the UV divergences (34) of the diagrams contributing to the electron's 2-point 1PI amplitude can be canceled by suitable values of the counterterm couplings δ_m and δ_2.

3. Now we turn from the good news to the bad news. Consider the 4-photon 1PI amplitude $\mathcal{V}_{\kappa\lambda\mu\nu}(k_1,k_2,k_3,k_4)$. The superficial degree of divergence of diagrams contributing to this amplitude is

$$\mathcal{D}[\mathcal{V}] \;=\; 4 \;-\; \tfrac{3}{2}(E_e=0) \;-\; (E_\gamma=4) \;=\; 0, \tag{36}$$

while derivatives WRT the photon's momenta have

$$\mathcal{D}[\partial\mathcal{V}/\partial k_i^\mu] \;=\; \mathcal{D}[\mathcal{V}] \;-\; 1 \;=\; -1. \tag{37}$$

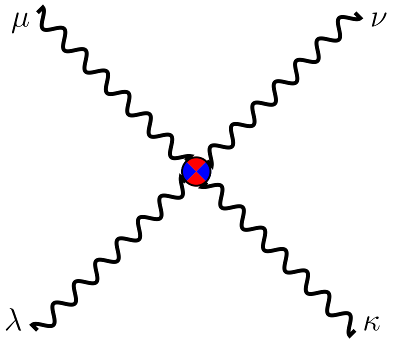
Hence, the diagrams themselves diverge as $O(\log\Lambda)$ but their derivatives are finite, so the divergence of $\mathcal{V}$ is a momentum-independent constant. Or rather, it's a constant tensor with 4 Lorentz indices $(\kappa.\lambda,\mu,\nu)$. By the Lorenz symmetry — and by the Bose symmetry of the 4 photons — this k-independent tensor must be proportional to

$$g_{\kappa\lambda}g_{\mu\nu} \;+\; g_{\kappa\mu}g_{\lambda\nu} \;+\; g_{\kappa\nu}g_{\lambda\mu}\,, \tag{38}$$

thus

$$\begin{aligned}\mathcal{V}_{\kappa\lambda\mu\nu}(k_1,k_2,k_3,k_4) \;&=\; [O(\log\Lambda)\text{ constant}]\times\big(g_{\kappa\lambda}g_{\mu\nu}+g_{\kappa\mu}g_{\lambda\nu}+g_{\kappa\nu}g_{\lambda\mu}\big)\\ &\qquad+\; \mathit{finite}_{\kappa\lambda\mu\nu}(k_1,\ldots,k_4).\end{aligned} \tag{39}$$

Naively, in the counterterm perturbation theory this divergence should be canceled by the counterterm vertex



$$= i\delta_{4\gamma} \times \bigl(g_{\kappa\lambda}g_{\mu\nu} + g_{\kappa\mu}g_{\lambda\nu} + g_{\kappa\nu}g_{\lambda\mu}\bigr) \qquad (40)$$

stemming from the counterterm

$$\mathcal{L}^{\text{counter}}_{\text{terms}} \supset \frac{\delta_{4\gamma}}{8} \times \bigl(A^\mu A_\mu\bigr)^2. \qquad (41)$$

However, QED's bare Lagrangian does not have this counterterm! Worse, such a counterterm would break the gauge symmetry of QED, so we cannot possibly add it to the Lagrangian without destroying the basic structure of the abelian gauge theory.

4. Finally, more bad news from the photon's 2-point 1PI amplitude $\Sigma^\gamma_{\mu\nu}(k)$. The diagrams contributing to this amplitude have superficial degree of divergence

$$\mathcal{D}[\Sigma^\gamma_{\mu\nu}] = 4 - \tfrac{3}{2}(E_e = 0) - (E_\gamma = 2) = +2, \qquad (42)$$

so it's derivatives WRT the photon's momentum k^λ have

$$\begin{aligned} \mathcal{D}[\partial\Sigma^\gamma_{\mu\nu}/\partial k^\lambda] &= \mathcal{D}[\Sigma^\gamma_{\mu\nu}] - 1 = +1, \\ \mathcal{D}[\partial^2\Sigma^\gamma_{\mu\nu}/\partial k^\kappa\,\partial k^\lambda] &= \mathcal{D}[\Sigma^\gamma_{\mu\nu}] - 2 = 0, \\ \mathcal{D}[\partial^3\Sigma^\gamma_{\mu\nu}/\partial k^\kappa\,\partial k^\lambda\,\partial k^\rho] &= \mathcal{D}[\Sigma^\gamma_{\mu\nu}] - 3 = -1, \end{aligned} \qquad (43)$$

