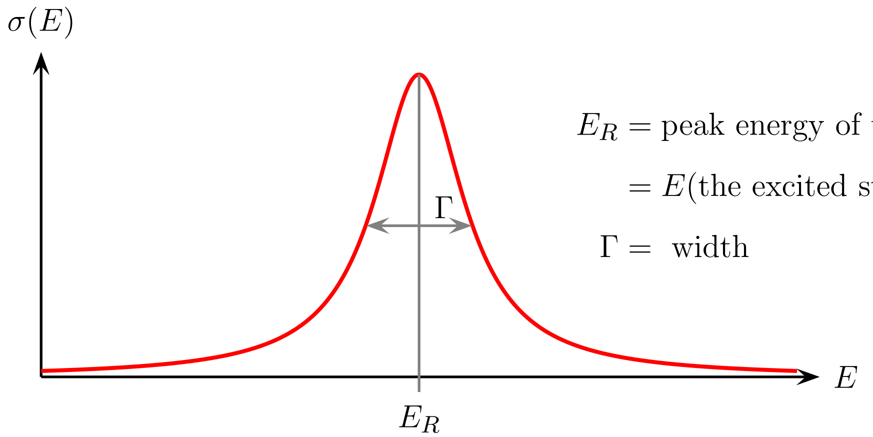
RESONANCES

In quantum mechanics, the resonances are closely related to the unstable quantum states. For example, consider an excited quantum state $|A^*\rangle$ of an atom. This excited state is unstable: After a while, the atom emits a photon and transitions to the ground state $|A\rangle$ or a less-excited state $|A'\rangle$. If the excited state lives long enough, we may observe it directly. But if its lifetime is too short, we see it only as a resonant peak in the scattering cross-section for the $A + \gamma \to A^* \to A + \gamma$ process, or perhaps also $A + \gamma \to A^* \to A' + \gamma'$:



$$\begin{aligned} E_R &= \text{peak energy of the resonance} \\ &= E(\text{the excited state}) \\ \Gamma &= \text{ width} \end{aligned} \tag{1}$$

Let's focus on the narrow resonances for which other interesting features of the $\sigma(E)$ curve — such as other resonances or thresholds — are much further away from E_R than the resonance's width Γ. Such narrow resonances generally have the *Breit–Wigner* profile of the scattering amplitude,

$$f(E) \;=\; \frac{\text{const}}{E - E_R + \frac{i}{2}\Gamma} \;+\; \text{smooth}(E), \tag{2}$$

and once they are made, they decay exponentially with lifetime $\tau = 1/\Gamma$. To see how this works, let the incoming state $|\text{in}\rangle$ before scattering be a wave packet of central energy $\langle E\rangle = E_R$ and energy width $\Delta E \gg \Gamma$. Thus, the time duration of the incoming wave packet — or rather, the time it spends in any particular place — is much shorter than $1/\Gamma$. Because of $\Delta E \gg \Gamma$, the energy profile of the outgoing scattered wave packet $|\text{out}\rangle$ follows the Breit–Wigner scattering amplitude (2),

$$\psi_{\text{out}}(E) \;=\; \frac{\text{slowly varying}(E)}{E - E_R + \frac{i}{2}\Gamma} \;\approx\; \frac{\text{const}}{E - E_R + \frac{i}{2}\Gamma}\,. \tag{3}$$

Fourier transforming this outgoing wave from energy-dependence to time-dependence, we obtain

$$\psi_{\text{out}}(t) \;=\; \exp(-iE_Rt)\times\begin{cases}0 & \text{for } t<0,\\ \exp(-\Gamma t/2) & \text{for } t>0.\end{cases} \tag{4}$$

In terms of probability $|\psi|^2$, this means that the scattering creates an excited state which then decays exponentially with time

$$\textit{survival\_probability}(t) \;=\; \exp(-\Gamma t) \;=\; \exp(-t/\tau), \quad \text{lifetime } \tau=\frac{1}{\Gamma}\,. \tag{5}$$

In quantum field theory, the resonances are related to unstable particles. Such particles are created in some collisions, live for a short time, and then decay back to the original particles or perhaps to some other decay products. If the unstable particle lives long enough we may actually detect it as a particle: it travels a couple of meters through the particle detector from the collision point to the calorimeter, and if it's charged we can actually observe its path in the ionization chamber. For shorter lived particles such as D mesons (made from a c quark and a $\bar{u}$, $\bar{d}$, or $\bar{s}$ antiquark) with picosecond lifetimes, we can observe a short displacement of a point where the particle decays from the point it was created. But for much shorter lifetimes, we never see the unstable particle as a particle, we only see the resonant cross-section peak in processes like $A+B\to R\to A+B$ or $A+B\to R\to X+Y+\cdots$. For example, the Δ baryon is observed as a strong resonance in pion-nucleon elastic scattering at peak center-of-mass energy $E_{\text{cm}}\approx 1220$ MeV.