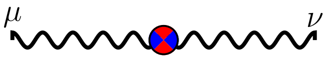
etc., *etc.* Thus, the amplitude $\Sigma^\gamma_{\mu\nu}(k)$ diverges as $O(\Lambda^2)$, it's first derivative as $O(\Lambda)$, the second derivative as $O(\log\Lambda)$, but the third and the higher derivatives are finite. Consequently, the divergent part of the $\Sigma^\gamma_{\mu\nu}(k)$ must be a quadratic polynomial of the photon's momentum k. Furthermore, by Lorentz symmetry such quadratic polynomial must gave form

$$A \times g_{\mu\nu} \; + \; B \times k^2 \times g_{\mu\nu} \; + \; C \times k_\mu k_\nu \tag{44}$$

for some $O(\Lambda^2)$ constant A and $O(\log \Lambda)$ constants B and C, thus

$$\Sigma^\gamma_{\mu\nu}(k) \;=\; A \times g_{\mu\nu} \; + \; B \times k^2 \times g_{\mu\nu} \; + \; C \times k_\mu k_\nu \; + \; \mathit{finite}_{\mu\nu}(k). \tag{45}$$

Naively, in the counterterm perturbation theory these divergences should be canceled by the counterterm vertex



$$= -i\delta_{3A} \times k^2 g_{\mu\nu} \; + \; i\delta_{3B} \times k_\mu k_\nu \; - \; i\delta^\gamma_m \times g_{\mu\nu} \tag{46}$$

stemming from the counterterms

$$\mathcal{L}^{\text{counter}}_{\text{terms}} \supset = \; -\frac{\delta_{3A}}{2} \times (\partial_\mu A_\nu)(\partial^\mu A^\nu) \; + \; \frac{\delta_{3B}}{2} \times (\partial_\nu A^\nu)^2 \; + \; \frac{\delta^\gamma_m}{2} \times A_\nu A^\nu \tag{47}$$

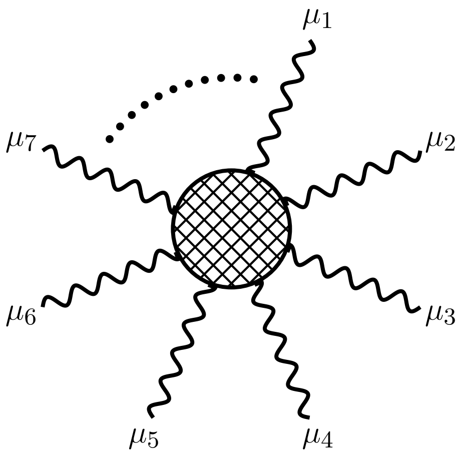
with suitably adjusted couplings δ_{3A}, δ_{3B}, and δ^γ_m. However, QED does not have the photon's mass[2] counterterm $\delta^\gamma_m A_\mu A^\mu$, while the photon's kinetic counterterms come only in a fixed combination $-\frac{1}{4}\delta_3 \times F_{\mu\nu}F^{\mu\nu}$ which corresponds to $\delta_{3A} = \delta_{3B} = \delta_3$. Thus, QED has only one two-photon counterterm instead of 3 counterterms needed to cancels all the apparent divergences (45) of the $\Sigma^\gamma_{\mu\nu}(k)$ amplitude. Worse, the missing counterterms are not gauge invariant, so we cannot add them to the bare QED Lagrangian without breaking the gauge symmetry of the theory.

The bottom line is: QED has a finite number of divergent amplitudes which can be canceled by a finite number of counterterms with order-by-order-adjusted coefficients. However, the gauge symmetry of QED forbids some of these counterterms, which questions QED's renormalizability.

Renormalizability (II):
Missing Counterterms and Ward–Takahashi Identities

In the previous section we saw that QED lacks some of the counterterms it seems to need for the UV divergence cancellation, namely the photon mass[2] counterterm δ_m^γ, the 4-photon counterterm $\delta_{4\gamma}$ and the *separate* photon kinetic counterterms δ_{3A} and δ_{3B}. Nevertheless, QED is a renormalizable theory because the divergences the missing counterterms are supposed to cancel actually cancel themselves. That is, the UV divergences of the loop diagrams contributing to the 2-photon and 4-photon amplitudes cancel each other order-by-order in perturbation theory!

The keys to this cancellation are the Ward–Takahashi identities, which we shall study in painful detail later in class. (For the impatient, here are my notes on the subject.) There is a whole series of Ward–Takahashi identities, but for the moment we need only the identities for the purely photonic amplitudes



$$= \; i\mathcal{M}^{\mu_1,\mu_2,\ldots,\mu_n}(k_1,\ldots,k_n). \qquad (48)$$

The n external photonic lines here could be truly external corresponding to the incoming or outgoing photos, or they could be propagators in a bigger Feynman graph. We allow for both possibilities, so the photon momenta $k_1,\ldots,k_n$ can be on-shell or off-shell. Also, the amplitude $\mathcal{M}^{\cdots}(k_1,\ldots,k_n)$ does not include the polarization vectors for the truly external photons or propagator factors for the photonic lines connected to other parts of a bigger graph. However, inside the cross-hatched disk we should total up all the diagrams with appropriate external legs up to some order in the QED coupling e. In particular, we should sum over all the non-trivial permutations of the n external photons. Remember: only the complete amplitudes should obey the Ward–Takahashi identities, but the individual Feynman diagrams generally don't.

Anyhow, the WT identity for an n-photon amplitude (48) says that when we contract the Lorentz index μ_i of any photon with the momentum vector $(k_i)_{\mu_i}$ of the same photon, we must get zero,

$$\forall i=1,\ldots,n: \quad (k_i)_{\mu_i}\times\mathcal{M}^{\mu_1,\mu_2,\ldots,\mu_n}(k_1,\ldots,k_n) \;=\; 0. \tag{49}$$

Now let's apply this identity to the 4-photon amplitude

$$\mathcal{V}_{\kappa\lambda\mu\nu}(k_1,k_2,k_3,k_4) \;=\; C\times\big(g_{\kappa\lambda}g_{\mu\nu}+g_{\kappa\mu}g_{\lambda\nu}+g_{\kappa\nu}g_{\lambda\mu}\big) \;+\; \mathit{finite}_{\kappa\lambda\mu\nu}(k_1,\ldots,k_4) \tag{39}$$

where C is a UV-divergent $O(\log\Lambda)$ constant. By the Ward–Takahashi identity $k_1^\kappa\times V_{\kappa\lambda\mu\nu}$ must vanish, thus

$$\begin{aligned} 0 \;&=\; k_1^\kappa\times\mathcal{V}_{\kappa\lambda\mu\nu}(k_1,k_2,k_3,k_4)\\ &=\; C\times\big((k_1)_\lambda g_{\mu\nu}\;+\;(k_1)_\mu g_{\lambda\nu}\;+\;(k_1)_\nu g_{\lambda\mu}\big)\;+\;k_1^\kappa\times\mathit{finite}_{\kappa\lambda\mu\nu}(k_1,\ldots,k_4). \end{aligned} \tag{50}$$

Moreover, since the finite term here cannot possibly cancel the UV-divergent term, the divergent term must vanish all by itself, thus $C=0$. In other words, the Ward–Takahashi identity for the 4-photon amplitude does not allow for its ultraviolet divergence. While individual Feynman diagrams for the 4-photon amplitude may suffer from a logarithmic UV divergence, once we total up all such diagrams at any given loop order, the net UV divergence must cancel out! And that's why QED does not need the 4-photon counterterm $\delta_{4\gamma}$ in order to keep the net 4-photon amplitude finite.