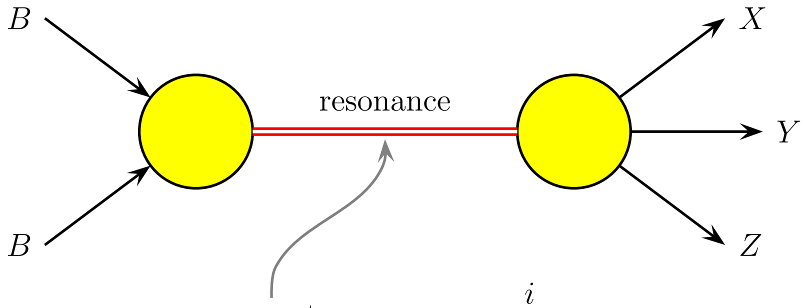
The relativistic form of a Breit–Wigner resonance in some $A+B\to R\to X+Y+\cdots$ process is

$$\mathcal{M}(E_{\text{cm}}) \;=\; \frac{\text{const}}{E_{\text{cm}}^2-M_R^2+iM_R\Gamma} \;+\; \text{smooth}(E_{\text{cm}}), \tag{6}$$

where the resonant term has a pole at complex

$$E_{\text{cm}}^2 \;=\; s \;=\; M_R^2\;-\;iM_R\Gamma \;\approx\; (M_R-\tfrac{i}{2}\Gamma)^2. \tag{7}$$

In terms of Feynman diagrams, this pole stems from the unstable particle's propagator,



$$\text{propagator} \;=\; \frac{i}{q^2 - M_R^2 + iM_R\Gamma}$$

For a stable particle's propagator in the s-channel, the amplitude would have a pole at a real $s = M^2$, but for an unstable particle AKA resonance, the pole moves below the real axis (in the complex s plane) to $s = M_R^2 - iM_R\Gamma$, so it is often said that *a resonance has a complex* mass$^2 = M_R^2 - iM_R\Gamma \approx (M_R - \frac{i}{2}\Gamma)^2$.

Now consider making an unstable particle / resonance R in some collision $A + B \to R$. Given the amplitude $\langle R|\,\mathcal{M}\,|A + B\rangle$, the cross-section obtains through the phase-space integral

$$\sigma \;=\; \frac{1}{4P_{\rm cm}E_{\rm cm}} \int \frac{d^3\mathbf{p}_R}{(2\pi)^3}\,\frac{1}{2E_R}\,|\mathcal{M}|^2\,(2\pi)^4\delta^{(4)}(p_R - p_A - p_B). \tag{8}$$

In the center of mass frame — which is also the rest frame of the final-state resonance — the delta-function becomes $(2\pi)^3\delta^{(3)}(\mathbf{p}_R) \times (2\pi)\delta(E_R - E_{\rm cm})$, so the momentum integral (8) evaluates to

$$\frac{|\mathcal{M}|^2}{2E_R} \times (2\pi)\delta(E_R - E_{\rm cm}) \;=\; \frac{|\mathcal{M}|^2}{2M_R} \times (2\pi)\delta(E_{\rm cm} - M_R), \tag{9}$$

hence

$$\sigma(A + B \to R) \;=\; \frac{|\mathcal{M}|^2}{8p_{\rm cm}M_R^2} \times (2\pi)\delta(E_{\rm cm} - M_R). \tag{10}$$

For a finite width resonance, the delta function here becomes a finite-width peak WRT energy,

$$
\begin{aligned}
(2\pi)\delta(E_{\rm cm} - M_R) \;&\to\; \frac{\Gamma}{(E_{\rm cm} - M_R)^2 + \frac{1}{4}\Gamma^2} \\
&\to\; \frac{4M_R^2\Gamma}{(s - M_R^2)^2 + (M_R\Gamma)^2} \;=\; \frac{4M_R^2\Gamma}{|s - M_R^2 + iM_R\Gamma|^2}\,.
\end{aligned}
\tag{11}
$$

thus

$$
\sigma(s) \;=\; \frac{|\mathcal{M}|^2}{2p_{\rm cm}M_R} \times \frac{M_R\Gamma}{|s - M_R^2 + iM_R\Gamma|^2}\,. \tag{12}
$$