Next, consider the 2-photon amplitude $\Sigma^\gamma_{\mu\nu}(k)$. By Lorentz symmetry, a two-index-tensor valued function of a single vector k^λ must have form

$$\Sigma^\gamma_{\mu\nu}(k) \;=\; \Xi(k^2)\times g_{\mu\nu} \;-\; \Pi(k^2)\times k_\mu k_\nu \tag{51}$$

for some scalar functions Ξ and Π of k^2. The WT identities for this 2-photon amplitude read

$$\begin{aligned} 0 \;&=\; k^\mu \times \Sigma^\gamma_{\mu\nu} \;=\; \Xi(k^2) \times k_\nu \;-\; \Pi(k^2) \times k^2 k_\nu \,, \\ 0 \;&=\; k^\nu \times \Sigma^\gamma_{\mu\nu} \;=\; \Xi(k^2) \times k_\mu \;-\; \Pi(k^2) \times k^2 k_\mu \,, \end{aligned} \tag{52}$$

which calls for

$$\Xi(k^2) \;=\; \Pi(k^2) \times k^2, \tag{53}$$

and therefore

$$\Sigma^\gamma_{\mu\nu}(k) \;=\; \Pi(k^2) \times \bigl(k^2 g_{\mu\nu} \;-\; k_\mu k_\nu\bigr). \tag{54}$$

Note: this identity applies to both finite and infinite parts of the 2-photon amplitude.

In the previous section, we saw that the UV-divergent part of the 2-photon amplitude has form

$$\bigl[\Sigma^\gamma_{\mu\nu}(k)\bigr]_{\text{divergent}} \;=\; A \times g_{\mu\nu} \;+\; B \times k^2 \times g_{\mu\nu} \;+\; C \times k_\mu k_\nu \tag{45}$$

for some $O(\Lambda^2)$ constant A and $O(\log\Lambda)$ constants B and C. Applying the WT identity to this divergent part of the amplitude, we get

$$0 \;=\; k^\mu \times \bigl[\Sigma^\gamma_{\mu\nu}(k)\bigr]_{\text{divergent}} \;=\; A \times k_\nu \;+\; B \times k^2 k_\nu \;+\; C \times k^2 k_\nu \,, \tag{55}$$

hence $A = 0$, $C = -B$, and consequently

$$\bigl[\Sigma^\gamma_{\mu\nu}(k)\bigr]_{\text{divergent}} \;=\; B \times \bigl(k^2 g_{\mu\nu} \;-\; k_\mu k_\nu\bigr). \tag{56}$$

In other words, the divergent part of the 2-photon amplitude must respect eq. (54).

Altogether, we have

$$\Sigma^\gamma_{\mu\nu}(k) \;=\; \Pi(k^2) \times \bigl(k^2 g_{\mu\nu} \;-\; k_\mu k_\nu\bigr) \tag{54}$$

$$\text{for} \quad \Pi(k^2) \;=\; [O(\log\Lambda)\ \text{constant}] \;+\; \textit{finite}(k^2). \tag{57}$$

Note that while individual diagrams contributing to the two-photon amplitude may suffer from quadratic UV divergences, those leading divergences must cancel out from the net amplitude once we total up the diagrams. Only the sub-leading logarithmic divergence may survive this cancellation, and it must have a rather restrictive form (54)+(57).

To cancel this remaining divergence, we do not need the photon mass^2 counterterm δ_m^γ, or separate photon kinetic counterterms δ_{3A} and δ_{3B}, or any other counterterm missing from QED by reasons of gauge symmetry. Instead, all we need is the δ_3 counterterm which does exist in QED. Indeed,

$$\left[\Sigma^\gamma_{\mu\nu}(k)\right]^{\rm net} \;=\; \left[\Sigma^\gamma_{\mu\nu}(k)\right]^{\rm loops} \;-\; \delta_3\times(k^2g_{\mu\nu}-k_\mu k_\nu), \tag{58}$$

which agrees with eq. (54) for the net amplitude and makes for

$$\begin{aligned} \left[\Sigma^\gamma_{\mu\nu}(k)\right]^{\rm net} \;&=\; \Pi^{\rm net}(k^2)\times\left(k^2g_{\mu\nu}\,-\,k_\mu k_\nu\right)\\ \text{for}\quad \Pi^{\rm net}(k^2)\;&=\;\Pi^{\rm loops}(k^2)\;-\;\delta_3\,. \end{aligned} \tag{59}$$

Consequently, the UV-divergent constant term in eq. (57) for the $\Pi^{\rm loops}(k^2)$ can be canceled by a suitably adjusted δ_3 counterterms order-by-order in perturbation theory.

So here is the bottom line of this section: Thanks to the Ward–Takahashi identities, QED does not need the missing counterterms to cancel all of its UV divergences; all it needs are the counterterms δ_1, δ_2, δ_m, and δ_3 that the theory does have. And that's what makes QED a renormalizable theory.

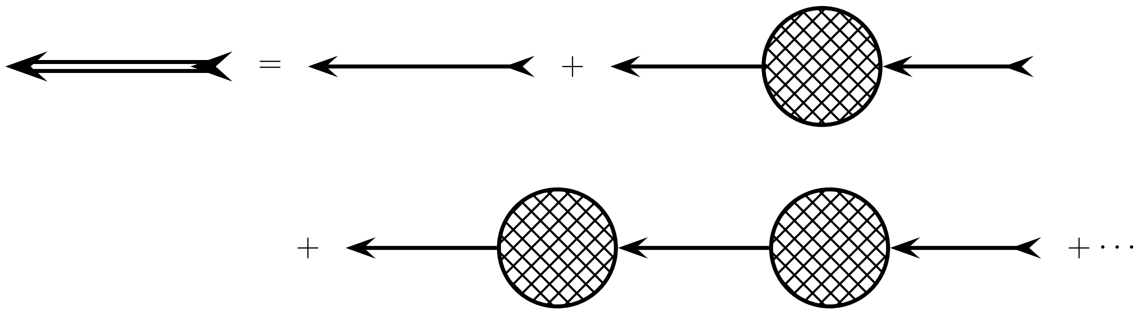
Dressed Propagators and Finite Parts of Counterterms

The infinite parts of QED counterterms δ_1, δ_2, δ_3, and δ_m follow from the requirement that they cancel the UV divergences of the loop graphs. The finite parts of the counterterms follow from the more subtle *renormalization conditions,* namely: m in the electron propagators is the physical electron mass, $-e$ is the physical electron charge, and $\hat\Psi(x)$ and $\hat A^\mu(x)$ are the renormalized quantum fields which create the physical electrons and photons from the vacuum with strength $=1$. Diagrammatically, the δ_2 and δ_m counterterms are related to the electron's *dressed propagator* — *i.e.*, 2-point correlation function, — the δ_3 counterterm is related to the photon's dressed propagators, and the δ_1 counterterm is related to the dressed electron-photon vertex. In this section, I shall address the dressed propagators; the dressed vertex and the δ_1 counterterm will be addressed in separate sets of notes — first here and then here.

Let's start with the electron's dressed propagators AKA Fourier transform of the 2-point correlation function

$$= \mathcal{F}_2^e(p) \;=\; \int d^4(x-y)\, e^{ip(x-y)} \mathcal{F}_2^e(x-y). \tag{60}$$

Diagrammatically, in terms of the 2-electron 1PI amplitude $\Sigma^e(\not{p})$,



(61)

which translates to

$$\begin{aligned} \mathcal{F}_2^e(p) \;=\; & \frac{i}{\not{p}-m+i0} \;+\; \frac{i}{\not{p}-m+i0}(-i\Sigma^e(\not{p}))\frac{i}{\not{p}-m+i0} \\ & +\; \frac{i}{\not{p}-m+i0}(-i\Sigma^e(\not{p}))\frac{i}{\not{p}-m+i0}(-i\Sigma^e(\not{p}))\frac{i}{\not{p}-m+i0} \;+\; \cdots . \end{aligned} \tag{62}$$