When the unstable particle R is not detected as a particle but only as a resonance, what we measure are the cross-sections of processes like $A + B \to R \to X + Y + \cdots$. At the resonant energies, all such cross-sections follow from eq. (12) as

$$
\sigma_{\rm res}(A + B \to R \to X + Y + \cdots) \;=\; \sigma(A + B \to R) \times B(R \to X + Y + \cdots) \tag{13}
$$

where $B(R \to X + Y + \cdots)$ is the *branching ratio* for the resonance R decaying into the $X + Y + \cdots$ channel,

$$
B(R \to X + Y + \cdots) \;=\; \frac{\Gamma(R \to X + Y + \cdots)}{\Gamma_{\rm total}(R \to \text{anything})}\,. \tag{14}
$$

- Note: the branching ratios of resonance's decays do not depend on how the resonance is made, the processes $A + B \to R \to X + Y + \cdots$ and $C + D \to R \to X + Y + \cdots$ have exactly the same branching ratios $B(R \to X + Y + \cdots)$. (However, the overall cross-section $\sigma(A + B \to R \to X + Y + \cdots)$ and $\sigma(C + D \to R \to X + Y + \cdots)$ would be different due to different cross-sections $\sigma(A + B \to R)$ and $\sigma(C + D \to R)$ for making the resonance in the first place.)
- All collisions producing the resonance R by itself (rather than accompanied by other particles, $A + B \to R + Q + \cdots$) have the same resonant dependence on the net energy,

$$
\sigma(s) \;\propto\; \frac{1}{|s - M_R^2 + iM_R\Gamma|^2}\,, \tag{15}
$$

although the factors multiplying this resonant peak differ from channel to channel due to different amplitudes $\mathcal{M}(A + B \to R) \neq \mathcal{M}(C + D \to R)$.

- Making a resonance in some collision $A + B \to R$ and its decay back to the original particles are related by the time reversal symmetry. By the CPT theorem, this means that

$$\mathcal{M}(A + B \to R) \;=\; \mathcal{M}^*(\overline{R} \to \overline{A} + \overline{B}), \tag{16}$$

and if the interactions responsible for these processes happen to be **C** or **CP** symmetric, then

$$\mathcal{M}(A + B \to R) \;=\; \mathcal{M}^*(R \to A + B). \tag{17}$$

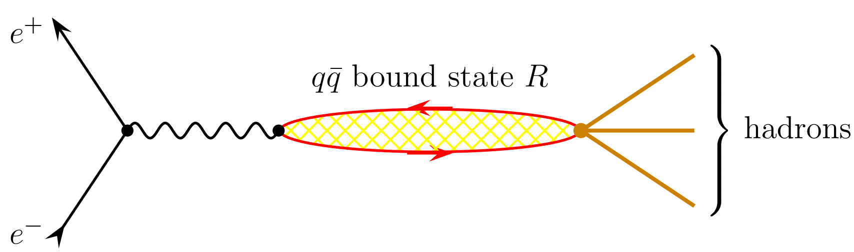
Thus, the cross-section $\sigma(A+B \to R)$ is related to the partial decay rate $\Gamma(R \to A+B)$ of the resonance back to the original particles, and hence to the branching ratio $B(R \to A+B)$. Putting all the phase-space factors together, we get the *Breit–Wigner formula* for the resonant cross-section,

$$\begin{aligned}\sigma(A + B \to R \to X + Y + \cdots) \;=\; &\frac{4\pi}{p_{\rm cm}^2} \times \frac{(2J_R + 1)}{(2J_A + 1)(2J_B + 1)} \times \frac{(M\Gamma)^2}{|s - M^2 + iM\Gamma|^2} \times \\ &\times B(R \to A + B) \times B(R \to X + Y + \cdots),\end{aligned} \tag{18}$$

where M is the resonance's mass and Γ is its *total* width.

Vector Resonances in e^-e^+ Collisions

Consider neutral mesons — bound states of a quark and and antiquark of the same flavor. Such a $q\bar{q}$ pair can be created in an electron-positron collision, and when q and $\bar{q}$ appear in a bound state (rather than parts of separate hadrons), this gives rise to a resonance in the $e^- + e^+ \to R \to$ hadrons reaction:



Note that the $q\bar{q}$ bound state in this diagram must have similar quantum numbers to the virtual photon which creates it. In particular, the virtual photon is a true vector (*i.e.*, has spin = 1 and negative parity) and is odd WRT charge conjugation, so the $q\bar{q}$ bound state must also have $J^{PC} = 1^{--}$. This does not forbid bound states with different quantum numbers — to the contrary, there are $q\bar{q}$ bound states with all kinds of J, P, and C. But only the states with $J^{PC} = 1^{--}$ may appear as resonances in electron-positron collisions; the states with different quantum numbers have to be made in other processes.

For a specific example, consider the J/ψ resonance at $E_{\rm cm} = 3097$ MeV.[⋆] J/ψ is a *charmonium* — a heavy meson made from a c (charm) quark and a $\bar{c}$ antiquark; specifically, J/ψ has net spin $S = 1$ while the spatial wave function $\psi(\mathbf{x}_{\rm relative})$ is in 1S state (lowest energy for $L = 0$). Together, $L = 0$ and $S = 1$ lead to $J^{PC} = 1^{--}$, which allows J/ψ to appear as a resonance in $e^- + e^+$ collisions. In general, the heavier the meson is, the faster it decays, but J/ψ is extraordinary narrow for its mass: its total decay rate $\Gamma = 93$ keV is 4.5 orders of magnitude smaller than the mass, $\Gamma/M \approx 3 \cdot 10^{-5}$. Indeed, the decay rate of J/ψ via strong interactions — basically, the annihilation of c and $\bar{c}$ into 3 gluons — is so small that the electromagnetic decays (via a virtual photon) to lepton pairs become competitive:

$$B(J/\psi \to e^- + e^+) \ \approx\ B(J/\psi \to \mu^- + \mu^+) \ \approx\ 6\% \tag{19}$$

while

$$B(J/\psi \to \text{hadrons}) \ \approx\ 88\%. \tag{20}$$

Let's take a closer look at the J/ψ resonance production in $e^- + e^+$ collisions. In the non-relativistic quark model approximation to the J/ψ meson, it has a wave function

$$\psi(\mathbf{x}_c - \mathbf{x}_{\bar{c}}, s_c, s_{\bar{c}}) \ =\ \psi_{\rm space}(r) \times \psi_{\rm spin}(s_x, s_{\bar{c}}), \tag{21}$$

so Fourier transforming the spatial wave function to the (reduced) momentum space, we may write the $|J/\psi\rangle$ state at rest as a superposition of free quark and antiquark with opposite

⋆ The double name J/ψ is due to simultaneous discovery in 1974 by two very different experiments: Samuel Ting's group — who called the resonance J — at the AGS proton accelerator at Brookhaven National Laboratory, and Burton Richter's group — who called the resonance ψ — at the SPEAR electron-positron collider at the Stanford Linear Accelerator Center.

momenta:

$$|J/\psi\rangle \;=\; \int\!\frac{d^3\mathbf{p}_{\rm red}}{(2\pi)^3}\,\tilde\psi(\mathbf{p}_{\rm red}) \sum_{s_c,s_{\bar c}} \psi_S(s_c,s_{\bar c}) \sum_i^{\rm colors} \frac{1}{\sqrt3}\times \left|c(+\mathbf{p}_{\rm red},s_c,i),\bar c(-\mathbf{p}_{\rm red},s_{\bar c},i)\right\rangle . \qquad (22)$$

Consequently, the QED amplitude for making this bound state in an electron-positron collision obtains by combining the QED amplitude for making a free quark+antiquark pair with the wave function of the bound state. Correcting for the relativistic normalization of the amplitudes, we have

$$\begin{aligned}\langle J/\psi|\,\mathcal{M}\,\big|e^-,e^+\big\rangle \;=\; \frac{\sqrt{2M_{J/\psi}}}{2m_c} \int\!\frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{\rm spins}\sum_i^{\rm colors} \tilde\psi(\mathbf{p})\times\psi_S({\rm spins})\times\frac{1}{\sqrt3}\,\times \\ \times \left\langle c(+\mathbf{p},s_c,i),\bar c(-\mathbf{p},s_{\bar c},i)\right|\mathcal{M}\,\big|e^-,e^+\big\rangle .\end{aligned} \qquad (23)$$