Note that every propagator or $\Sigma^e(\not{p})$ factor in this formula is a 4×4 matrix in Dirac indices, and their products are matrix products *in the order they are written down in eq.* (62) Fortunately, the 1PI amplitude $\Sigma^e(\not{p})$ is a power series in $\not{p}$, so as a matrix it commutes with $\not{p}$ and every function of $\not{p}$ such as the free propagator $i/(\not{p}-m+i0)$. This allows us to take products in eq. (62) in any order, so an N–bubble term amounts to

$$\left(\frac{i}{\not{p}-m+i0}\right)^{N+1} \times (-i\Sigma^e(\not{p}))^N \;=\; \frac{i}{\not{p}-m+i0} \times \left(\frac{\Sigma^e(\not{p})}{\not{p}-m+i0}\right)^N . \tag{63}$$

Totaling up all such N-bubble terms, we obtain the complete dressed propagator as

$$\begin{aligned} \mathcal{F}_2^e(p) \;&=\; \sum_{N=0}^{\infty} \frac{i}{\not{p}-m+i0} \times \left(\frac{\Sigma^e(\not{p})}{\not{p}-m+i0}\right)^N \\ &=\; \frac{i}{\not{p}-m+i0} \times \left[1 - \frac{\Sigma^e(\not{p})}{\not{p}-m+i0}\right]^{-1} \\ &=\; \frac{i}{\not{p}-m-\Sigma^e(\not{p})+i0} . \end{aligned} \tag{64}$$

Now consider the poles of the electron propagator. The free Dirac propagator

$$\frac{i}{\not p - m + i0} \;=\; \frac{i(\not p + m)}{p^2 - m^2 + i0} \tag{65}$$

has a pole at $p^2 = m^2$ with the residue $(\not p + m) = 2m \times$ (projection matrix onto the positive eigenvalue $+m$ of the Dirac matrix $\not p$). In other words, this is a pole at (eigenvalue of $\not p$) $= +m$ with residue $= 1$, — or less formally, a pole at $\not p = +m$ with unit residue. And that's the general behavior of the Dirac fields' propagators, free or dressed.

In particular, the dressed propagator for the electron field should have a pole at $\not p =$ physical electron mass, with the residue being the electron field strength factor. In the counterterm perturbation theory, the mass m in the free electron propagator should be equal to the electron's physical mass, so the dressed propagator should have a pole at the same point $\not p = m$ as the free propagator. Likewise, this pole should have a unit residue since the renormalized electron field should create physical electrons with strength $= 1$, thus

$$\mathcal{F}^e_2(p) \;=\; \frac{i}{\not p - m + i0} \;+\; \text{smooth@}(\not p = m). \tag{66}$$

In terms of the denominator of

$$\mathcal{F}^e_2(p) \;=\; \frac{i}{\not p - m - \Sigma^e(\not p) + i0} \tag{64}$$

this means

$$\text{for } \not p \text{ near } m: \;\; \not p - m - \Sigma^e(\not p) \;=\; 0 + 1\times(\not p - m) + O\big((\not p - m)^2\big) \quad\Longrightarrow\quad \Sigma^e(\not p) \;=\; O\big((\not p - m)^2\big) \tag{67}$$

and therefore

$$@(\not p = m) \quad \text{both} \quad \Sigma^e \;=\; 0 \quad \text{and} \quad \frac{d\Sigma^e}{d\not p} \;=\; 0. \tag{68}$$

The first condition here ($\Sigma^e(\not p = m) = 0$) is needed to keep the dressed propagator's pole at the same point $\not p = m$ as in the free propagator, while the second condition ($d\Sigma^e/d\not p = 0$) provides for the unit residue of that pole.

To be precise, the 1PI 2-electron amplitude in all the above formulae is the net amplitude comprising both the loop diagrams and the counterterms,

$$\Sigma^e_{\rm net}(\not p) \;=\; \Sigma^e_{\rm loops} \;+\; \delta_m \;-\; \delta_2 \times \not p. \tag{69}$$

And it is this net 1PI amplitude which must vanish — together with its first derivative — at $\not p = m$. Thus, we should have

$$\Sigma^e_{\rm loops} @ (\not p = m) \;+\; \delta_m \;-\; \delta_2 \times m \;=\; 0 \quad\text{and}\quad \frac{d\Sigma^e_{\rm loops}}{d\not p} @ (\not p = m) \;-\; \delta_2 \;=\; 0. \tag{70}$$

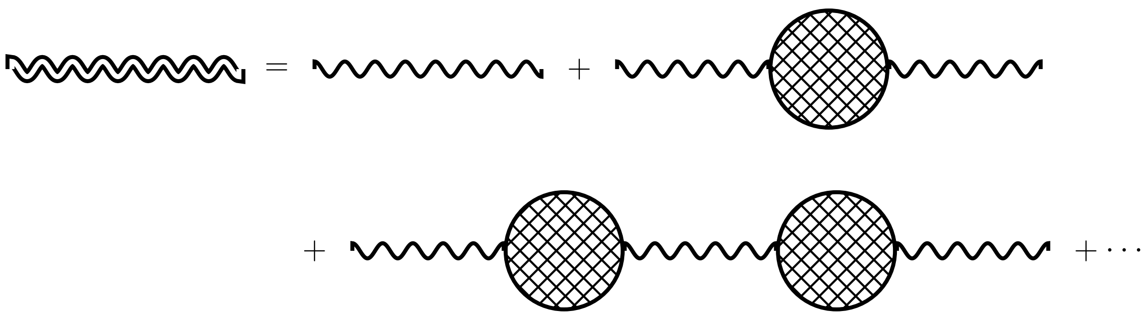
Together, these two equations completely determine the counterterms δ_2 and δ_m, including their finite parts.

⋆ ⋆ ⋆

Finally, consider the photon's dressed propagator, *i.e.*the Fourier transform of the photon's 2-point correlation function

$$= \mathcal{F}^{\mu\nu}_{2\gamma}(k) \;=\; \int\! d^4(x-y)\, e^{ik(x-y)} \mathcal{F}^{\mu\nu}_{2\gamma}(x-y). \tag{71}$$

Diagrammatically, in terms of the 2-photon 1PI amplitude $\Sigma^\gamma_{\mu\nu}(k)$,

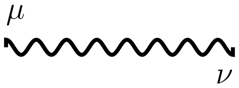


(72)

Note that every propagator and every 1PI bubble here bears two Lorentz indices, so multiplying them together involves a lot of index contractions. To simplify the algebra, let's treat the free propagators and the 1PI bubbles as 4×4 matrices and multiply them as matrices.

For consistency we raise the first Lorenz index of any such matrix and lower the second index, thus

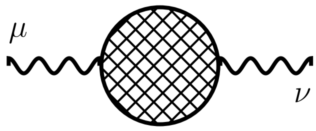
$$\text{prop}^{\mu}_{\ \nu} \ = $$



$$= \ \frac{-i}{k^2+i0}\left[\delta^\mu_\nu \,+\, (\xi-1)\frac{k^\mu k_\nu}{k^2+i0}\right] \tag{73}$$

and

$$[i\Sigma^\gamma(k)]^\mu_{\ \nu} \ = $$



$$= \ i\Pi(k^2)\times\bigl(k^2\delta^\mu_\nu \,-\, k^\mu k_\nu\bigr). \tag{74}$$

In these matrix notations, the series (72) amounts to

$$\|\mathcal{F}_{2\gamma}\| \ = \ \text{prop} \,+\, \text{prop}\times(i\Sigma^\gamma)\times\text{prop} \,+\, \text{prop}\times(i\Sigma^\gamma)\times\text{prop}\times(i\Sigma^\gamma)\times\text{prop} \,+\, \cdots. \tag{75}$$

To evaluate the matrix products here, let's introduce the *projector matrices*

$$[P_{\|}(k)]^\mu_{\ \nu} \ = \ \frac{k^\mu k_\nu}{k^2+i0} \tag{76}$$

which projects 4-vectors onto their components parallel to the k^μ, and

$$[P_{\perp}(k)]^\mu_{\ \nu} \ = \ \delta^\mu_\nu \ - \ \frac{k^\mu k_\nu}{k^2+i0} \tag{77}$$

which projects 4-vectors onto their components perpendicular to the k^μ. These two matrices are complementary projectors, which obey

$$P_{\|}P_{\|} \ = \ P_{\|}\,, \quad P_{\perp}P_{\perp} \ = \ P_{\perp}\,, \quad P_{\|}P_{\perp} \ = \ P_{\perp}P_{\|} \ = \ 0, \quad P_{\perp} \,+\, P_{\|} \ = \ 1. \tag{78}$$