On the second line here is the QED amplitude for pair-production of a free quark and a free antiquark, which is exactly similar to the amplitude of the $\mu^-+\mu^+$ pair production we have studied in class. The only difference is the electric charge of the charm quark being $+\frac{2}{3}e$ instead of $-e$ charge of the muon; also, the quark and the antiquark must have matching colors. Thus,

$$\langle c,\bar c|\,\mathcal{M}\,\big|e^-,e^+\big\rangle \;=\; -\frac{2e^2}{3s}\times\bar v(e^+)\gamma_\nu u(e^-)\times\bar u(c)\gamma^\nu v(\bar c)\times\delta_{c,\bar c\,{\rm colors}} . \qquad (24)$$

Since the electron is much lighter than the charm quark, the initial electron and positron must be ultrarelativistic. Hence, as you should have seen in a recent homework, for opposite helicities of the electron and the positron

$$\bar v(e^+)\gamma^\nu u(e^-) \;=\; 2E(0,\pm i,1,0)^\nu \qquad (25)$$

while for similar helicities $\bar v(e^+)\gamma^\nu u(e^-)=0$. On the other hand, the quark and the antiquark are non-relativistic, hence

$$u(c)\;\approx\;\sqrt{m_c}\begin{pmatrix}\xi\\ \xi\end{pmatrix}, \quad v(\bar c)\;\approx\;\sqrt{m_c}\begin{pmatrix}+\eta\\ -\eta\end{pmatrix}, \qquad (26)$$

and therefore

$$\bar{u}(c)\gamma^0\, v(\bar{c}) \;=\; u^\dagger(c)v(\bar{c}) \;\approx\; 0 \tag{27}$$

while

$$\bar{u}(c)\vec{\gamma}\, v(\bar{c}) \;=\; u^\dagger(c)\gamma^0\vec{\gamma}v(\bar{c}) \;\approx\; m_c\,(\,\xi^\dagger \quad \xi^\dagger\,)\begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}\begin{pmatrix}+\eta\\ -\eta\end{pmatrix} \;=\; 2m_c(\xi^\dagger\vec{\sigma}\eta), \tag{28}$$

which depends on the quark's and antiquark's spins but not their momenta. In terms of the net spin vector $\mathbf{S}$ of the quark-antiquark system

$$\xi^\dagger(s_c)\vec{\sigma}\eta(s_{\bar{c}}) \;=\; -\sqrt{2}i\mathbf{S},$$

hence altogether, the amplitude for creating a free non-relativistic quark-antiquark pair is

$$\langle c,\bar{c}|\,\mathcal{M}\,\big|e^-,e^+\big\rangle \;=\; -\frac{2e^2}{3}\times\sqrt{2}(\pm S_x - iS_y)\times\delta_{c,\bar{c}\,\text{colors}}\,. \tag{29}$$

Plugging this formula into eq. (23) for the amplitude of creating the J/ψ resonance, we get

$$\begin{aligned}\langle J/\psi|\,\mathcal{M}\,\big|e^-,e^+\big\rangle \;&=\; \frac{2\sqrt{2}e^2}{3\sqrt{m_c}}\times(\mp S_x + iS_y)\times\left(\frac{3}{\sqrt{3}}\right)_{\text{colors}}\times\\ &\times\int\frac{d^3\mathbf{p}}{(2\pi)^2}\,\tilde{\psi}(\mathbf{p})\times 1\end{aligned} \tag{30}$$

where the integral on the second line is simply the spatial wave function at the $\mathbf{x}_{\text{relative}} = 0$. Altogether,

$$\langle J/\psi|\,\mathcal{M}\,\big|e^-,e^+\big\rangle \;=\; \frac{2\sqrt{2}e^2}{\sqrt{3m_c}}\times\psi_{\text{space}}(0)\times(\mp S_x + iS_y)\times\begin{cases}0 & \text{for } \lambda(e^+) = \lambda(e^-),\\ 1 & \text{for } \lambda(e^+) \neq \lambda(e^-).\end{cases} \tag{31}$$

Finally, summing this amplitude or rather $|\mathcal{M}|^2$ over the J/ψ spin states and averaging over the electron's and positron's helicities, we get

$$\overline{|\mathcal{M}|^2} \;=\; \frac{8e^4}{3m_c}\,|\psi_{\text{space}}(0)|^2, \tag{32}$$

and hence cross-section

$$\sigma(e^- + e^+ \to J/\psi) \;=\; \frac{2e^4|\psi(0)|^2}{3m_c^3}\times\frac{M\Gamma}{|s - M^2 + iM\Gamma|^2}\,. \tag{33}$$