In terms of these projector matrices, the free photon propagator matrix is

$$\text{prop} \ = \ \frac{-i}{k^2+i0}\times\bigl(P_{\perp} \,+\, \xi P_{\|}\bigr) \tag{79}$$

while the 1PI 2-photon bubble is

$$i\Sigma^\gamma(k) \ = \ i\Pi(k^2)\times k^2 P_{\perp}\,. \tag{80}$$

Note that the free propagator and the 1PI bubble commute with each other as matrices, so the N-bubble term in the series (72) amounts to

$$\begin{aligned}
N^{\text{th}}\text{ term} &= (\text{prop})^{N+1}\times(i\Sigma^\gamma)^N \\
&= \left(\frac{-i}{k^2+i0}\right)^{N+1}\left(P_\perp+\xi P_\|\right)^{N+1}\times\left(ik^2\Pi(k^2)\right)^N(P_\perp)^N \\
&= \frac{-i}{k^2+i0}\times\left(\Pi(k^2)\right)^N\times\left(P_\perp\,+\,\xi^{N+1}P_\|\right)\times\begin{cases}P_\perp & \text{for } N>0,\\ 1 & \text{for } N=0,\end{cases} \\
&= \frac{-i}{k^2+i0}\times\left(\Pi(k^2)\right)^N\times\begin{cases}P_\perp & \text{for } N>0,\\ P_\perp+\xi P_\| & \text{for } N=0.\end{cases}
\end{aligned}\tag{81}$$

Consequently, we may sum the whole series (72) as

$$\begin{aligned}
\|\mathcal{F}_{2\gamma}\| &= \sum_{N=0}^{\infty}\frac{-i}{k^2+i0}\times\left(\Pi(k^2)\right)^N\times\left(P_\perp\,+\,\xi\delta_{N,0}P_\|\right) \\
&= \frac{-i}{k^2+i0}\times\left(P_\perp\times\sum_{N=0}^{\infty}\left(\Pi(k^2)\right)^N\,+\,\xi P_\|\right) \\
&= \frac{-i}{k^2+i0}\times\left(\frac{P_\perp}{1-\Pi(k^2)}\,+\,\xi P_\|\right) \\
&= \frac{-i}{k^2+i0}\times\frac{1}{1-\Pi(k^2)}\times\left(P_\perp\,+\,\tilde{\xi}P_\|\right)
\end{aligned}\tag{82}$$

where $\tilde{\xi}=\xi\times(1-\Pi(k^2))$ is the quantum-corrected gauge fixing parameter.

Note that thanks to the Ward–Takahashi identity leading to $\Sigma^\gamma=\Pi\times k^2\times P_\perp$, the dressed photon propagator (82) automagically has a pole at $k^2=0$ (*cf.* the first factor on the bottom line of eq. (82)), the same place as the free photon propagator. In other words, the photon remains exactly massless, and we do not need extra counterterms to prevent the loop corrections from giving it a non-zero mass^2.

On the other hand, the residue of the pole at $k^2=0$ stems from the second factor on the bottom line of eq. (82),

$$\text{Residue}\ =\ \frac{1}{1-\Pi(k^2=0)}\,.\tag{83}$$

Thus, to make sure the renormalized EM field $\hat{A}^\mu(x)$ creates photons with strength $=1$, we need $\Pi(k^2)$ to vanish at $k^2 = 0$. Or rather,

$$\Pi^{\text{net}}(k^2) \;=\; \Pi^{\text{loops}}(k^2) \;-\; \delta_3 \;\longrightarrow\; 0 \text{ for } k^2 = 0, \tag{84}$$

and this is the equation which determines the finite part of the δ_3 counterterm. We shall see how this works at the one-loop level in the next set of my notes.